Inhomogeneous Linear Second Order Equations; Continued

Method of Variation of Parameters; the Wronskian Determinant; Fundamental Sets Let \( w(x) \) and \( z(x) \) be two solutions of the homogeneous second order linear differential equation

\[
\frac{d^2 z}{dx^2} + p(x) \frac{dz}{dx} + q(x) z = 0.
\]

If we take a linear combination of these:

\[
c_1 w(x) + c_2 z(x),
\]

and want to satisfy initial conditions \( y(x_0) = y_0, \ y'(x_0) = y_1 \) at a point \( x = x_0 \), then we need

\[
c_1 w(x_0) + c_2 z(x_0) = y_0,

c_1 w'(x_0) + c_2 z'(x_0) = y_1.
\]

The condition for unique solvability, as always, is that the determinant

\[
\det \begin{pmatrix} w(x_0) & z(x_0) \\ w'(x_0) & z'(x_0) \end{pmatrix} = w(x_0) z'(x_0) - w'(x_0) z(x_0) \neq 0.
\]

The determinant expression

\[
W(w, z, x) = w(x) z'(x) - w'(x) z(x)
\]

is called the Wronskian (determinant) of the solution pair \( w(x), z(x) \).

**Proposition** On any interval \( a < x < b \) where \( p(x) \) and \( q(x) \) are continuous, the Wronskian \( W(w, z, x) \), as a function of \( x \), is either never zero or is identically zero.

**Proof** We compute \( \frac{dW(w, z, x)}{dx} = \frac{d}{dx} (w(x) z'(x) - w'(x) z(x)) = w'(x) z'(x) + w(x) z''(x) - w''(x) z(x) - w'(x) z'(x) \).
\[ w(x) = w(x) (-p(x) z'(x) - q(x) z(x)) - (-p(x) w'(x) - q(x) w(x)) z(x) \]
\[ = -p(x) \det \begin{pmatrix} w(x) & z(x) \\ w'(x) & z'(x) \end{pmatrix} + q(x) \det \begin{pmatrix} w(x) & z(x) \\ w(x) & z(x) \end{pmatrix} \]
\[ = -p(x) W(w, z, x) + 0 = -p(x) W(w, z, x), \]

where, to obtain the last expression, we have used the differential equation satisfied by both \( w(x) \) and \( z(x) \) and the fact that the determinant of a matrix with two identical rows is equal to zero. Thus, as a function of \( x \), the Wronskian \( W(w, z, x) \) satisfies a linear homogeneous first order differential equation

\[ \frac{d}{dx} W(w, z, x) = -p(x) W(w, z, x). \]

Given any point \( x_0 \) in the interval \( a < x < b \), solution of this differential equation gives

\[ W(w, z, x) = \exp \left( -\int_{x_0}^{x} p(s) \, ds \right) W(w, z, x_0), \quad x \in (a, b). \]

Thus, if the Wronskian is zero at any point in \( (a, b) \), taking \( x_0 \) to be that point, the formula shows that the Wronskian is zero at every other point \( x \in (a, b) \). On the other hand, if there is a point, call it \( x_0 \), where the Wronskian is not zero, then, since the exponential function is never zero, the formula shows that the Wronskian is non-zero at every other point, \( x \), in \( (a, b) \).

Thus the vanishing, or non-vanishing of the Wronskian \( W(w, z, x) \) associated with two solutions \( w(x) \) and \( z(x) \) of the homogeneous equation \( \frac{d^2 z}{dx^2} + p(x) \frac{dz}{dx} + q(x) z = 0 \) is a property of the two solution functions and not a property of the particular point \( x \) in question. Two solutions \( w(x) \) and \( z(x) \) for which the Wronskian is different from zero are said to constitute a fundamental (solution) set. When we have a fundamental solution set, \( w(x) \) and \( z(x) \), we can say without any further checking that \( y(x, c, d) = c w(x) + d z(x) \) is the general solution of the homogeneous equation in question.
The Second Order Variation of Parameters Formula  Let us consider the inhomogeneous second order linear differential equation

\[
\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x) y = g(x)
\]

with the usual continuity assumptions on \(p(x), q(x)\) and \(g(x)\). Following much the same reasoning as in the first order case, we note that if \(w(x)\) and \(z(x)\) are a fundamental set of solutions, then \(c w(x) + d z(x)\) is the general solution of the homogeneous equation. From this we are led to try for a particular solution of the inhomogeneous equation by allowing the parameters \(c\) and \(d\) to vary:

\[
y_p(x) = c(x) w(x) + d(x) z(x).
\]

