BIFURCATING SOLUTIONS AT THE ONSET OF CONVECTION IN THE BÉNARD PROBLEM FOR TWO FLUIDS

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We consider two fluids with different thermal and mechanical properties arranged in parallel layers between two infinite horizontal plates. The bottom plate is kept at a higher temperature than the top plate. In the unbounded directions we impose periodic boundary conditions with the periods chosen such that the problem has hexagonal symmetry.

In contrast to the Bénard problem for one fluid, the onset of convection in the two-fluid Bénard problem considered here can be oscillatory. The oscillations are essentially due to the competition between the destabilizing temperature gradient and a stable interface between the two fluids. The hexagonal symmetry of the problem causes a sixfold degeneracy of the critical eigenvalues. On the "abstract" level, Hopf bifurcations with this type of symmetry-induced degeneracy were investigated by Roberts, Swift and Wagner. They showed that there are eleven qualitatively different types of bifurcating solutions and they identified the parameters which determine the stability of these solutions. In this paper, we apply their results to the two-fluid Bénard problem. Since the eigenfunctions at criticality are not explicitly known in this problem, we shall use a combination of analysis and numerical computation. In the examples we study, we find that most branches are subcritical and none are stable near the bifurcation point.

1. Introduction

The Bénard problem is concerned with the onset of thermal convection in a fluid filling the space between two infinite horizontal plates. Each of the plates has constant temperature, and the temperature of the lower plate is the higher one. In studying this problem mathematically, it is customary to look for solutions which are doubly periodic in the unbounded directions. Particular attention has been focussed on the case of solutions which are doubly periodic with respect to a hexagonal lattice. This case is of interest from a physical point of view because solutions of this nature are observed in experiments and from a mathematical point of view because the symmetry of the hexagonal lattice causes a sixfold degeneracy of the critical eigenvalue at the onset of convection. This degeneracy leads to an interesting problem of pattern selection. For a discussion of bifurcating solutions and their stability we refer to [2, 6].

In this paper, we consider the situation where the space between the plates is not filled by one fluid, but by two fluids with different thermal and mechanical properties. We study bifurcation from the trivial state where the fluids are arranged in parallel layers and the temperature profile in each fluid is linear. The essential new feature is that, in contrast to the Bénard problem for one fluid [4], the critical eigenvalues can now be purely imaginary [11, 12, 15] (see also [1] for a related problem). Essentially, the oscillatory nature
of the instability is due to the opposing influences of a destabilizing temperature gradient and a stable interface.

In [12, 13], the linear stability problem has been studied for the case when the two fluids have nearly equal properties. The results indicate that the factors which are most essential for the stability of the interface are surface tension, density difference across the interface, and thermal conductivity stratification. The effects of surface tension and density difference are rather obvious, but the influence of thermal conductivity stratification is more subtle. For disturbances of long to moderate wavelength, thermal conductivity stratification is stabilizing if the less conducting fluid is in a thin layer [13]. Indeed, thermal conductivity stratification becomes the dominant influence on stability of the interface as the thickness of the layer tends to zero. These results are very much reminiscent of the role of viscosity stratification in shearing flows (see [14] for a review). Roughly speaking, the temperature gradient assumes the role of shear, and thermal conductivity assumes the role of viscosity. For short wave disturbances, on the other hand, thermal conductivity stratification is always destabilizing, as will be shown in the appendix. This is also the case for viscosity stratification in shearing flows [7]. With \( \alpha \) denoting the wavenumber of the disturbance, the growth rates of short wave instabilities resulting from thermal conductivity stratification behave like \( \alpha^{-2} \). The effects of density difference and surface tension, on the other hand, are of orders \( \alpha^{-1} \) and \( \alpha \), respectively, and will thus dominate for large \( \alpha \) if they are present. We also mention the recent work of Yih [17] which shows that thermal conductivity stratification can destabilize the flow down an inclined heated plane. The results of [13] and the present paper show that the presence of motion in the fluid is in no way essential for the occurrence of such instabilities.

