Vector Functions of a Vector Variable

General Definition of a Vector Function A vector function F is a mathematical rule of correspondence, expressed via $Y = F(X)$, which assigns to each vector X in a region $\mathcal{D}(F) \subset \mathbf{R}^n$, called the **domain** of the function F, a vector Y in $\mathbf{R}^m$. The set of all vectors thus obtained as X varies in $\mathcal{D}(F)$ is the **range** of F, denoted by $\mathcal{R}(F)$. When it is not necessary to specify $\mathcal{D}(F)$ or $\mathcal{R}(F)$ precisely, we can write $F : \mathbf{R}^n \to \mathbf{R}^m$ to indicate the dimensions of the vectors involved.

The vector equation

$$Y = F(X)$$

is a mathematical "shorthand" expression for the more detailed relationships satisfied by the components of X and Y:

$$\begin{array}{rcl} y_1 & = & f_1(x_1, x_2, ..., x_n) \\ y_2 & = & f_2(x_1, x_2, ..., x_n) \\ & \vdots & \\ y_m & = & f_m(x_1, x_2, ..., x_n) \end{array} ,$$

which expresses the manner in which each component, y_i, of Y depends on the components $x_j,\ j = 1, 2, ..., n$, of X.

Example 1 A function $F : \mathbf{R}^3 \to \mathbf{R}^2$ can be defined by

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{x_1^2 + x_2^2 + x_3^2 - 1}\right) \\ \sin\left(\sqrt{x_1^2 + x_2^2 + x_3^2 - 1}\right) \end{pmatrix} = F(X).$$

In this case $m = 2,\ n = 3,\ \mathcal{D}(F) = \{X \in \mathbf{R}^3 \,|\, \|X\| \geq 1\}$ and $\mathcal{R}(F) = \{Y \in \mathbf{R}^2 \,|\, \|Y\| = 1\}$.

Important Special Cases

a) m = 1; this is the traditional **scalar-valued** function of several variables:

$$y(\in \mathbf{R}^1) = f(x_1, x_2, ..., x_n) = f(X).$$

An example, for $n = 3$, is $y = x_1^2 + x_2^2 + x_3^2 - (x_1 + x_2 + x_3)^2$. In the very special case where n also is equal to 1 we have the standard scalar function of a scalar variable studied in elementary calculus.

b) n = 1; here $X = x$ is scalar but, in general, Y can have any dimension. In this case it is common to replace x by t and write $Y = F(t)$, by which we mean

$$Y(\in \mathbf{R}^m) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_m(t) \end{pmatrix} = F(t).$$

This corresponds to a **parametrized curve** in $\mathbf{R}^m$, t being the parameter in question.

c) m = n > 1; in this case we have

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, ..., x_n) \\ f_2(x_1, x_2, ..., x_n) \\ \vdots \\ f_n(x_1, x_2, ..., x_n) \end{pmatrix} = F(X).$$

This case is referred to as a **vector field** in $\mathbf{R}^n$; the study of such vector fields is a central topic of these notes.

Example 2 $y_1 = x_1 x_2, \; y_2 = x_1^2 - x_2^2$.

Case b) again: Curves in Space An m-vector valued function $Y = F(t)$, defined for t in some interval $a \leq t \leq b$ (a or b could be replaced by $-\infty$ or ∞, respectively, and $\leq$ by $<$) is a **parametrized curve** in $\mathbf{R}^m$. The range

$$\mathcal{R}(F) = \{Y \in \mathbf{R}^m \,|\, Y = F(t), t \in [a,b]\}$$

is a **curve** in $\mathbf{R}^m$. In some special "degenerate" cases it may reduce to a single point.

Examples 3 We have already mentioned the first example:

i) $Y = F(t) = P + t\,(Q - P) = t\,Q + (1-t)\,P$; a *straight line* passing through the points P and Q.

ii) A *circle* in $\mathbf{R}^2$, centered at the origin:

$$\begin{array}{rcl} y_1 &=& r\,\cos(a\,t + b), \\ y_2 &=& r\,\sin(a\,t + b), \end{array} \quad a \neq 0, \;\; r \geq 0, \;\; b\,\text{real}.$$

Note that this circle collapses to a point when $r = 0$.

iii) A *helix* in $\mathbf{R}^3$:

$$y_1 = r\,\cos(a\,t + b); \;\; a \neq 0, \, r > 0, \; b\,\text{arbitrary};$$

$$y_2 = r\,\sin(a\,t + b);$$

$$y_3 = c\,t + d, \;\; c \neq 0, \; d\,\text{arbitrary}.$$

The same curve in $\mathbf{R}^m$ may have many different parametrizations. In i) the roles of P and Q can be interchanged, for example; in ii) we obtain the same circle for any choice of a and b, as long as $a \neq 0$.