Differentiating, we have

\[
y_p'(x) = c'(x) w(x) + d'(x) z(x) + c(x) w'(x) + d(x) z'(x).
\]

Let us arbitrarily set

\[
c'(x) w(x) + d'(x) z(x) = 0. \quad \text{(Equation I)}
\]

Then we have

\[
y_p''(x) = c'(x) w'(x) + d'(x) z'(x) + c(x) w''(x) + d(x) z''(x).
\]

Requiring that \(y_p(x)\) should satisfy the given inhomogeneous equation, we obtain

\[
(c'(x) w'(x) + d'(x) z'(x) + c(x) w''(x) + d(x) z''(x)) + p(x) (c(x) w'(x) + d(x) z'(x)) + q(x) (c(x) w(x) + d(x) z(x)) = g(x).
\]

Because \(w(x)\) and \(z(x)\) satisfy the homogeneous equation we readily see that the coefficients of \(c(x)\) and \(d(x)\) (undifferentiated) reduce to zero and we are left with

\[
c'(x) w'(x) + d'(x) z'(x) = g(x) \quad \text{(Equation II)}.
\]
Putting Equation I and Equation II together, we have
\[
\begin{align*}
  c'(x) w(x) + d'(x) z(x) &= 0 \\
  c'(x) w'(x) + d'(x) z'(x) &= g(x).
\end{align*}
\]
When we check the determinant of this system we see that it is the Wronskian, \( W(w, z, x) \), which is non-vanishing because we are assuming \( w(x) \) and \( z(x) \) to constitute a fundamental set of solutions. Solving this system for \( c(x) \) and \( d(x) \) we obtain
\[
\begin{align*}
  c'(x) &= -\frac{z(x) g(x)}{w(x) z'(x) - w'(x) z(x)}, \\
  d'(x) &= \frac{w(x) g(x)}{w(x) z'(x) - w'(x) z(x)}.
\end{align*}
\]
Given a point \( x_0 \in (a, b) \), we can obtain a particular solution \( y_p(x) \) with \( y_p(x_0) = 0 \). By integrating the equations for \( c'(x) \) and \( d'(x) \) from \( x_0 \) to an arbitrary point \( x \):
\[
y_p(x) = \int_{x_0}^{x} \left( \frac{w(s)z(x) - z(s)w(x)}{w(s)z'(s) - z(s)w'(s)} \right) g(s) \, ds.
\]
It should be noted that the denominator of the fraction in the integrand is the Wronskian determinant of \( w(s) \) and \( z(s) \) and thus does not vanish. When we compute the derivative of \( y_p(x) \) at \( x_0 \) we obtain
\[
y'_p(x_0) = \left( \frac{w(x_0)z(x_0) - z(x_0)w(x_0)}{w(x_0)z'(x_0) - z(x_0)w'(x_0)} \right) g(x_0) = 0.
\]
Thus all of the initial data for \( y_p(x) \) vanish at \( x = x_0 \).

**Example 1**  Let us try the method first in a case where we already know how to obtain the solution from earlier work. We consider the second order equation
\[
x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = x.
\]
Trying solutions of the form \( x^r \) for the homogeneous equation we obtain the indicial equation \( r^2 - 3r + 2 = 0 \) with solutions \( r_1 = 1, \ r_2 = 2 \). Thus we have solutions

\[
  w(x) = x, \ z(x) = x^2.
\]

for which the Wronskian is \( W(w, z, x) = x^2 \), which does not vanish on the interval \((0, \infty)\) where the coefficients of the normalized differential equation (the one obtained by dividing by \( x^2 \)) are continuous. Using our formula with \( x_0 = 1 \), and noting that we have to normalize so that the coefficient of the second derivative is 1, we obtain the solution

\[
  y_p(x) = \int_1^x \left( s \frac{x^2 - s^2 x}{s^2} \right) s^{-1} \, ds = \int_1^x \left( \frac{x^2}{s} - x \right) s^{-1} \, ds
\]

\[
  = x^2 \int_1^x s^{-2} \, ds - x \int_1^x s^{-1} \, ds = x^2 - x - x \log x.
\]

This is the solution whose value and derivative both vanish at \( x = 1 \). Since \( x \) and \( x^2 \) are already solutions of the homogeneous equation, we can see that \( -x \log x \) must also be a particular solution of the inhomogeneous equation. This is easily checked.

