Linear Systems of Differential Equations

A *first order linear* n-dimensional system of differential equations takes the form

$$Y^{'}(t) \;=\; \mathbf{A}(t)\, Y(t) \;+\; B(t),$$

or, in expanded form,

$$\begin{pmatrix} y_1^{'}(t) \\ y_2^{'}(t) \\ \vdots \\ y_n^{'}(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

As usual, we define a *solution* of this system to be a differentiable n-vector function $Y(t)$ which reduces the above to an identity upon substitution. The system is *homogeneous* if $B(t) \;\equiv\; 0$ (the zero vector), *inhomogeneous* otherwise.

In our discussion we will assume that the functions $a_{kj}(t)$ forming the entries of the matrix $\mathbf{A}(t)$ and the functions $b_k(t)$ forming the components of the vector function $B(t)$ are (at least) piecewise continuous functions of the independent variable t; most examples involve continuous functions of t.

Example 1 The system of equations

$$\begin{pmatrix} y_1^{'}(t) \\ y_2^{'}(t) \end{pmatrix} \;=\; \begin{pmatrix} \frac{1}{t}\, y_1(t) \;+\; y_2(t) \\ \frac{2}{t}\, y_2(t) \end{pmatrix}$$

constitutes a 2 dimensional linear first order homogeneous system of differential equations, $0 \;<\; t \;<\; \infty$. If we change the system to

$$\begin{pmatrix} y_1^{'}(t) \\ y_2^{'}(t) \end{pmatrix} \;=\; \begin{pmatrix} \frac{1}{t}\, y_1(t) \;+\; y_2(t) \;+\; t \\ \frac{2}{t}\, y_2(t) \;+\; t^2 \end{pmatrix}$$

we have a 2 dimensional linear first order inhomogeneous system of differential equations. Here we have

$$\mathbf{A}(t) = \begin{pmatrix} \frac{1}{t} & 1 \\ 0 & \frac{2}{t} \end{pmatrix}, \quad B(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

The *general solution* of a linear homogeneous system $Y'(t) = \mathbf{A}(t)\,Y(t)$ takes the form

$$Y(t, c_1, c_2, ..., c_n) = c_1\,Y_1(t) + c_2\,Y_2(t) + \cdots + c_n\,Y_n(t),$$

where in this formula the $Y_k(t),\ k = 1, 2, ..., n$, are n-vector solutions of the system; thus

$$Y_k(t) = \begin{pmatrix} y_{1k}(t) \\ y_{2k}(t) \\ \vdots \\ y_{nk}(t) \end{pmatrix}.$$

Further, these solutions should constitute a *fundamental set* of n- vector solutions, by which we mean that, given any value of t_0 in an interval (a, b) in which the system satisfies our basic assumptions (continuity, etc.), and given an initial vector

$$Y_0 = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{n0} \end{pmatrix},$$

there is a unique vector of constants $C = (c_1\ c_2\ \cdots\ c_n)^*$ such that, with $Y(t, c_1, c_2, ..., c_n)$ in the form given, $Y(t_0, c_1, c_2, ..., c_n) = Y_0$. If we define a matrix $\mathbf{Y}(t)$ by specifying its columns to be the solutions $Y_k(t)$;

$$\mathbf{Y}(t) = [Y_1(t)\,Y_2(t)\ \cdots\ Y_n(t)],$$

this is the same thing as saying that

$$\mathbf{Y}(t_0)\,C = Y_0$$

has a unique solution C for any choice of the vector Y_0. This is true, of course, just in case $\det \mathbf{Y}(t_0) \neq 0$. In that case we have

$$C = \mathbf{Y}(t_0)^{-1} Y_0.$$

Example 2 In the homogeneous instance of Example 1 given above we can verify that

$$Y_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad Y_2(t) = \begin{pmatrix} \frac{1}{2} t^3 - \frac{1}{2} t \\ t^2 \end{pmatrix},$$

are vector solutions. The general solution then takes the form

$$Y(t, c_1, c_2) = c_1 \begin{pmatrix} t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{2} t^3 - \frac{1}{2} t \\ t^2 \end{pmatrix}.$$

The corresponding matrix $\mathbf{Y}(t)$:

$$\mathbf{Y}(t) = \begin{pmatrix} t & \frac{1}{2} t^3 - \frac{1}{2} t \\ 0 & t^2 \end{pmatrix}$$

has determinant $\det \mathbf{Y}(t) = t^3$ which does not vanish in the interval $0 < t < \infty$, so we see that these form a pair of fundamental solutions on that interval.

