The Cross Product

Determinants are used in linear algebra to study the solvability of systems of linear equations. We will have more to say about them generally in a later section. For a $2 \times 2$ array we have

$$\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1;$$

e.g. $\det\begin{pmatrix} 2 & 3 \\ -2 & 5 \end{pmatrix} = 2 \times 5 - 3 \times (-2) = 10 + 6 = 16.$

Computation of higher determinants can be defined inductively. For a $3 \times 3$ array we can expand by any row or column in terms of $2 \times 2$ determinants taken from other rows or columns. Thus

$$d \equiv \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

can be expanded by the first row, using determinants taken from $2 \times 2$ arrays taken from the second and third rows to give

$$d = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - c_1 b_3) + a_3 (b_1 c_2 - c_1 b_2).$$

Cross Product In $R^3$ (not in $R^n$ for general $n$) it is possible to define a vector product or cross product of two vectors $X = (x_1 \ x_2 \ x_3)$, $Y = (y_1 \ y_2 \ y_3)$, by means of the symbolic determinant

$$X \times Y = \det\begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

$$= i(x_2 y_3 - y_2 x_3) - j(x_1 y_3 - y_1 x_3) + k(x_1 y_2 - y_1 x_2),$$

wherein we have used the symbols $i = (1 \ 0 \ 0), j = (0 \ 1 \ 0), k = (0 \ 0 \ 1)$. Thus, e.g., for $X = (3,1,-2)$ and $Y = (1,-1,1)$ we have

$$X \times Y = \det\begin{pmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} = i(1-2) - j(3+2) + k(-3-1) = (-1,-5,-4).$$
Properties of the Cross Product

i) \( X \times Y \) is a vector, not a scalar.

ii) \((\alpha X + \beta Y) \times Z = \alpha(X \times Z) + \beta(Y \times Z)\) (the cross product is distributive).

iii) \( Y \times X = -(X \times Y) \) i.e., the cross product does not commute; this can be seen from the fact that \( \det = i(x_2 y_3 - y_2 x_3) - j(x_1 y_3 - y_1 x_3) + k(x_1 y_2 - y_1 x_2) \) changes sign if we interchange the roles of \( X \) and \( Y \).

iv) \( X \cdot (X \times Y) = 0 = Y \cdot (X \times Y) \), i.e., \( X \times Y \) is orthogonal to both \( X \) and \( Y \).

Proof of iv): We note that
\[
X \cdot (X \times Y) = (ix_1 + jx_2 + kx_3) \cdot (i(x_2 y_3 - y_2 x_3) - j(x_1 y_3 - y_1 x_3) + k(x_1 y_2 - y_1 x_2)),
\]
\[= x_1 (x_2 y_3 - y_2 x_3) - x_2 (x_1 y_3 - y_1 x_3) + x_3 (x_1 y_2 - y_1 x_2).
\]
This expression collapses to zero when the individual terms are examined.

v) \( \|X \times Y\| = \|X\| \|Y\| \sin \theta \)

vi) The direction of \( X \times Y \) is determined by the right hand rule.

Proof of v):
\[
\|X \times Y\|^2 = \langle X \times Y, X \times Y \rangle
\]
\[= (x_2 y_3 - y_2 x_3)^2 + (x_1 y_3 - y_1 x_3)^2 + (x_1 y_2 - y_1 x_2)^2
\]
\[= x_2^2 y_3^2 + y_2^2 x_3^2 + x_1^2 y_3^2 - 2x_2 y_3 y_2 x_3 - x_1^2 y_1^2
\]
\[+ x_1^2 y_3^2 + y_1^2 x_3^2 + x_2^2 y_2^2 - 2x_1 y_3 y_1 x_3 - x_2^2 y_2^2
\]
\[+ x_1^2 y_2^2 + y_1^2 x_2^2 + x_3^2 y_3^2 - 2x_1 y_2 y_1 x_2 - x_3^2 y_3^2.
\]
The nine terms with "plus" signs can be seen to add to $\|X\|^2 \|Y\|^2$ while those with "minus" signs add to $-(X \cdot Y)^2$. Thus we have

$$\|X\|^2 \|Y\|^2 - (X \cdot Y)^2 = \|X\|^2 \|Y\|^2 - (\|X\| \|Y\| \cos \theta)^2$$

$$= \|X\|^2 \|Y\|^2 (1 - \cos^2 \theta) = \|X\|^2 \|Y\|^2 \sin^2 \theta$$

and then taking the square root we have

$$\|X \times Y\| = \|X\| \|Y\| \sin \theta.$$

**Interpretation of $\|X \times Y\|$ as an Area**

![Figure 3.2](image)

Area = $\|X\| \|Y\| \sin \theta = \|X \times Y\|$ 

**The Scalar Triple Product** This is the scalar quantity $(X \times Y) \cdot Z$, which also equals $(Y \times Z) \cdot X$ and $(Z \times X) \cdot Y$.

**Proposition** If the vectors $X, Y, Z$ form a right hand system, i.e., if the direction of $Z$ agrees with that indicated by the right hand rule applied to $X, Y$, in that order, then

$$(X \times Y) \cdot Z = \text{the volume of the rectangular solid subtended by } X, Y, Z.$$

**Proof** The volume just described equals the area of the base parallelogram, subtended by $X$ and $Y$, times the height from that base
subtended by $Z$. The height indicated is the norm of the vector component of $Z$ in the direction of $X \times Y$. Thus the volume is

$$\|X \times Y\| \| (Z \cdot (X \times Y)) \cdot X \times Y\| / \|X \times Y\|^2 = |Z \cdot (X \times Y)|.$$  

The absolute value of $Z \cdot (X \times Y)$ is the same as this quantity itself if $X$, $Y$ and $Z$ form a right hand system.

**Equations of Certain Planes** Let us suppose that $P$, $Q$ and $R$ are non-collinear points in $\mathbb{R}^3$. Then there is a unique two-dimensional plane $\mathcal{P}$ passing through these points. To find the equation of $\mathcal{P}$ we note that the vectors $Q - P$ and $R - P$ are parallel to this plane, so the vector $(Q - P) \times (R - P)$ is perpendicular to the plane. An arbitrary point $X$ in $\mathcal{P}$ will have the property that $X - P$ is parallel to $\mathcal{P}$, hence orthogonal to $(Q - P) \times (R - P)$. So a defining equation for $\mathcal{P}$ is $((Q - P) \times (R - P)) \cdot (X - P) = 0$.

**Figure 3.2**

![Figure 3.2](image)

**Example** Let us take $P = (1 \ 1 \ 1)$, $Q = (1 \ 2 \ 2)$, $R = (2 \ 0 \ -1)$. Then

$$(Q - P) \times (R - P) = (0 \ 1 \ 1) \times (1 \ -1 \ -2)$$

$$= \det \begin{pmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} i \ -2 + 1 \ j \ 0 \ -1 \ k \ 0 \ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$  

Thus the equation of the plane $(Q - P) \times (R - P) \cdot (X - P) = 0$ is

$$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x - 1 \\ y - 1 \\ z - 1 \end{pmatrix} = 0 \text{ or } x - y + z - 1 = 0.$$  

4