In a recent paper, Roberts, Swift and Wagner [16] discuss Hopf bifurcations with the symmetry of a hexagonal lattice. They establish the existence of eleven qualitatively different types of bifurcating solutions and identify the parameters determining their stability. The equations discussed in [16] are given in a reduced canonical form, and in order to apply the results of [16] to a concrete problem one must first go through the reduction process. This reduction process involves the determination of a center manifold and the theory of normal forms (see [3, 5]). In this paper, we shall proceed formally. A rigorous justification of the center manifold approach requires coercive estimates for the underlying partial differential equations; such estimates were derived in [10]. After deriving the reduced form of the equations, we shall discuss three cases suggested by the above discussion of interfacial stability;

1. All fluid properties are equal, but there is surface tension between the two fluids.
2. The thermal expansion coefficients differ in such a way that a stabilizing density difference across the interface results.
3. The two fluids have a thermal conductivity stratification and depth ratio which is stabilizing for disturbances of long to moderate wavelengths. We include enough surface tension to stabilize short wave disturbances, but keep surface tension small enough so that thermal conductivity stratification remains the dominant influence at the critical wavelength.

For each of these cases, we shall discuss the stability of the bifurcating solutions.

The paper is organized as follows. In section 2, we introduce the equations governing the two-fluid Bénard problem. In section 3, we discuss the relevant results concerning linear stability. Section 4 gives a summary of the results of [16]. In sections 5 and 6, we go through the reduction process that is necessary to transform the equations for the two-fluid Bénard problem to the form used in [16]. The results for the three cases outlined above are then given in section 7. None of the branches are found to be stable near the bifurcation point.
2. Formulation of the problem

The domain occupied by the two fluids is the space between two infinite horizontal plates located at \( z^* = 0 \) and \( z^* = 1^* \). The location of the interface is given by \( z^* = h^*(x^*, y^*, t^*) \). We refer to the fluid below the interface as fluid 1 and to the fluid above the interface as fluid 2. Fluid \( i \) has viscosity \( \mu_i \), density \( \rho_i \), thermal conductivity \( k_i \), thermal diffusivity \( \kappa_i \) and thermal expansion coefficient \( \alpha_i \). In each fluid, the governing equations are Fourier’s law of heat conduction and the Navier–Stokes equations. We use the Boussinesq approximation, which consists of assuming that all fluid properties are independent of temperature, except the density. The density is treated as a linear function of temperature in the gravity term, but is considered constant elsewhere in the equations. Let \( \theta^* \) denote the temperature, \( \mathbf{v}^* = (u^*, v^*, w^*) \) the velocity and \( p^* \) the pressure.

We nondimensionalize the problem by introducing the following rescaled variables:

\[
(x, y, z) = (x^*, y^*, z^*)/l^*, \quad t = \kappa^* t^*/(l^*)^2,
\]

\[
\mathbf{v} = \mathbf{v}^*/\kappa_1, \quad \theta = (\theta^* - \theta_0)/(\theta_1 - \theta_0), \quad p = p^*/(\rho_1 \kappa_1^2),
\]

where \( \theta_0 \) and \( \theta_1 \) are the temperatures of the upper and lower plates, respectively. Expressed in the new variables, the equations in fluid 1 read

\[
\begin{align*}
\dot{\theta} + (\mathbf{v} \cdot \nabla) \theta &= \Delta \theta, \\
\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= P \Delta \mathbf{v} - \nabla p + (R \mathbf{\theta} - G) \mathbf{e}_z, \\
\text{div} \: \mathbf{v} &= 0.
\end{align*}
\]

The dimensionless parameters contained in this set of equations are the Rayleigh number,

\[
R = \frac{g \alpha_1 (\theta_1 - \theta_0) (l^*)^3 \rho_1}{\kappa_1 \mu_1},
\]

the Prandtl number,

\[
P = \frac{\mu_1}{\kappa_1 \rho_1},
\]

and a dimensionless measure of gravity,

\[
G = \frac{g (l^*)^3}{\kappa_1^2}.
\]

We denote the ratios of fluid properties as follows:

\[
m = \mu_1/\mu_2, \quad r = \rho_1/\rho_2, \quad \gamma = \kappa_1/\kappa_2, \quad \zeta = k_1/k_2, \quad \beta = \alpha_1/\alpha_2.
\]
Then the dimensionless equations in fluid 2 are as follows:

\[ \dot{\theta} + (v \cdot \nabla) \theta = \frac{1}{\gamma} \Delta \theta, \]

\[ \dot{\theta} + (v \cdot \nabla) v = \frac{r}{m} \Delta v - r \nabla p + \left( \frac{RP}{\beta^*} \theta - G \right) e_z, \]

\[ \text{div} \, v = 0. \]

