The Cross Product

Determinants are used in linear algebra to study the solvability of systems of linear equations. We will have more to say about them generally in a later section. For a 2 × 2 array we have

$$det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = a_1\, b_2 \,-\, a_2\, b_1;$$

$$\text{e.g. } det\begin{pmatrix} 2 & 3 \\ -2 & 5 \end{pmatrix} = 2 \times 5 \,-\, 3 \times (-2) \,=\, 10 \,+\, 6 \,=\, 16.$$

Computation of higher determinants can be defined inductively. For a 3 × 3 array we can expand by any row or column in terms of 2 × 2 determinants taken from other rows or columns. Thus

$$d \equiv \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

can be expanded by the first row, using determinants taken from 2 × 2 arrays taken from the second and third rows to give

$$d \,=\, a_1\,(b_2c_3 \,-\, c_2b_3) \,-\, a_2\,(b_1c_3 \,-\, c_1b_3) \,+\, a_3\,(b_1c_2 \,-\, c_1b_2).$$

Cross Product In R^3 (not in R^n for general n) it is possible to define a *vector product* or *cross product* of two vectors $X \,=\, (\, x_1 \quad x_2 \quad x_3\,)$, $Y \,=\, (\, y_1 \quad y_2 \quad y_3\,)$, by means of the *symbolic determinant*

$$X \times Y \,=\, \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

$$= \mathbf{i}\,(x_2y_3 \,-\, y_2x_3) \,-\, \mathbf{j}\,(x_1y_3 \,-\, y_1x_3) \,+\, \mathbf{k}\,(x_1y_2 \,-\, y_1x_2),$$

wherein we have used the symbols $\mathbf{i} \,=\, (\,1 \quad 0 \quad 0\,)\,, \mathbf{j} \,=\, (\,0 \quad 1 \quad 0\,)\,, \mathbf{k} \,=\, (\,0 \quad 0 \quad 1\,)$. Thus, e.g., for $X \,=\, (3, 1, -2)$ and $Y \,=\, (1, -1, 1)$ we have

$$X \times Y \,=\, \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \,=\, \begin{matrix} \mathbf{i}\,(1-2) \,-\, \mathbf{j}\,(3+2) \\ +\,\mathbf{k}\,(-3-1) \,=\, (-1, -5, -4). \end{matrix}$$

Properties of the Cross Product

i) $X \times Y$ is a *vector*, not a scalar.

ii) $(\alpha X + \beta Y) \times Z = \alpha(X \times Z) + \beta(Y \times Z)$ (the cross product is distributive).

iii) $Y \times X = -(X \times Y)$ i.e., the cross product does not commute); this can be seen from the fact that det $= \mathbf{i}\,(x_2y_3 - y_2x_3) - \mathbf{j}\,(x_1y_3 - y_1x_3) + \mathbf{k}\,(x_1y_2 - y_1x_2)$ changes sign if we interchange the roles of X and Y.

iv) $X \cdot (X \times Y) = 0 = Y \cdot (X \times Y)$, i.e., $X \times Y$ is orthogonal to both X and Y.

Proof of iv): We note that

$$X \cdot (X \times Y) = (\mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3) \cdot (\mathbf{i}\,(x_2y_3 - y_2x_3) - \mathbf{j}\,(x_1y_3 - y_1x_3) + \mathbf{k}\,(x_1y_2 - y_1x_2)).$$

$$= x_1\,(x_2y_3 - y_2x_3) - x_2\,(x_1y_3 - y_1x_3 + x_3\,(x_1y_2 - y_1x_2).$$

This expression collapses to zero when the individual terms are examined.

v) $\|X \times Y\| = \|X\|\,\|Y\|\,|\sin\theta|$

vi) The *direction* of $X \times Y$ is determined by the *right hand rule.*

Proof of v):

$$\|X \times Y\|^2 = \langle X \times Y, X \times Y \rangle$$

$$= (x_2y_3 - y_2x_3)^2 + (x_1y_3 - y_1x_3)^2 + (x_1y_2 - y_1x_2)^2$$

$$= x_2^2y_3^2 + y_2^2x_3^2 + x_1^2y_1^2 - 2x_2y_3y_2x_3 - x_1^2y_1^2$$

$$+ x_1^2y_3^2 + y_1^2x_3^2 + x_2^2y_2^2 - 2x_1y_3y_1x_3 - x_2^2y_2^2$$

$$+ x_1^2y_2^2 + y_1^2x_2^2 + x_3^2y_3^2 - 2x_1y_2y_1x_2 - x_3^2y_3^2.$$

The nine terms with "plus" signs can be seen to add to $\|X\|^2 \|Y\|^2$ while those with "minus" signs add to $-(X \cdot Y)^2$. Thus we have

