Vector Spaces; the Space R^n

Vector Spaces A *vector space* (over the real numbers) is a set $\mathcal{V}$ of mathematical entities, called *vectors*, U, V, W, etc., in which an addition operation "+" is defined and in which multiplication of vectors by *scalars* i.e., real numbers, is defined, with the following properties:

- $U,\, V \in \mathcal{V} \Rightarrow U + V \in \mathcal{V}$;
- There is a unique zero vector in $\mathcal{V}$, which we denote by $\mathbf{0}$, such that, for each $U \in \mathcal{V},\ U + \mathbf{0} = U$;
- $U \in \mathcal{V}$ and α is a real number $\Rightarrow \alpha U \in \mathcal{V}$;
- If U, V are two vectors in $\mathcal{V}$ and α, β are two scalars, then we have the *distributive properties*:

$$\alpha\,(U + V) = \alpha U + \alpha V, \quad (\alpha + \beta)\,U = \alpha U + \beta U.$$

There are many examples of vector spaces but we will be concerned with just a few. The vector space R^n consists of all *ordered "n-tuples"* of real numbers

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},\ \text{etc.,}$$

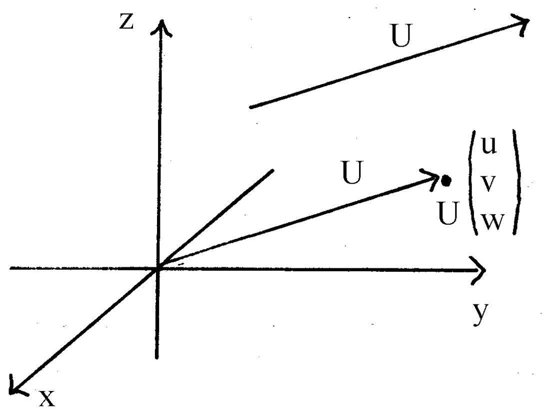
where $x_i,\ y_i,$ are scalars (real or complex numbers), called the *components* of X and Y, respectively. The qualification *ordered* means that, e.g.,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

The most important cases for our work are R^2 and R^3; as a vector space R^1 is essentially trivial. However, the general *Cartesian* or *Euclidean* space R^n, for n any natural number, is also important.

Vectors in R^n (R^3 shown in the diagrams) can be thought of either as *points in space* or as *directed magnitudes*, the latter represented graphically by arrows. In the latter context vectors serve to represent, e.g., velocities, accelerations, forces, etc.

Figure 1.1



Quantities representable by a single number α (e.g., speed, time, weight, mass, density) are *scalar* quantities. In these notes scalars are always real numbers but in other contexts they could be complex numbers, e.g.. Real numbers have *magnitude*, $|\alpha|$, and *sign* (positive or negative).

If $X = (x_1, x_2, ..., x_n)$ is an n-dimensional vector, its *magnitude* or *norm* is

$$\|X\| = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}.$$

This is the *length* of a vector in the ordinary, *Euclidean*, sense. Thus

$$\left\| \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\| = \sqrt{(-2)^2 + (1)^2 + (3)^2} = \sqrt{14} = \sqrt{2}\sqrt{7}.$$

The *distance* between two points (i.e., vectors) U, V is defined to be $d(U, V) = \|(U - V)\|$. We have the symmetry property $d(U, V) = d(V, U)$ and it is clear that $U = V$ if and only if $d(U, V) = 0$.

Vectors do not have an unambiguous sign like real numbers but they do have *direction cosines.* If $X = (x_1, x_2, ..., x_n)$ is an n-dimensional vector, i.e., a vector in R^n, the real numbers

$$c_i = \frac{x_i}{\|X\|} = \frac{x_i}{\sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}},$$

which clearly satisfy $-1 \le c_i \le 1$, are called the *direction cosines* of X. They are easily seen to be the cosines of the angles between X and the positively directed coordinate axes. Clearly $\sum_{k=1}^{n} c_k^2 = 1$.

Algebraic Operations in $\mathbf{R^n}$

Addition and subtraction of vectors are performed *componentwise*; the result is another vector of the same size or *dimension*:

$$Z = X \pm Y : \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{pmatrix},$$

i.e., $z_i = x_i \pm y_i,\ i = 1, 2, ..., n$. Thus, for example,

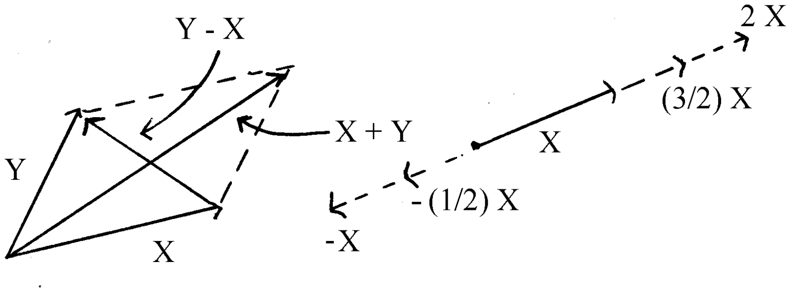
$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-3 \\ 0-1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

Addition or subtraction of two vectors of unequal dimension is *not defined.*

While it is clearly possible to define a component by component multiplication of vectors, this notion has very little signficance in the present course. We will soon define other types of vector multiplication which are significant.