(7)

The boundary conditions at the plates are

\[ v = 0, \quad \theta = 1 \quad \text{at} \quad z = 0, \]

\[ v = 0, \quad \theta = 0 \quad \text{at} \quad z = 1. \]

(8)

At the interface, we must have continuity of velocity, temperature and heat flux, and balance of tractions. To formulate these conditions, we first introduce some notations. With \( l_1^* \) denoting the average value of \( h^* \), we define a dimensionless depth ratio by \( l_1 = l_1^*/l^* \); we shall use the notation \( l_2 = 1 - l_1 \). We set

\[ h(x, y, t) = h^*(x^*, y^*, t^*)/l^* - l_1. \]

(9)

The interface is then at \( z = l_1 + h(x, y, t) \). Let \( t_1, t_2 \) denote two unit vectors parallel to the interface, and let \( n \) be a unit normal to the interface:

\[ t_1 = (1, 0, \frac{\partial h}{\partial x})/\sqrt{1 + \left( \frac{\partial h}{\partial x} \right)^2}, \]

\[ t_2 = (0, 1, \frac{\partial h}{\partial y})/\sqrt{1 + \left( \frac{\partial h}{\partial y} \right)^2}, \]

\[ n = \left(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1\right)/\sqrt{1 + \left( \frac{\partial h}{\partial x} \right)^2 + \left( \frac{\partial h}{\partial y} \right)^2}. \]

(10)

By \([ \cdot \cdot \cdot ]\) we denote the jump of a quantity across the interface, i.e. its value in fluid 1 minus its value in fluid 2. By \( T \) we denote the dimensionless stress tensor, i.e. \( T = P[\nabla v + (\nabla v)^T] - pI \) in fluid 1, and \( T = (P/m)[\nabla v + (\nabla v)^T] - pI \) in fluid 2. The dimensionless surface tension parameter is denoted by \( S \). The conditions to be satisfied at the interface are

\[ [v] = 0, \quad [t_i \cdot T \cdot n] = 0, \quad i = 1, 2, \]

\[ [n \cdot T \cdot n] = \frac{S\left( \frac{\partial^2 h}{\partial x^2} \left( 1 + \left( \frac{\partial h}{\partial x} \right)^2 \right) + \frac{\partial^2 h}{\partial y^2} \left( 1 + \left( \frac{\partial h}{\partial y} \right)^2 \right) - 2 \frac{\partial^2 h}{\partial x \partial y} \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \right)}{(1 + (\nabla h)^2)^{3/2}}, \]

(11)

\[ [\theta] = 0, \quad [kn \cdot \nabla \theta] = 0, \]

\[ \dot{h} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} = w. \]
A trivial solution to the problem is given by

\[ h = 0, \quad v = 0, \quad \vartheta = 1 - A_1 z \quad \text{for} \quad 0 \leq z \leq l_1, \quad \vartheta = A_2 (1 - z) \quad \text{for} \quad l_1 \leq z \leq 1, \]

\[ p = p_1 - Gz + RPz - \frac{RPA_1}{2} z^2, \quad 0 \leq z \leq l_1, \]

\[ p = p_2 - \frac{G}{r} z + \frac{RPA_2}{\beta} z^2 - \frac{RPA_2}{2r\beta} z^2, \quad l_1 \leq z \leq 1. \]  

(12)

Here we have set

\[ A_1 = \frac{1}{l_1 + \xi l_2}, \quad A_2 = \xi A_1, \]  

(13)

and \( p_1 - p_2 \) must be chosen such that \( p \) is continuous at \( z = l_1 \).

