Name ______________________

Exam 1; Math 4426. Due: 02-26-08

1. A matrix $\mathbf{A}$ is antisymmetric (equivalently, antihermitian) if $\mathbf{A}^* = -\mathbf{A}$.

i) Show that if $\mathbf{A}$ is antisymmetric and λ is an eigenvalue of $\mathbf{A}$, then λ is purely imaginary or zero; i.e., either $\lambda = 0$ or $\lambda = i\,\omega$ for some real number ω. Show that if $\mathbf{A}$ is real, $n \times n$, and n is odd, then there is a zero eigenvalue.

ii) Find the eigenvalues λ and eigenvectors Φ of $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$

by solving, analytically, $\det(\lambda\mathbf{I} - \mathbf{A}) = 0, \quad (\lambda\mathbf{I} - \mathbf{A})\,\Phi = 0$.

2. Let $\mathcal{D}$ be the differential operator given, for $x \in [0,1]$, by $(\mathcal{D}y)(x) = y'''(x)$. Consider the boundary conditions:

$$y(0) = 0,\ \ y(1) = 0,\ \ y'(0) = y'(1).$$

Show $\mathcal{D}$ is antisymmetric with respect to the standard inner product in $L^2[0,1]$. Compute the eigenvalues and eigenfunctions.

3. Let $\mathcal{D}$ be the differential operator with domain $\mathcal{S}$:

$$(\mathcal{D}y)(\xi) = \xi^2 \frac{d^2y}{d\xi^2} + \xi \frac{dy}{d\xi}; \ \ \mathcal{S} = \Big\{ y \in PC^2[1,e] \,\Big|\, y(1) = y(e) = 0 \Big\}.$$

Find the eigenvalues and eigenfunctions of $\mathcal{D}$. Hint: make the change of independent variable $x = \log(\xi)$.

4. Consider the eigenvalue - eigenfunction problem

$$\frac{d}{dx}\left(p(x)\,\frac{dy}{dx}\right) + (q(x) + \lambda\,\rho(x))\,y = 0, \ \ x \in [0, 2\pi], \ \ y(0) = 0, \ \ y(2\pi) = 0,$$

with $q(x) \equiv 1$, $p(x) = \rho(x) = 2 + \sin x$. Use the Sturm Comparison Theorem to find $\Lambda > 0$ for which we can be sure there is an eigenvalue λ with $0 < \lambda < \Lambda$. You should make Λ as small as you can; pure guessing will not be accepted.

5. Find the eigenvalues and eigenfunctions for the differential operator

$$(\mathcal{D}y)(x) = y^{(iv)}(x),$$

with domain

$$\mathcal{S} = \Big\{ y \in PC^4[0,1] \,\Big|\, y'(0) = y'''(0) = 0, \ \ y(1) = y''(1) = 0 \Big\}.$$