The “Dot”, or “Inner”, Product

Definition and Properties If X and Y are vectors in R^n, then the *dot product* or *inner product* of X and Y is (showing the various notations used)

$$X \cdot Y = \langle X, Y \rangle = Y' X = \sum_{k=1}^{n} x_k y_k \quad \text{(which is scalar!)}$$

Thus, e.g., $(2, -1, 3) \cdot (-1, 2, 1) = 2(-1) + (-1)2 + 3(1) = -1$. Notice that

$$X \cdot X = (x_1)^2 + (x_2)^2 + ... + (x_n)^2 = \|X\|^2 \geq 0, \; X \neq 0.$$

That is, the *norm* is naturally expressed in terms of the dot product.

Properties of the Dot Product

- The *distributive* laws:

$$(\alpha X + \beta Y) \cdot Z = \alpha(X \cdot Z) + \beta(Y \cdot Z),$$

$$(\alpha X) \cdot Z = \alpha (X \cdot Z);$$

- The *commutative* law:

$$X \cdot Y = Y \cdot X;$$

- The *non-negativity* and *coercivity* properties:

$$X \cdot X \geq 0, \; X \in R^n; \quad X \cdot X = 0 \Rightarrow X = 0;$$

- The *Schwarz inequality*:

$$|X \cdot Y| \leq \|X\| \, \|Y\|$$

with equality holding if and only if X, Y are *collinear*, i.e., there are scalars α, β, not both zero, such that $\alpha X + \beta Y = 0$.

The first three properties are clear from the definition of the dot product; the last will be proved shortly. From the definition of $\|X\|$, given earlier, it is clear that, for all $X \in R^n$,

$$\|X\| = \sqrt{X \cdot X} \geq 0; \quad \|X\| = 0 \Rightarrow X = \mathbf{0}.$$

Proof of the Schwarz Inequality If either X or Y is the zero vector then $X \cdot Y = 0$ and either $\|X\| = 0$ or $\|Y\| = 0$; in either case the Schwarz inequality is trivially valid. Otherwise, assuming neither X nor Y is the zero vector,

$$0 \leq \left\| \frac{X}{\|X\|} \pm \frac{Y}{\|Y\|} \right\|^2 = \frac{\|X\|^2}{\|X\|^2} + \frac{\|Y\|^2}{\|Y\|^2} \pm \frac{2\,X \cdot Y}{\|X\|\,\|Y\|} = 2 \pm \frac{2\,X \cdot Y}{\|X\|\,\|Y\|}$$

$$\Rightarrow \pm \frac{X \cdot Y}{\|X\|\,\|Y\|} \leq 1 \Rightarrow |X \cdot Y| \leq \|X\|\,\|Y\|.$$

If the initial inequality is actually an equality then $\|Y\|\,X \pm \|X\|\,Y = 0$ and we have collinearity with $\alpha = \|Y\|,\ \beta = \pm\|X\|$.

Properties of the Norm

$$\|\alpha X + \beta Y\|^2 = \langle \alpha X + \beta Y, \alpha X + \beta Y \rangle = \alpha^2 \|X\|^2 + \beta^2 \|Y\|^2 + 2\alpha\beta \langle X, Y \rangle;$$

$$\|\alpha X\| = |\alpha|\,\|X\|;$$

$$\|X\| \geq 0; \ \|X\| = 0 \rightarrow X = 0 \text{ (already stated)};$$

$$\|X + Y\| \leq \|X\| + \|Y\|.$$

Again the first three properties are evident from the definitions. The last property is called the *triangle inequality*; its proof depends on the Schwarz inequality satisfied by the inner product:

$$\|X + Y\|^2 = \langle X + Y, X + Y \rangle = \|X\|^2 + 2\langle X, Y \rangle + \|Y\|^2$$

$$\leq \|X\|^2 + 2\,\|X\|\,\|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2.$$

Taling the square root of both sides, the triangle inequality follows.

The triangle inequality is so named because, for any three points (forming the vertices of a triangle) in R^n,

$$d(X,\, Y) \;=\; \|X \,-\, Y\| \;=\; \|X \,-\, Z \,+\, Z \,-\, Y\|$$
$$\leq\; \|X \,-\, Z\| \,+\, \|Z \,-\, Y\| \;=\; d(X,\, Z) \,+\, d(Z,\, Y)$$

states that the length of one side must be less than or equal to the sum of the lengths of the other two sides, equality occurring if and only if $X - Z$ and $Z - Y$ are collinear, i.e., if and only if the triangle is degenerate.