Addition and Subtraction of vectors can be interpreted geometrically via the *parallelogram rule.*

Figure 1.2



Multiplication of vectors by scalars, the latter commonly denoted by Greek letters (α, β, γ..., etc.) is also defined componentwise:

$$\alpha X = \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}, \text{ e.g. } 3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix}.$$

In particular, $-1\,X = -1\,(x_1, x_2, ..., x_n) \equiv -X$ is the *additive inverse* of X and $0\,X = 0\,(x_1, x_2, ..., x_n) \equiv (0, 0, ...0) \equiv \mathbf{0}$ is the *additive identity*.

Linear Combinations of Vectors Formation of *linear combinations* of vectors is the most important vector operation. In general, if α, β, γ, ... are scalars and X, Y, Z, ... are vectors of the same dimension, we have

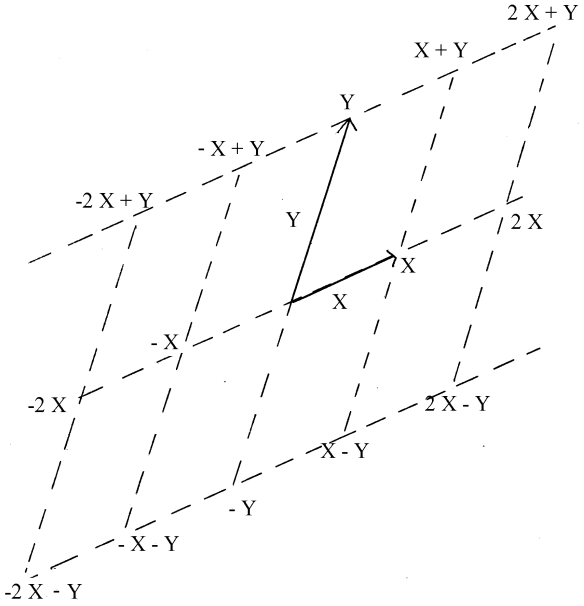
$$\alpha X + \beta Y + \ldots + \gamma Z = \begin{pmatrix} \alpha x_1 + \beta y_1 + \ldots + \gamma z_1 \\ \alpha x_2 + \beta y_2 + \ldots + \gamma z_2 \\ \vdots \\ \alpha x_n + \beta y_n + \ldots + \gamma z_n \end{pmatrix}.$$

Thus, e.g.,

$$-1 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2+2+3 \\ -1+2+0 \\ -3+2-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}.$$

We can represent various linear combinations geometrically.

Figure 1.3

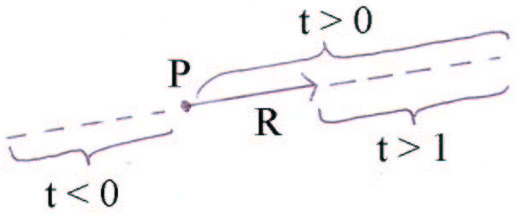


Various Lines and their Representations

i) **The line through the point (vector) P in the direction of the vector R** All points on such a line can be represented parametrically, with a scalar parameter t, in the form

$$X = P + t\,R, \;\; -\infty < t < \infty.$$

Figure 1.4



Example: The line through $P = (1, -1, 3)$ in the direction of $R = (1, 1, 1)$ takes the form

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+t \\ -1+t \\ 3+t \end{pmatrix}.$$

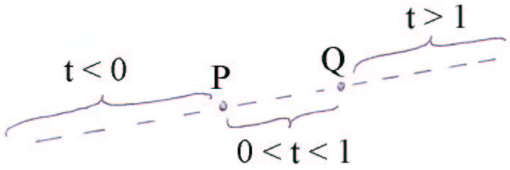
We can eliminate t to obtain the equivalent representation in terms of two linear algebraic equations:

$$x - 1 = y + 1 = z - 3 \text{ (all equal to } t\text{)}.$$

ii) **The line through two points (vectors) P, Q.** This goes much as in i) except that we use $R = Q - P$. Thus we have

$$X(t) = P + t\,(Q - P) = (1-t)\,P + t\,Q, \;\; -\infty < t < \infty.$$

Figure 1.5



Example: The line through $P = (-3, 0, 2)$ and $Q = (1, -2, 1)$ has the parametric representation

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = (1-t) \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3+4t \\ -2t \\ 2-t \end{pmatrix}.$$

Suggested Exercise In R^2 find a parametric representation for the line passing through $P = (\,1 \quad -2\,)$ and $Q = (\,-2 \quad 1\,)$. Plot the points on the line corresponding to the integer values of t between -3 and 3. For what value of r does the point $R = (\,5 \quad r\,)$ belong to this line? For which values of a, b, c does the equation $a\,x + b\,y + c = 0$ describe the points on this line?