Definition An $n \times m$ matrix function $\mathbf{Y}(t)$ whose columns are vector solutions of the system $Y'(t) = \mathbf{A}(t)\, Y(t)$ is called a *matrix solution* of that system. If $\mathbf{Y}(t)$ is $n \times n$ and the columns are a fundamental set of solutions, i.e., if $\det \mathbf{Y}(t) \neq 0$, then $\mathbf{Y}(t)$ is called a *fundamental matrix solution.*

In either case, if we agree that the derivative of a matrix function $\mathbf{Y}(t)$ is the matrix function $\mathbf{Y}'(t)$ whose entries are the derivatives of the corresponding entries of $\mathbf{Y}(t)$, we have

$$\mathbf{Y}'(t) = \mathbf{A}(t)\, \mathbf{Y}(t).$$

Proposition 1 *Let* $\mathbf{Y}(t)$ *be an* $n \times n$ *matrix solution of the system* $Y^{'}(t) = \mathbf{A}(t)\,Y(t)$ *on an interval* $a < t < b$ *where* $\mathbf{A}(t)$ *is continuous (i.e., its entries* $a_{ij}(t)$ *are continuous there). Then the Wronskian determinant* $W(t, \mathbf{Y}) = \det\,\mathbf{Y}(t)$ *is either identically zero on* $a < t < b$ *or is never zero on that interval.*

Remark Thus the property of being a fundamental matrix solution of $Y^{'}(t) = \mathbf{A}(t)\,Y(t)$ is independent of the choice of t in any interval $a < t < b$ where the system matrix $\mathbf{A}(t)$ is a continuous function of t.

Proof The proof will require some properties of determinants. Suppose

$$\mathbf{M} = [M_1 \;\cdots\; M_j \;\cdots\; M_n]$$

is an $n \times n$ matrix with columns as indicated. Let $\hat{\mathbf{M}}$ be obtained from $\mathbf{M}$ by replacing the column M_j by $\hat{M}_j$;

$$\hat{\mathbf{M}} = [M_1 \;\cdots\; \hat{M}_j \;\cdots\; M_n].$$

Then

$$\textbf{i)}\ \det\,[M_1 \;\cdots\; \alpha\,M_j + \beta\,\hat{M}_j \;\cdots\; M_n] = \alpha\,\det\,\mathbf{M} + \beta\,\det\,\hat{\mathbf{M}}.$$

Further, if $\hat{M}_j = M_k$ for some $k \neq j$, then

$$\textbf{ii)}\ \det\,[M_1 \;\cdots\; \alpha\,M_j + \beta\,M_k \;\cdots\; M_n] = \alpha\,\det\,\mathbf{M}.$$

If $\mathbf{M} = \mathbf{M}(t)$ is differentiable (i.e., all of its entries are differentiable), then

$$\textbf{iii)}\ \frac{d\,(\det \mathbf{M}(\mathbf{t}))}{dt} = \sum_{j=1}^{n} \det\,[M_1(t) \;\cdots\; \hat{M}_j^{'}(t) \;\cdots\; M_n(t)].$$

Furthermore, all three of these properties remain true if, instead of working with the *columns* of $\mathbf{M}$, we work with the *rows* of $\mathbf{M}$.