We denote by \( \tilde{\vartheta} \) the difference between \( \vartheta \) and the trivial solution (12), and by \( \tilde{p} \) the difference between \( p \) and the trivial solution. This changes the equations in fluid 1 to

\[ \dot{\vartheta} + (v \cdot \nabla) \vartheta - A_1 w = \Delta \tilde{\vartheta}, \]

\[ \dot{v} + (v \cdot \nabla) v = P \Delta v - \nabla \tilde{p} + RP \tilde{e}_z, \]  

(14)

\[ \text{div} \, v = 0. \]

In fluid 2 the equations of motion become

\[ \dot{\vartheta} + (v \cdot \nabla) \vartheta - A_2 w = \frac{1}{r} \Delta \tilde{\vartheta}, \]

\[ \dot{v} + (v \cdot \nabla) v = \frac{r}{m} P \Delta v - r \nabla \tilde{p} + \frac{RP}{\beta} \tilde{e}_z, \]

(15)

\[ \text{div} \, v = 0. \]

The boundary conditions on both walls are now

\[ v = 0, \quad \tilde{\vartheta} = 0, \]  

(16)

and the interface conditions are

\[ [v] = 0, \quad [t_i \cdot \tilde{T} \cdot n] = 0, \quad i = 1, 2, \]

\[ [n \cdot \tilde{T} \cdot n] = M_1 h + M_2 h^2 + \frac{S}{(1 + (\nabla h)^2)^{3/2}} \left\{ \frac{\partial^2 h}{\partial x^2} \left( 1 + \left( \frac{\partial h}{\partial y} \right)^2 \right) + \frac{\partial^2 h}{\partial y^2} \left( 1 + \left( \frac{\partial h}{\partial x} \right)^2 \right) - 2 \frac{\partial^2 h}{\partial x \partial y} \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \right\}. \]

(17)

\[ [\tilde{\vartheta}] = h (A_1 - A_2), \quad [kn \cdot \nabla \tilde{\vartheta}] = 0, \]

\[ \dot{h} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} = w. \]
Here we have set $\tilde{T} = P[\nabla v + (\nabla v)^T] - \tilde{\rho} \mathbf{I}$ in fluid 1, and $\tilde{T} = (P/m)[\nabla v + (\nabla v)^T] - \tilde{\rho} \mathbf{I}$ in fluid 2. Moreover, we have

$$M_1 = G \left( \frac{1}{r} - 1 \right) + RPA_2 (1 - l_1) \left( 1 - \frac{1}{r\beta} \right),$$

$$M_2 = \frac{RP}{2} \left( \frac{A_2}{r\beta} - A_1 \right).$$

(18)

In this paper, we are concerned with solutions bifurcating from the trivial solution. One of the features which make the two-fluid Bénard problem complicated is the fact that the domain occupied by each fluid is unknown and the interface must be determined as part of the solution. In theoretical studies, it is customary to introduce a mapping which transforms the domain occupied by each fluid to a fixed domain. Thus the partial differential equations are posed on a known domain; however, the coefficients in the transformed equations become rather complicated. This makes it cumbersome to apply the method in actual calculations. In the situation considered in this paper, the perturbation $h$ of the interface position is small, and we are only concerned with the terms of leading order in an asymptotic expansion which exploits this smallness. This makes a much simpler approach possible. We shall solve eqs. (14) and (15) on the unperturbed domains $0 \leq z \leq l_1$ and $l_1 \leq z \leq 1$, respectively. We think of the solutions on the unperturbed domains as being smoothly extended up to the interface $z = l_1 + h$. We then pose interface conditions at $z = l_1$ which result from a Taylor expansion of the conditions (17) with respect to $h$, e.g.

$$\hat{\theta}(z = l_1 + h) = \left[ \hat{\theta}(z = l_1) + h \left[ \hat{\theta}_z(z = l_1) + \frac{h^2}{2} \left[ \hat{\theta}_{zz}(z = l_1) + \cdots \right] = h(A_1 - A_2). \right]$$

(19)

These interface conditions are then not satisfied exactly, but only to within a certain order of the small parameter. For more information on domain perturbations we refer to the articles of Joseph [8] and Lebovitz [9].

In the $x$- and $y$-direction, we assume that the solution is doubly periodic, i.e. we have $\hat{\theta}(x + n_1x_1 + n_2x_2, t) = \hat{\theta}(x, t)$ for every pair of integers $(n_1, n_2)$. Here $x = (x, y, z)$ and the vectors $x_1$ and $x_2$ span a hexagonal lattice of period $L$:

$$x_1 = L \cdot \left( \sqrt{3}/2, \frac{1}{2}, 0 \right), \quad x_2 = L \cdot (0, 1, 0).$$

(20)