Proposition Let X, Y be vectors in R^n and θ the angle between them. Then

$$\langle X,\, Y\rangle \;=\; \|X\|\; \|Y\|\; \cos\theta.$$

Proof From the *Law of Cosines* in trigonometry we have

$$\|X \,-\, Y\|^2 \;=\; \|X\|^2 \,+\, \|Y\|^2 \,-\, 2\; \|X\|\; \|Y\| \cos\theta.$$

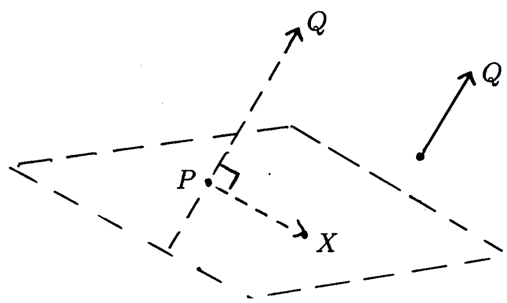
The result then follows by combining the above with

$$\|X \,-\, Y\|^2 \;=\; \langle X \,-\, Y,\, X \,-\, Y\rangle \;=\; \langle X,\, X\rangle \,+\, \langle Y,\, Y\rangle \,-\, 2\,\langle X,\, Y\rangle.$$

Use of the Dot Product in Defining Certain Planes If $X \cdot Y = 0$ then $\cos\theta = 0 \rightarrow X, Y$ are *orthogonal*, i.e., lie at an angle of $90°$ to each other. This property allows us to define the plane in R^3 through a point P orthogonal to a vector $Q \neq 0$. This plane consists of all vectors X with the property

$$Q \cdot (X \,-\, P) \;=\; 0, \;\text{ i.e., }\; Q \cdot X \;=\; Q \cdot P.$$

Figure 2.1



Example Find the equation of the plane through $(1, 2, 1)$ orthogonal to $(2, -1, 3)$.

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow 2\,x - y + 3\,z = 2 - 2 - 3 = -3 \rightarrow 2\,x - y + 3\,z + 3 = 0.$$

Unit Vectors A vector U is a *unit vector* if $\|U\| = 1$. Given any non-zero vector Q we can construct a unit vector in the direction of Q. We set $U_Q = \alpha\, Q$ and require

$$1 = \|U_Q\|^2 = \langle U_Q,\, U_Q \rangle = \langle \alpha Q,\, \alpha Q \rangle = |\alpha|^2\, \|Q\|^2$$

so that $|\alpha| = 1/\,\|Q\|$, i.e., $\alpha = \pm 1/\,\|Q\|$, and, selecting the positive sign to maintain the direction, we have

$$U_Q = \alpha\, Q = \frac{Q}{\|Q\|}.$$

We say that U_Q is the *normalization* of Q. If we select the - sign we obtain a unit vector in the direction opposite to Q.

A set of vectors $U_1,\, U_2,\, ...\, U_m$ is an *orthogonal* set if $\langle U_i,\, U_j \rangle = U_i \cdot U_j = 0,\ i \neq j$. If, in addition, we have $\|U_i\| = 1, i = 1, 2, ..., m$, then the vectors form an *orthonormal* set.

Examples of Orthonormal Sets

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

$$\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}.$$

Component of One Vector with Respect to Another Let $X, Q \in R^n$, $Q \neq \mathbf{0}$. The *vector component* of X in the direction of Q is a scalar multiple of Q, $r\,Q$, such that the angle, shown as ψ in the diagram, is a right angle. Thus

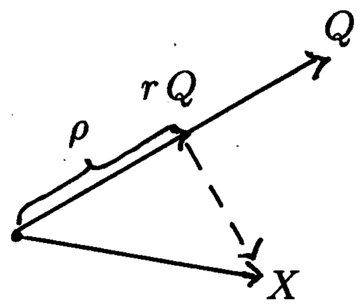
$$Q \cdot (X - r\,Q) = 0 \Rightarrow r\,\|Q\|^2 = X \cdot Q$$
$$\Rightarrow r = \frac{X \cdot Q}{\|Q\|^2} \Rightarrow r\,Q = \left(\frac{X \cdot Q}{\|Q\|^2}\right) Q.$$

Another way to write this is

$$r\,Q = \left(X \cdot \frac{Q}{\|Q\|}\right) \frac{Q}{\|Q\|} = (X \cdot U_Q)\,U_Q.$$

The scalar quantity $\rho = X \cdot U_Q$ is the *numerical component* of X with respect to Q.

Figure 2.2



Example The vector and numerical components of $(2,1,3)$ in the direction of $(-1,2,1)$ are

$$\frac{\begin{pmatrix}2\\1\\3\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\1\end{pmatrix}}{(-1)^2+2^2+1^2} \begin{pmatrix}-1\\2\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\2\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\\3\end{pmatrix} \cdot \frac{\begin{pmatrix}-1\\2\\1\end{pmatrix}}{\sqrt{(-1)^2+2^2+1^2}} = 3/\sqrt{6}.$$