We will complete the theorem working with the three dimensional case; the general n dimensional case is treated in essentially the same way. Thus we suppose that we have the system

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

and three solution vectors

$$Y_j(t) = \begin{pmatrix} y_{1j}(t) \\ y_{2j}(t) \\ y_{3j}(t) \end{pmatrix}, \quad j = 1, 2, 3.$$

forming the columns of a 3×3 matrix solution $Y(t)$. Differentiating $\det \mathbf{Y}(t)$ by rows we have (suppressing (t) now for brevity)

$$\frac{d\Big(\det \mathbf{Y}(t)\Big)}{dt} = \det \begin{pmatrix} y_{11}' & y_{12}' & y_{13}' \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

$$+ \det \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21}' & y_{22}' & y_{23}' \\ y_{31} & y_{32} & y_{33} \end{pmatrix} + \det \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31}' & y_{32}' & y_{33}' \end{pmatrix}.$$

Looking at just the first of these three matrices and using the differential equations implied by $\mathbf{Y}'(t) = A(t)\,\mathbf{Y}(t)$ we have

$$\begin{pmatrix} y_{11}' & y_{12}' & y_{13}' \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}y_{11} + a_{12}y_{21} + a_{13}y_{31} & a_{11}y_{12} + a_{12}y_{22} + a_{13}y_{32} & a_{11}y_{13} + a_{12}y_{23} + a_{13}y_{33} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}.$$

In the first row the second and third terms of each entry are just a_{12} times the corresponding entries of the second row of $\mathbf{Y}(t)$ plus a_{13}

times the corresponding entries of the third row of $\mathbf{Y}(t)$. Using the row versions of **i**) and **ii**), we get just $a_{11} \det \mathbf{Y}(t)$. The other two matrices in the formula for $\det \mathbf{Y}(t)$, manipulated in the same way, yield $a_{22} \det \mathbf{Y}(t)$ and $a_{33} \det \mathbf{Y}(t)$. Thus the final result becomes

$$\frac{d\Big(\det \mathbf{Y}(t)\Big)}{dt} = (a_{11}(t) + a_{22}(t) + a_{33}(t)) \det \mathbf{Y}(t) \equiv (Tr\, \mathbf{A}(t)) \det \mathbf{Y}(t).$$

(The trace of a square matrix $\mathbf{M}$ is the sum of its diagonal entries and is written $Tr\, \mathbf{M}$.) This is a first order scalar linear homogeneous equation and we thus have

$$\det \mathbf{Y}(t_1) = \exp\Big(\int_{t_0}^{t_1} Tr\, \mathbf{A}(s)\, ds\Big) \det \mathbf{Y}(t_0)$$

for any values of t_0 and t_1 in the interval (a, b). Since the exponential function is never zero, we see that $\det \mathbf{Y}(t_1) = 0$ if and only if $\det \mathbf{Y}(t_0) = 0$ and the proposition follows from this.

Proposition 2 *If* $\mathbf{Y}(t)$ *is an* $n \times n$ *matrix solution for* $Y'(t) = \mathbf{A}(t)\, Y(t)$ *and* $\mathbf{C}$ *is a constant* $n \times m$ *matrix, then* $\mathbf{Y}(t)\, \mathbf{C}$ *is an* $n \times m$ *matrix solution for* $Y'(t) = \mathbf{A}(t)\, Y(t)$.

Remark: $\mathbf{C}$ could be an $n \times 1$ matrix; i.e., a column vector C; then $\mathbf{Y}(t)\, C$ is a vector solution.

Proof We just multiply the matrix equation $\mathbf{Y}'(t) = \mathbf{A}(t)\, \mathbf{Y}(t)$ on the right by $\mathbf{C}$ and use $\mathbf{Y}'(t)\mathbf{C} = (\mathbf{Y}(t)\mathbf{C})'$.

Now suppose we want to find the solution of an initial value problem

$$Y'(t) = \mathbf{A}(t)\, Y(t), \quad Y(t_0) = Y_0,$$

where t_0 is a given value of the independent variable and Y_0 is a given vector. Suppose we have a fundamental matrix solution $\mathbf{Y}(t)$; i.e., one for which $\det \mathbf{Y}(t) \neq 0$. Let us try to find a solution of the initial value problem in the form $Y(t) = \mathbf{Y}(t)\, C$, where C is a (constant) column vector. Our proposition shows that $Y(t)$, thus defined, is a solution. Evaluating at $t = t_0$ we have

$$Y(t_0) = \mathbf{Y}(t_0)\, C = Y_0 \longrightarrow C = \mathbf{Y}(t_0)^{-1}\, Y_0.$$

From this we see that the initial value problem has the (unique) solution

$$Y(t) = \mathbf{Y}(t)\, \mathbf{Y}(t_0)^{-1}\, Y_0.$$

Thus we can solve initial value problems with ease once we have a fundamental matrix solution.

