Contour Surfaces, Streamlines

Definition Let $\phi(X)$ be a continuously differentiable function of the n-vector X. Then the $(n-1)$-dimensional *surfaces*

$$\phi(X) = \phi(x_1, x_2, ..., x_n) = c,$$

where c is an arbitrary constant, are the *contour surfaces*, or *level surfaces* of the function ϕ. If $n = 2$ these surfaces reduce to curves in 2-dimensional space. For some values of c these surfaces or curves may be empty (if there are no solutions of the equation for that value of c) or may reduce to a finite set of discrete points.

Example 1 Let $\phi(x, y, z) = x^2 + 4y^2 + 9z^2$. Then for each $c \geq 0$ the equation

$$x^2 + 4y^2 + 9z^2 = c$$

describes an elliptical contour surface in $\mathbf{R}^3$. When $c = 0$ the surface degenerates to the single point $(0, 0, 0)$.

It can be shown that if X_0 is a point where $\phi(X_0) = c$, for a given constant c, and if the gradient $\nabla\phi(X_0)$ is not the zero vector, then the solutions X of $\phi(X) = c$ near X_0 trace out a genuine $n-1$-dimensional surface in $\mathbf{R}^n$ near the point X_0. In the example above the gradient of $\phi(x, y, z)$ reduces to the zero vector only at the point $(0, 0, 0)$.

Definition Let $Y = F(X)$ denote a vector field in $\mathbf{R}^n$. Then the solutions $X(t)$ of the vector differential equation $X'(t) = F(X(t))$ are curves in $\mathbf{R}^n$, called *streamlines* of the vector field $F(X)$.

Remark The vector differential equation $X'(t) = F(X(t))$ is, of course, just an abbreviation for the system of n (in general, coupled) differential equations

$$x_i'(t) = f_i(x_1(t), x_2(t), ..., x_n(t)),\ i = 1, 2, ..., n.$$

Examples 2 In the case of the gradient field described earlier, where

$$F(x,y,z) = \nabla\phi\,(x,y,z)^* = \begin{pmatrix} 2x \\ 8y \\ 18z \end{pmatrix},$$

the streamlines are the solutions of the (uncoupled) differential equations

$$x' = 2x,\ y' = 8y,\ z' = 18z.$$

On the other hand, for

$$F(x,y,z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix},$$

the streamlines satisfy the highly coupled differential equations

$$x' = yz,\ y' = xz,\ z' = xy.$$

Proposition The streamlines of a scalar function $\phi(X)$, also called the paths of steepest ascent for $\phi(X)$, i.e., the curves in $\mathbf{R}^n$ which are solutions of the differential equation $X' = \nabla\phi(X)^*$, are normal to the contour surfaces $\phi(X) = c$ at the points where they meet those surfaces.

Proof Let $\Xi(s) = (\xi_1(s), \xi_2(s), ..., \xi_n(s))$ be a differentiable parametrized curve, with parameter s (not necessarily arc length here) lying in a contour surface $\mathcal{C} : \phi(X) = c$. Then we have, for all s,

$$\phi(\Xi(s)) \equiv c.$$

Differentiating this identity and using the *Chain Rule* for functions of several variables we obtain

$$\frac{\partial\phi}{\partial x_1}(\Xi(s))\xi_1'(s) + \frac{\partial\phi}{\partial x_2}(\Xi(s))\xi_2'(s), ..., \frac{\partial\phi}{\partial x_n}(\Xi(s))\xi_n'(s) \equiv 0,$$

that is:

$$\nabla\phi(\Xi(s)) \cdot \Xi^{'}(s) \equiv 0.$$

The vector $\Xi'(s)$ is everywhere tangent to the curve traced out by $\Xi(s)$; hence everywhere tangent to $\mathcal{C}$. Considering all of the possible curves $\Xi(s)$ passing through a particular point X_0 on $\mathcal{C}$, the vector $\nabla\phi(X_0)$ must be orthogonal to the tangent vectors generated at X_0 by all of these curves; hence $\nabla\phi(X_0)$ is orthogonal to the surface $\mathcal{C}$ at $X_0 \in \mathcal{C}$.

The Tangent Plane to a Surface Let $\phi(X)$ be a continuously differentiable scalar valued function of the n-vector variable X. Let us suppose X_0 is a point on the contour surface $\mathcal{C} : \phi(X) = c$ and suppose further that $\nabla\phi\,(X_0) \neq 0$. We define the *tangent plane* to $\mathcal{C}$ at X_0 to be the set of points

$$\{X \,|\, \nabla\phi(X_0)\,(X - X_0)\} = 0.$$

Example 3 We return to $\phi(x, y, z) = x^2 + 4y^2 + 9z^2$ and consider the contour surface $\phi(x, y, z) = 14$. Clearly the point $(1, 1, 1)$ lies on this surface. The gradient at this point is

$$\nabla\phi(1, 1, 1) = \begin{pmatrix} 2x & 8y & 18z \end{pmatrix}_{x=y=z=1} = \begin{pmatrix} 2 & 8 & 18 \end{pmatrix}.$$

Thus the tangent plane to the surface $\phi(x, y, z) = 14$ at $(1, 1, 1)$ is described by

$$\begin{pmatrix} 2 & 8 & 18 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = 0,$$

or $2x + 8y + 18z = 28$, i.e., $x + 4y + 9z = 14$.