Practice Exam 2: Math 2224, Fall, 2009

1. Let $f(x,y) = \sqrt{x^2+4y^2-1}$, $g(z) = \sqrt{9-z^2}$.

a) Describe precisely which points (x,y) lie in the domain of $f(x,y)$ and which numbers z lie in the domain of $g(z)$.
b) Describe precisely which numbers lie in the range of $f(x,y)$ and which numbers lie in the range of $g(z)$.

2. For $f(x,y)$ as in **1.** sketch: **a)** the contour curve $\mathcal{C}(f) = \{(x,y)\,|f(x,y) = 2\}$; **b)** the contour curve of $f(x,y)$ which passes through the point $(1,2)$.

3. For each of the following, determine **i)** if $lim_{(x,y)\to(0,0)} f(x,y)$ exists; **ii)** if $f(x,y)$ is continuous at $(0,0)$; **c)** if $f(x,y)$ is continuously differentiable at $(0,0)$. Justify your conclusions in each case.

$$\textbf{a) } f(x,y) = \frac{x^4-y^4}{x^4+y^4}; \quad \textbf{b) } f(x,y) = \frac{x^2-y^2}{|x|+|y|},\ (x,y)\neq(0,0),\ f(0,0) = 0.$$
$$\textbf{c) } f(x,y) = \frac{x^2y^2}{x^2+y^2},\ (x,y)\neq(0,0),\ f(0,0) = 0.$$

4. Let $f(x,y)$ be defined as in Problem 1. **a)** Compute the linear approximation to $f(x,y)$ at the point $(x_0,y_0) = (1,2)$; **b)** Use that approximation to estimate $f(.9,2.1)$ and compare with the actual value $f(.9,2.1)$.

5. A roller-coaster track traces out the curve $X(t) = (x(t)\,y(t)\,z(t))^*$ in R^3, with $x(t) = \cos t$, $y(t) = 3\sin t$, $z(t) = g(x(t),y(t))$, where $g(x,y) = (x+y)^2$. Use the Chain Rule to find the slope of the track at the point corresponding to $t = \pi/4$. Where is the highest point on the track and how high is it at that point?

6. A certain function $h(x,\,y,\,z)$ has gradient

$$\nabla h(x,\,y,\,z) = \begin{pmatrix} y-z \\ x+z \\ -x+y \end{pmatrix}. \qquad \text{Let } P = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix},\quad Q = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}.$$

Assume $h(P) = 4$. Use the Chain rule to compute

$$h(Q) = h(P) + \int_0^1 \frac{d}{dt}(h(X(t)))\,dt, \quad X(t) = (1-t)\,P + t\,Q.$$

7. Find a and b for which $y = ax+b$ provides the least squares linear fit to the values of $\sin x$ at $x_1 = 0$, $x_2 = \pi/6$, $x_3 = \pi/4$, $x_4 = \pi/3$ and $x_5 = \pi/2$. Verify that the values of a and b which you obtain are a global minimum for the function

$$\ell(a,\,b) = \sum_{k=1}^{5} (a\,x_k + b - \sin(x_k))^2\,.$$

8. List all of the critical points (i.e., points where the gradient equals 0), for the function $f(x,y) = (x^3-x)\,(y^3-y)$. Find at least one of these which is neither a maximum nor a minimum.