Some Properties of Curves

Let us suppose we have a parametrized curve in R^3 given by

$$X(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

We recall then that the velocity and acceleration vectors are

$$X'(t) = (x'(t), y'(t), z'(t)) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k},$$

$$X''(t) = (x''(t), y''(t), z''(t)) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}.$$

We have seen that when $X'(t) \neq 0$ the vector $X'(t)$ is tangent to the curve $\mathcal{C}$ which is the range of $X(t)$. The corresponding *unit tangent vector* is

$$T(t) = X'(t)/\left\|X'(t)\right\|.$$

If we fix an initial value of t, say $t = a$, and define, for general t,

$$s(t) = \int_a^t \|X'(u)\|\, du,$$

then $s(t)$ is the *arc length* on $\mathcal{C}$ between $X(a)$ and $X(t)$. Much of the work to follow relies on the use of s as an alternative parameter for $\mathcal{C}$. By the *Fundamental Theorem of Calculus*, if we assume $X(t)$ continuously differentiable with respect to t, $s(t)$ is also continuously differentiable with respect to t and

$$\frac{ds}{dt} = \left\|X'(t)\right\|, \ \frac{dt}{ds} = \frac{1}{\|X'(t)\|}, \ X'(t) \neq 0.$$

Then

$$T(t) = X'(t)\frac{1}{\|X'(t)\|} = \frac{dX(t)}{dt}\frac{dt}{ds} = \frac{d}{ds}X(t(s))$$

is the rate of change of $X(t) = X(t(s))$ with respect to the arc length parameter s. It should be noted that this is always a *unit vector*.

We use the following example to motivate our definition of *curvature*.

Example 1 Let

$$X(t) = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \\ 0 \end{pmatrix}; \; X'(t) = \begin{pmatrix} -\omega R\sin(\omega t) \\ \omega R\cos(\omega t) \\ 0 \end{pmatrix}.$$

Clearly $\|X'(t)\| = \omega R$, so

$$T(t) = \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}.$$

Since we then also have $\frac{ds}{dt} = \|X'(t)\| = \omega R$, taking the initial point for arc length computation to be $X(0)$ (i.e., $a = 0$), we have

$$s = \omega R t, \; t = \frac{s}{\omega R}$$

and thus, as a function of s,

$$T(s) = \begin{pmatrix} -\sin(\frac{s}{R}) \\ \cos(\frac{s}{R}) \\ 0 \end{pmatrix} \rightarrow \frac{dT(s)}{ds} = \frac{1}{R}\begin{pmatrix} -\cos(\frac{s}{R}) \\ -\sin(\frac{s}{R}) \\ 0 \end{pmatrix}$$

and therefore $\|\frac{dT(s)}{ds}\| = \frac{1}{R}$. Thus in the case of this circle in the x, y plane, $\|\frac{dT(s)}{ds}\|$ works out to be the *reciprocal of the radius* of the circle. By definition, in the case of circles, the reciprocal of the radius is the *curvature* of the circle. For general curves in R^3 it is not so easy to see what we mean by the radius. We therefore use the left hand side of the equation $\|\frac{dT(s)}{ds}\| = \frac{1}{r}$, rather than the (undefined) right hand side to define curvature in general.

Definition If $X(t)$ is a parametrized curve in R^3 and $s(t) = \int_a^t \|X'(u)\| \, du$, for an arbitrary value of a, is the arc length parameter along the curve $\mathcal{C}$ represented by $X(t)$, the *curvature* of $\mathcal{C}$ at a point $X(t)$ on $\mathcal{C}$ is defined by

$$\kappa(t) = \|\frac{dT(s)}{ds}\|_{s=s(t)}.$$

To obtain a more usable expression, we note that, since $\frac{ds}{dt} = \|X'(t)\|$, we have

$$\Big\|\frac{dT(s)}{ds}\Big\|_{s\,=\,s(t)} = \Big\|\frac{d}{ds}\Big(\frac{X'(t(s))}{\|X'(t(s))\|}\Big)\Big\|$$

$$= \Big\|\frac{1}{\|X'(t)\|}\frac{d}{dt}\Big(\frac{X'(t)}{\|X'(t)\|}\Big)\Big\|.$$

It is clear that

$$\frac{d}{dt}\Big(\frac{X'(t)}{\|X'(t)\|}\Big) = \frac{\|X'(t)\|\,X''(t) - X'(t)\frac{d}{dt}\,\|X'(t)\|}{\|X'(t)\|^2}.$$

A rather tedious calculation shows the norm of the numerator to be $\|X'(t) \times X''(t)\|$, yielding the result

$$\kappa(t) = \frac{\|X'(t) \times X''(t)\|}{\|X'(t)\|^3}.$$

Example 2 Let us consider the helix $X(t) = (\cos\, t, \sin\, t, at)$. Then

$$X'(t) = \begin{pmatrix} -\sin\, t \\ \cos\, t \\ a \end{pmatrix},\; X''(t) = \begin{pmatrix} -\cos\, t \\ -\sin\, t \\ 0 \end{pmatrix}$$

and therefore

$$X'(t) \times X''(t) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\, t & \cos\, t & a \\ -\cos\, t & -\sin\, t & 0 \end{pmatrix}$$

$$= a\,\sin\, t\mathbf{i} - a\,\cos\, t\mathbf{j} + (\sin^2 t + \cos^2 t)\mathbf{k} = a\,\sin\, t\mathbf{i} - a\,\cos\, t\mathbf{j} + \mathbf{k}.$$