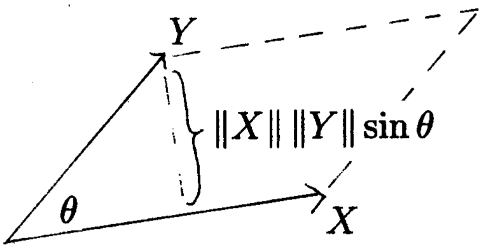
$$\|X\|^2 \|Y\|^2 - (X \cdot Y)^2 = \|X\|^2 \|Y\|^2 - (\|X\| \, \|Y\| \cos\theta)^2$$
$$= \|X\|^2 \|Y\|^2 (1 - \cos^2\theta) = \|X\|^2 \|Y\|^2 \sin^2\theta$$

and then taking the square root we have

$$\|X \times Y\| = \|X\| \, \|Y\| \, |\sin\theta| \, .$$

Interpretation of $\|X \times Y\|$ as an Area

Figure 3.2



$$\text{Area} = \|X\| \, \|Y\| \sin\theta = \|X \times Y\|$$

The Scalar Triple Product This is the scalar quantity $(X \times Y) \cdot Z$, which also equals $(Y \times Z) \cdot X$ and $(Z \times X) \cdot Y$.

Proposition If the vectors X, Y, Z form a *right hand system*, i.e., if the direction of Z agrees with that indicated by the right hand rule applied to X, Y, in that order, then

$(X \times Y) \cdot Z =$ the volume of the rectangular solid subtended by X, Y, Z.

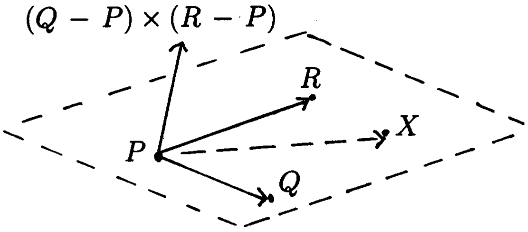
Proof The volume just described equals the area of the base parallelogram, subtended by X and Y, times the height from that base subtended by Z. The height indicated is the norm of the vector component of Z in the direction of $X \times Y$. Thus the volume is

$$\|X \times Y\| \ \|(Z \cdot (X \times Y)) \ X \times Y\| \,/\, \|X \times Y\|^2 \ = \ |Z \cdot (X \times Y)| \,.$$

The absolute value of $Z \cdot (X \times Y)$ is the same as this quantity itself if X, Y and Z form a right hand system.

Equations of Certain Planes Let us suppose that P, Q and R are non-collinear points in R^3. Then there is a unique two-dimensional plane $\mathcal{P}$ passing through these points. To find the equation of $\mathcal{P}$ we note that the vectors $Q - P$ and $R - P$ are parallel to this plane, so the vector $(Q - P) \times (R - P)$ is perpendicular to the plane. An arbitrary point X in $\mathcal{P}$ will have the property that $X - P$ is parallel to $\mathcal{P}$, hence orthogonal to $(Q - P) \times (R - P)$. So a defining equation for $\mathcal{P}$ is $((Q - P) \times (R - P)) \cdot (X - P) = 0$.

Figure 3.2



Example Let us take $P = (\,1 \quad 1 \quad 1\,)$, $Q = (\,1 \quad 2 \quad 2\,)$, $R = (\,2 \quad 0 \quad -1\,)$. Then

$$(Q - P) \times (R - P) = (\,0 \quad 1 \quad 1\,) \times (\,1 \quad -1 \quad -2\,)$$

$$= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \mathbf{i}\,(-2+1) - \mathbf{j}\,(0-1) + \mathbf{k}\,(0-1) = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Thus the equation of the plane $(Q - P) \times (R - P) \cdot (X - P) = 0$ is

$$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x - 1 \\ y - 1 \\ z - 1 \end{pmatrix} = 0 \ \text{or} \ x - y + z - 1 = 0.$$