Arc Length; Integration with Respect to Arc Length

Arc Length Let us suppose we have a curve in $\mathbb{R}^n$:

$$C : X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad a \leq t \leq b,$$

$X(t)$ being continuously differentiable with respect to $t$. If we fix an initial value of $t$, say $t = a$, and define, for general $t$,

$$s(t) = \int_a^t \|X'(\tau)\| \, d\tau,$$

then $s(t)$ is the arc length on $C$ between $X(a)$ and $X(t)$. Much of the work to follow relies on the use of $s$ as an alternative parameter for $C$. By the Fundamental Theorem of Calculus $s(t)$ is also continuously differentiable with respect to $t$ and

$$\frac{ds}{dt} = \|X'(t)\|, \quad \frac{dt}{ds} = \frac{1}{\|X'(t)\|}, \quad X'(t) \neq 0.$$

The unit tangent vector $T(t)$, defined where $X'(t) \neq 0$, is given by

$$T(t) = X'(t) \frac{1}{\|X'(t)\|} = \frac{dX(t)}{dt} \frac{dt}{ds} = \frac{d}{ds} X(t(s))$$

and is thus the rate of change of $X(t) = X(t(s))$ with respect to the arc length parameter $s$. It should be noted that this is always a unit vector.

Integration with Respect to Arc Length Suppose further that we have a scalar valued function $\rho(X)$ defined and continuous in a region $\mathcal{R} \subset \mathbb{R}^n$ which contains $C$. When we write

$$\int_C \rho(X) \, ds,$$
what we mean is the integral of $\rho(X)$ over the curve $C$ with respect to the arc length parameter, $s$, on $C$. We recall that if the curve $C$ is parametrized by $t$ as indicated, and if we arbitrarily set the arc length equal to zero at $t = a$, i.e., $s(a) = 0$, then for $t \geq a$ we have

$$s(t) = \int_a^t \|X'(u)\| \, du$$

or

$$s'(t) = \|X'(t)\| = \sqrt{(x'_1(t))^2 + (x'_2(t))^2 + \cdots + (x'_n(t))^2}.$$

Then we define the arc length integral:

$$\int_C \rho(X(s)) \, ds = \int_a^b \rho(X(t)) \frac{ds}{dt} \, dt = \int_a^b \rho(X(t)) \|X'(t)\| \, dt.$$

Generally the last formula appearing here is the more suitable one for practical computation because the formulae defining the curve $C$ are ordinarily given in terms of the original parameter, $t$, rather than in terms of $s$. It can be seen without great difficulty that the integral with respect to arc length is independent of the particular parametrization of $C$ as $X(t)$.

**Example 1** Suppose a wire, weighing three grams per centimeter of length, is coiled into a helix

$$x(t) = 4 \cos t$$

$$\begin{align*}
C : \quad y(t) &= 4 \sin t, \quad 0 \leq t \leq 4\pi. \\
z(t) &= t
\end{align*}$$

What is the weight of the wire coil?

**Solution** What we need here is the arc length integral $\int_C \rho(X) \, ds$, where $\rho(X) \equiv 3$ and $s$ denotes the arc length in centimeters, measured from one end, say the end corresponding to $t = 0$. Since

$$\frac{ds}{dt} = \sqrt{16 \sin^2 t + 16 \cos^2 t + 1} = \sqrt{17},$$

we have
the required weight is
\[ \int_0^{4\pi} 3 \cdot \sqrt{17} \, dt = 12\pi\sqrt{17}. \]

**Example 2**  Consider another wire, now bent into a planar curve in \( \mathbb{R}^2 \) described by
\[ y = x^2, \quad 0 \leq x \leq 4 \]
and tapered so that the density is \( \rho = \rho(x) = x \).

**Solution**  Here we will take the coordinate \( x \) to be the parameter, so we can write
\[ C : \quad x = x, \quad y = x^2. \]
Clearly then
\[ s(x) = \int_0^x \sqrt{1 + (2\xi)^2} \, d\xi; \quad \frac{ds}{dx} = \sqrt{1 + 4x^2}. \]
Then the weight is
\[ \int_0^4 x\sqrt{1 + 4x^2} \, dx. \]
Taking \( r = x^2 \) we have \( x = \frac{1}{2} \frac{dr}{dx} \) and the integral becomes
\[ \frac{1}{2} \int_0^{16} \sqrt{1 + 4r} \, dr = \frac{1}{12} (1 + 4r)^{3/2} \Big|_0^{16} = \frac{1}{12} (65^{3/2} - 1^{3/2}). \]

Suppose a wire has a density equal to \( \rho(X) \) mass units per unit length and is bent into the shape of a curve \( C : X = X(t) \). The center of gravity of such a wire is the point \( \hat{X} \) such that
\[ \int_C \rho(X)(X - \hat{X}) \, ds = 0 \longrightarrow M\hat{X} = \int_C X\rho(X) \, ds, \]
where \( M = \int_C \rho(X) \, ds \) is the mass of the wire. In particular, then, in those cases where the density \( \rho(X) \) can be written in terms of the arc length parameter \( s \) as \( \rho(s) \), we have
\[ \hat{x} = \frac{1}{M} \int_{s=0}^{s=L} x(s)\rho(s) \, ds, \quad \hat{y} = \frac{1}{M} \int_{s=0}^{s=L} y(s)\rho(s) \, ds. \]
Example 3  Consider a wire of unit density per unit length bent into a planar semicircle of radius 2. Where is its center of gravity?

Solution  It is clear that the mass of the wire is equal to its length, $2\pi$. Taking $x(\theta) = 2\cos \theta$, $y(\theta) = 2\sin \theta$, we have

$$\hat{x} = \frac{2}{2\pi} \int_0^\pi \cos \theta \cdot 2 \, d\theta = 0$$

while

$$\hat{y} = \frac{2}{2\pi} \int_0^\pi \sin \theta \cdot 2 \, d\theta = \frac{2}{\pi} \int_0^\pi \sin \theta \, d\theta = \frac{4}{\pi}.$$