The norm of this vector is $\sqrt{1 + a^2 \sin^2 t + a^2 \cos^2 t} = \sqrt{1 + a^2}$. On the other hand $\|X'(t)\| = \sqrt{\sin^2 t + \cos^2 t + a^2} = \sqrt{1 + a^2}$ and therefore

$$\kappa(t) \equiv \frac{\sqrt{1 + a^2}}{\sqrt{1 + a^2}^3} = \frac{1}{1 + a^2}.$$

It should be noted in this case that the *speed* $\|X'(t)\|$ is constant but the *velocity vector* $X'(t)$ is not. As a result the acceleration vector is not 0.

Proposition The unit tangent vector $T(t) = X'(t)/\|X'(t)\|$ has the property $T(t) \cdot T'(t) = 0$.

Proof We simply note that

$$T(t) \cdot T(t) = \|T(t)\|^2 \equiv 1 \Rightarrow \frac{d}{dt}(T(t) \cdot T(t)) = T(t) \cdot \frac{dT(t)}{dt} + \frac{dT(t)}{dt} \cdot T(t)$$
$$= 2\,T(t) \cdot T'(t) = 0 \Rightarrow T(t) \cdot T'(t) = 0.$$

Thus $T'(t)$ is *normal* to the curve $\mathcal{C}$ at the point $X(t)$ in the sense that it is orthogonal to the tangent vector $T(t)$ there. We define

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

to be the *unit normal* to $\mathcal{C}$ at the point $X(t)$ on $\mathcal{C}$. Then we note that

$$N(t) = \frac{\frac{dT(s)}{ds}\frac{ds}{dt}}{\left\|\frac{d}{dt}\left(\frac{X'(t)}{\|X'(t)\|}\right)\right\|} = \frac{\frac{dT(s)}{ds}\|X'(t)\|}{\left\|\frac{d}{dt}\left(\frac{X'(t)}{\|X'(t)\|}\right)\right\|}$$
$$= \left\|\frac{1}{\|X'(t)\|}\frac{d}{dt}\left(\frac{X'(t)}{\|X'(t)\|}\right)\right\|^{-1}\frac{dT}{ds} = \frac{1}{\kappa}\frac{dT}{ds} = \rho\,\frac{dT}{ds}.$$

Theorem The acceleration vector has the decomposition

$$X''(t) = \frac{dv(t)}{dt}\,T(t) + \kappa(t)\,v(t)^2\,N(t),$$

where $v(t)$ is the speed, $\|X'(t)\|$.

Proof Since $T(t) = \frac{X'(t)}{\|X'(t)\|} = \frac{1}{v(t)}\,X'(t)$ we have $X'(t) = v(t)\,T(t)$. Then

$$X''(t) = \frac{d}{dt}X'(t) = \frac{d}{dt}(v(t)T(t)) = \frac{dv}{dt}T + v\,\frac{dT}{dt} = \frac{dv}{dt}T + v\,\frac{ds}{dt}\,\frac{dT}{ds}$$

$$= \frac{dv}{dt} T + v \, \|X'(t)\| \, \frac{dT}{ds} = \frac{dv}{dt} T + v^2 \frac{dT}{ds} = \frac{dv}{dt} T + \frac{v^2}{\rho} N,$$

which is the desired result.

Example 3 A roulette wheel is two feet in diameter. A ball is rolling along its rim, in the counter-clockwise direction, at a speed of 10 ft. per second and is slowing down at the rate of 2 ft. per sec.^{2}. Describe the acceleration vector.

Solution Let us assume coordinates are chosen so that, at the instant in question, the ball is located at the point $(1, 0)$ in the x, y plane. Then the unit tangent vector in the (assumed) counterclockwise direction of motion is $T(0) = (0, 1)$. The x component of the acceleration vector is clearly negative, so we must have $N(0) = (-1, 0)$. Since

$$\frac{dv}{dt}(0) = -2, \;\; v(0) = 10, \;\; \kappa = \frac{1}{R} = 1, \;\; \rho = \frac{1}{\kappa} = 1,$$

we have, from the formula given earlier,

$$X''(0) = \frac{dv}{dt}(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{10^2}{1} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} -100 \\ 0 \end{pmatrix} = - \begin{pmatrix} 100 \\ 2 \end{pmatrix}.$$

Another vector sometimes used is the *binormal* which is defined as

$$B(t) = T(t) \times N(t).$$

Since $T(t)$ and $N(t)$ are orthogonal, $\|B(t)\| = \|T(t)\| \, \|N(t)\| \, |\cos \theta| = 1^3 = 1$ and we see that $B(t)$ is also a unit vector. The orthonormal vector set $\{T(t), N(t), B(t)\}$ moves continuously along the curve $\mathcal{C}$ as t varies throughout its range.