**Example 2** We take the same homogeneous part but a different inhomogeneous term:

\[
  x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2 y = x^4 e^x.
\]

Here, again remembering that we have to divide the equation by \( x^2 \) in order to get the correct right hand side, we obtain the integral

\[
  y_p(x) = \int_1^x \left( \frac{s x^2 - s^2 x}{s^2} \right) s^2 e^s \, ds = \int_1^x \left( s x^2 - s^2 x \right) e^s \, ds = x^2 \int_1^x s e^s \, ds
\]

\[
  - x \int_1^x s^2 e^s \, ds = x^2 \left( x e^x - e^x \right) - x \left( x^2 e^x - 2x e^x + e^x - e \right)
\]

\[
  = x^2 e^x - x e^x + x e.
\]
Since $x$, and hence $xe^x$, is already a solution of the homogeneous equation, we see that $x^2 e^x - xe^x$ must also be a particular solution of the inhomogeneous equation.

**More on the Method of Undetermined Coefficients** In this section we will consider some instances of use of the method of undetermined coefficients not treated in our main section on that topic. These instances arise when the inhomogeneous term $g(x)$ appearing on the right hand side of the differential equation is already a solution of the corresponding homogeneous equation. When this is the case the method of undetermined coefficients has to be modified before it will work properly. We will begin with

**Example 1** We consider the second order linear differential equation with constant coefficients

$$\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6 y = e^{-2x}.$$  

The finite family generated by $e^{-2x}$ under differentiation consists of just that function itself. The original method of undetermined coefficients would then indicate that we should try a particular solution of the form $y_p(x) = ce^{-2x}$ with a coefficient $c$ to be determined. However, when we substitute this solution form into the differential equation we obtain

$$c \left(4e^{-2x} - 10e^{-2x} + 6e^{-2x}\right) = e^{-2x} \Rightarrow c0 = e^{-2x},$$

which is clearly impossible.

In the corresponding section on $n$-th order linear differential equations with constant coefficients we will indicate a systematic approach to difficulties of this type. Here we will simply indicate the remedial technique. To motivate it, we use the *Variation of Parameters* method to solve the differential equation. Letting $w(x) = e^{-2x}, z(x) = e^{-3x},$
that method gives the particular solution

\[ y_p(x) = \int^x e^{-2s} e^{-3x} - e^{-3s} e^{-2x} \, ds \]
\[ = \int^x e^{-2s} e^{-3x} - e^{-3s} e^{-2x} \, ds \]
\[ = e^{-2x} \int^x ds - e^{-3x} \int^x e^s \, ds = x e^{-2x} - e^{-2x}. \]

Since \( e^{-2x} \) is already a solution of the homogeneous equation we can ignore that term in forming the general solution, which is now seen to be

\[ y(x, c_1, c_2) = c_1 e^{-2x} + c_2 e^{-3x} + x e^{-2x}. \]

This result suggests the remedial technique for the method of undetermined coefficients, which in this example is to replace \( e^{-2x} \) in \( y_p(x) = c e^{-2x} \) with \( x e^{-2x} \), i.e., we try \( y_p(x) = c x e^{-2x} \). Then

\[ y_p'(x) = c e^{-2x} - 2c x e^{-2x}, \quad y_p''(x) = -4c e^{-2x} + 4c x e^{-2x}. \]

Substitution gives

\[ (-4c e^{-2x} + 4c x e^{-2x}) + 5 (c e^{-2x} - 2c x e^{-2x}) + 6c x e^{-2x} = e^{-2x}. \]

We readily see that the terms involving \( x e^{-2x} \) on the left hand side all cancel out, leaving us with

\[ -4c e^{-2x} + 5c e^{-2x} = e^{-2x} \Rightarrow c = 1 \]

and we conclude that \( y_p(x) = x e^{-2x} \) is a particular solution of the differential equation. The general solution is then

\[ y(x, c_1, c_2) = c_1 e^{-2x} + c_2 e^{-3x} + x e^{-2x}. \]

However, the expedient of multiplying by \( x \) is not enough in all cases. The general procedure consists in taking the finite family generated
by the inhomogeneous term \( g(x) \) under repeated differentiation and modifying it by multiplying the functions in that family by powers of \( x \), finally discarding functions which are solutions of the homogeneous equation. The first two rules for this procedure are:

**i)** Assuming the finite family for \( g(x) \) includes \( m \) functions, augment those functions by repeatedly multiplying them by powers of \( x \), i.e., \( x, x^2, \ldots \), etc., until the augmented family includes \( m \) functions which are not solutions of the homogeneous equation.

**ii)** Discard from the augmented family all functions which are solutions of the corresponding homogeneous equation, express the particular solution \( y_p(x) \) as a linear combination of the remaining \( m \) functions with coefficients to be determined and substitute into the inhomogeneous differential equation to determine those coefficients.