Example 3 Let us again consider the system

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{t}\, y_1(t) + y_2(t) \\ \frac{2}{t}\, y_2(t) \end{pmatrix}.$$

Suppose we want the solution corresponding to $y_1(2) = 1$, $y_2(2) = -1$. From our earlier example

$$\mathbf{Y}(t) = \begin{pmatrix} t & \frac{1}{2}\, t^3 - \frac{1}{2}\, t \\ 0 & t^2 \end{pmatrix}$$

is a matrix solution; since its determinant is t^3, it is a fundamental matrix solution for $t > 0$. Accordingly, the solution of the given initial value problem at $t = 2$ is given by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} t & \frac{1}{2}\, t^3 - \frac{1}{2}\, t \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} 2 & \frac{1}{2}\, 2^3 - \frac{1}{2}\, 2 \\ 0 & 2^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 2 & \frac{1}{2}\, 2^3 - \frac{1}{2}\, 2 \\ 0 & 2^2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{8} \\ 0 & \frac{1}{4} \end{pmatrix}$$

the solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} t & \frac{1}{2}\, t^3 - \frac{1}{2}\, t \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{3}{8} \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} t - \frac{1}{8} t^3 \\ -\frac{1}{4} t^2 \end{pmatrix}.$$

The fact that the solution of the initial value problem $Y(t_0) = Y_0$ can be written in the form

$$Y(t) = \mathbf{Y}(t)\,\mathbf{Y}(t_0)^{-1}\,Y_0$$

for a (.e., any) fundamental matrix solution $\mathbf{Y}(t)$ justifies referring to $\mathbf{Y}(t)\,C$, for an arbitrary n-vector C, as a *general (vector) solution* for the system $Y'(t) = \mathbf{A}(t)\,Y(t)$; the choice $C = \mathbf{Y}(t_0)^{-1}Y_0$ yields $Y(t)$ with $Y(t_0) = Y_0$.

We have seen that for a matrix solution $\mathbf{Y}(t)$ and an $n\times m$ constant matrix $\mathbf{C}$, $\mathbf{Y}(t)\,\mathbf{C}$ is also a matrix solution. If $\mathbf{Y}(t)$ is a fundamental matrix solution, thus $n \times n$ and nonsingular for each t under consideration, then, given t_0, $\mathbf{Y}(t_0)^{-1}$ exists and

$$\mathbf{Y}(t, t_0) \equiv \mathbf{Y}(t)\mathbf{Y}(t_0)^{-1}$$

is again a fundamental matrix solution. This particular fundamental solution has the special property $\mathbf{Y}(t_0, t_0) = \mathbf{I}$, the $n \times n$ identity matrix. In general, for arbitrary t, τ, $\mathbf{Y}(t,\, \tau)$ is the fundamental matrix solution such that

$$\mathbf{Y}(\tau,\, \tau) = \mathbf{Y}(t,\, t) = \mathbf{I}.$$

Example 4 For the system in Example 3 we have

$$\mathbf{Y}(t) = \begin{pmatrix} t & \frac{1}{2}\,t^3 - \frac{1}{2}\,t \\ 0 & t^2 \end{pmatrix}$$

for which the inverse matrix at $t = \tau$ and $Y(t,\, \tau)$ are, respectively,

$$\mathbf{Y}(\tau)^{-1} = \begin{pmatrix} \frac{1}{\tau} & \frac{1}{2}\,\frac{1}{\tau^2} - \frac{1}{2} \\ 0 & \frac{1}{\tau^2} \end{pmatrix},\ \mathbf{Y}(t,\, \tau) = \mathbf{Y}(t)\mathbf{Y}(\tau)^{-1} = \begin{pmatrix} \frac{t}{\tau} & \frac{t^3}{2\,\tau^2} - \frac{t}{2} \\ 0 & \frac{t^2}{\tau^2} \end{pmatrix}.$$