Differentiation of Vector Functions of a Scalar Variable

Suppose we have a parametric curve in $\mathbf{R}^n$:

$$Y = F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

We say that $F(t)$ is differentiable at t_0 if the vector limit

$$\lim_{t \to t_0} \frac{1}{t - t_0}(F(t) - F(t_0))$$

exists. If this is the case, the limit is denoted by $F^{'}(t_0)$; it is the (vector) derivative of $F(t)$ at $t = 0$. Clearly, this is the case if and only if

$$\lim_{t \to t_0} \frac{1}{t - t_0}(f_k(t) - f_k(t_0))$$

exists for $k = 1, 2, ..., n$, i.e., each of the component functions of $F(t)$ is differentiable at $t = t_0$. Thus differentiation of vector functions of a scalar variable is performed **componentwise**.

Example 4 If

$$F(t) = \begin{pmatrix} r\,\cos(a\,t + b) \\ r\,\sin(a\,t + b) \\ c\,t + d \end{pmatrix} \text{ then } F'(t) = \begin{pmatrix} -a\,r\,\sin(a\,t + b) \\ a\,r\,\cos(a\,t + b) \\ c \end{pmatrix}.$$

If a curve, $\mathcal{C}$, in $\mathbf{R}^m$, is given parametrically by $Y = F(t)$, and if $Y_0 = F(t_0)$ is a point on $\mathcal{C}$ where $F'(t_0) \neq 0$, we say that the vector $F'(t_0)$ is **tangent** to $\mathcal{C}$ at Y_0.

Proposition If $F'(t_0) \neq 0$, then the tangent line

$$Y = F(t_0) + F'(t_0)\,(t - t_0)$$

approximates the curve $\mathcal{C}$ near Y_0 in the sense that

$$\lim_{t \to t_0} \frac{\|F(t) - (F(t_0) + F'(t_0)\,(t - t_0))\|}{t - t_0} = 0.$$

Proof This is just the same as saying

$$\lim_{t \to t_0} \left\| \frac{F(t) - F(t_0)}{t - t_0} - F'(t_0) \right\| = 0,$$

which is just the definition of $F'(t_0)$ when $F(t)$ is differentiable at t_0.

Example 5 Consider the helix

$$F(t) = \begin{pmatrix} \cos\left(2t + \frac{\pi}{2}\right) \\ \sin\left(2t + \frac{\pi}{2}\right) \\ 3t \end{pmatrix}.$$

Taking $t_0 = \frac{\pi}{6}$, we can compute

$$F'\left(\frac{\pi}{6}\right) = \begin{pmatrix} -2\sin(\frac{5\pi}{6}) \\ 2\cos\left(\frac{5\pi}{6}\right) \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \\ 3 \end{pmatrix}.$$

The tangent line at $t = \frac{\pi}{6}$ is then

$$\mathcal{L} : \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \\ \pi/2 \end{pmatrix} + (t - \pi/6)\begin{pmatrix} -1 \\ -\sqrt{3} \\ 3 \end{pmatrix}.$$

Velocity, Speed, Acceleration Vector, (scalar) Acceleration

If $Y = F(t)$ is the path of a particle in space, then $F'(t)$ is the **velocity vector** of the particle at time t. The **speed** of the particle is the scalar quantity $\|F'(t)\|$. The unit tangent vector $\frac{1}{\|F'(t)\|}F'(t)$ gives the **direction of motion** .

Thus, in the previous example, the direction of motion at $t = \pi/6$ is given by the unit vector

$$\frac{1}{\sqrt{1^2 + (-\sqrt{3})^2 + 3^2}}\begin{pmatrix} -1 \\ -\sqrt{3} \\ 3 \end{pmatrix} = \frac{1}{\sqrt{13}}\begin{pmatrix} -1 \\ -\sqrt{3} \\ 3 \end{pmatrix};$$

the speed at $t = \pi/6$ is $\sqrt{13}$.

If $F(t)$ is twice differentiable at t_0 we can proceed to define the second (vector) derivative of $F(t)$ at t_0:

$$F^{''}(t_0) = \begin{pmatrix} f_1^{''}(t_0) \\ f_2^{''}(t_0) \\ \vdots \\ f_m^{''}(t_0) \end{pmatrix}.$$

If $Y = F(t)$ is the path of a particle, this vector is the **vector acceleration** of the particle at $t = t_0$. The norm of this vector, $\|F''(t_0)\|$, is the (scalar) acceleration of the particle at $t = t_0$. In the earlier example of the helix we have

$$F''(\pi/6) = \begin{pmatrix} -4\cos(5\pi/6) \\ -4\sin(5\pi/6) \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} \\ -2 \\ 0 \end{pmatrix}$$

and the scalar acceleration is $\|F''(\pi/6)\| = \sqrt{4 \times 3 + 4} = 4$.