**Example 2** Consider the differential equation

\[
\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 4 y = x e^{-2x}.
\]

Here the finite family consists of the two functions \( x e^{-2x} \) and \( e^{-2x} \) - both of which are solutions of the homogeneous equation. Thus \( m = 2 \). If we try for a particular solution as a linear combination of those two it is easy to see that again obtain \( 0 = e^{-2x} \). Following the rules just outlined we construct the augmented set consisting of the functions

\[
e^{-2x}, \; x e^{-2x}, \; x^2 e^{-2x}, \; x^3 e^{-2x}.
\]

We drop from this list both of the functions \( e^{-2x} \) and \( x e^{-2x} \), because they are solutions of the homogeneous equation, and use, instead, the modified set of two functions consisting of \( x^2 e^{-2x} \) and \( x^3 e^{-2x} \). We then try a particular solution of the form

\[
y_p(x) = d_1 x^2 e^{-2x} + d_2 x^3 e^{-2x}.
\]
However, if we carry out the computations we see that when \( x^2 e^{-2x} \) is substituted into the equation, we obtain a term involving \( d_1 e^{-2x} \). Since no term \( e^{-2x} \) appears on the right hand side in the equation, it will be enough here to consider \( y_p(x) = d x^3 e^{-2x} \). When this is substituted into \( \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4 y = x e^{-2x} \) we obtain

\[
(4 d x^3 e^{-2x} - 12 d x^2 e^{-2x} + 6 d x e^{-2x}) + 4 (-2 d x^3 e^{-2x} + 3 d x^2 e^{-2x})
+ 4 d x^3 e^{-2x} = x e^{-2x}.
\]

The coefficients of \( x^2 e^{-2x} \) and \( x^3 e^{-2x} \) automatically reduce to 0 for any choice of \( d \) and we are left with

\[
6 d x e^{-2x} = x e^{-2x} \quad \rightarrow \quad d = \frac{1}{6}.
\]

We conclude that a particular solution is \( y_p(x) = \frac{1}{6} x^3 e^{-2x} \) and the general solution is

\[
y(x, c_1, c_2) = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{6} x^3 e^{-2x}.
\]

When the finite family generated by \( g(x) \) involves trigonometric functions it can be seen that rule i) set forth earlier implies that the pairing of \( \sin x \) with \( \cos x \) in the original finite family must be preserved in the modified family.

**Example 3** We consider the equation

\[
\frac{d^2 y}{dx^2} + y = x \cos x.
\]

Here the general solution of the homogeneous equation

\[
\frac{d^2 z}{dx^2} + z = 0
\]

is \( z(x, c_1, c_2) = c_1 \sin x + c_2 \cos x \). The finite family associated with \( x \cos x \) includes the four functions \( x \cos x, \ x \sin x, \ \cos x \) and \( \sin x \),

\[
\frac{d^2 y}{dx^2} + y = x \cos x.\]

which includes two functions, \( \cos x \) and \( \sin x \), which are solutions of the homogeneous equation. In obtaining the modified family of functions for use with the method of undetermined coefficients we need to a) retain four functions in the family, and b) retain the pairing of \( \sin x \) and \( \cos x \). We augment the family by multiplying by \( x \), obtaining 
\[ x^2 \cos x, \ x^2 \sin x, \ x \cos x, \ x \sin x, \ \cos x \text{ and } \sin x; \]
then we discard the last two, which are solutions of the homogeneous equation; the final modified collection consists of 
\[ x^2 \cos x, \ x^2 \sin x, \ x \cos x, \ x \sin x. \]
None of these functions are solutions of the homogeneous equation, so we try for a particular solution in the form 
\[ y_p(x) = d_1 x \sin x + d_2 x \cos x + d_3 x^2 \sin x + d_4 x^2 \cos x. \]

When we substitute this form into the differential equation we find that the coefficients of \( x^2 \sin x \) and \( x^2 \cos x \) vanish automatically, whatever the values of the \( d_k \) may be. We obtain
\[
-4d_4 x \sin x + 4d_3 x \cos x + (2d_3 - 2d_2) \sin x + (2d_1 + 2d_4) \cos x = x \cos x.
\]
Equating coefficients of similar terms on both sides we have
\[
-4d_4 = 0, \ 4d_3 = 1, \ 2d_3 - 2d_2 = 0, \ 2d_1 + 2d_4 = 0.
\]
The solution is easily seen to be \( d_1 = d_4 = 0, \ d_2 = d_3 = \frac{1}{4}. \) Thus the particular solution sought is \( y_p(x) = \frac{1}{4}(x \cos x + x^2 \sin x) \) and the general solution is
\[
y(x, c_1, c_2) = c_1 \sin x + c_2 \cos x + \frac{1}{4}(x \cos x + x^2 \sin x).
\]