The same periodicity condition holds for $v, \tilde{\rho}$ and $h$. We can then expand $\hat{\theta}$ in a Fourier series with modes proportional to $e^{i a \cdot x}$. Here, $e^{i a \cdot x}$ must satisfy the same periodicity condition as $\hat{\theta}(x, y, z, t)$. Thus, $e^{i a \cdot x_1}$ and $e^{i a \cdot x_2}$ must be 1, and therefore $a \cdot x_1$ and $a \cdot x_2$ must be multiples of $2\pi$. We set up the vectors $a_1$ and $a_2$ to span such vectors (the reciprocal lattice) as follows: $a_1 \cdot x_j = 2\pi \delta_{ij}$. This leads to

$$a_1 = \frac{4\pi}{L\sqrt{3}} (1, 0, 0), \quad a_2 = \frac{4\pi}{L\sqrt{3}} (-\frac{1}{2}, \sqrt{3}/2, 0),$$

(21)

$$a = ka_1 + la_2,$$

where $k, l$ are integers. The Fourier expansion of $\hat{\theta}$ reads

$$\hat{\theta}(x, y, z, t) = \sum_{k, l=-\infty}^{\infty} \hat{\theta}_{k,l}(z, t) e^{i k a_1 \cdot x + i l a_2 \cdot z}.$$  

(22)
3. The linearized problem

We now consider the linearization of problem (14)-(17). In this linearized problem, we can separate variables, i.e. we look for solutions in the form

\[ \tilde{\theta}(x, y, z, t) = e^{i\omega t} e^{i k a_1 x + i l a_2 z} \tilde{\theta}(z), \]

and similarly for \( \tilde{v} \), \( \tilde{p} \) and \( h \). This leads to an eigenvalue problem for \( \omega \). Due to symmetry under rotation about the \( z \)-axis, the eigenvalues do not depend on the direction of the vector \( ka_1 + la_2 \), but only on its magnitude \( \alpha \):

\[ \alpha = |ka_1 + la_2|^2 = \frac{16 \pi^2}{3 L^2} \left( k^2 + l^2 - kl \right). \]

For a specific wavenumber \( \alpha \), this equation determines the period \( L \) of the lattice. Since \( k \) and \( l \) are arbitrary integers, the factor \( k^2 + l^2 - kl \) can be 0, 1, 3, 4, 7, \ldots. The mode \( k = l = 0 \) is not of interest in the linear problem, but will enter into the nonlinear interactions later. The smallest nonzero value of \( k^2 + l^2 - kl \) is 1 for which \( L = L_1 = 4 \pi / \sqrt{3} \alpha \). This occurs for six possible pairs \((k, l)\): \((\pm 1, 0), (0, \pm 1), (1, 1), \) and \((-1, -1)\). Thus, there is a sixfold degeneracy of the corresponding eigenvalue. We note that the solutions with period \( L_1 \) are also permitted as solutions on lattices with larger periods. At a value of \( k^2 + l^2 - kl \) larger than one, two possibilities exist:

(i) There is again a sixfold degeneracy of the eigenvalue. For example, \( k^2 + l^2 - kl = 3 \) occurs for \((k, l) = (1, 2), (2, 1), (-1, -2), (-2, -1), \) and \((-1, 1)\). Here, the value \( L \) from (24) is \( L_1 \sqrt{3} = 4 \pi / \alpha \). However, the solutions with lattice period \( L_1 \) are also solutions with lattice period \( L_1 \sqrt{3} \) with respect to the spanning vectors \( L_1 \sqrt{3} (1, 0, 0) \) and \( L_1 \sqrt{3} (\frac{1}{2}, \sqrt{3} / 2, 0) \). Thus the case \( k^2 + l^2 - kl = 3 \) will yield the same physical solutions as the case \( k^2 + l^2 - kl = 1 \); only the orientation of the patterns with respect to the lattice is different.

(ii) There is a higher than sixfold degeneracy. For example, the case \( k^2 + l^2 - kl = 7 \) leads to the possibilities \((3, 2), (2, 3), (-3, -2), (-2, -3), (3, 1), (1, 3), (-3, -1), (-1, -3), (-2, -1), (1, -2), (2, -1), \) and \((-1, 2)\). In this case, it is possible that new patterns arise, in addition to the solutions for \( k^2 + l^2 - kl = 1 \). The classification of such patterns is an open problem.

We now set \( k^2 + l^2 - kl = 1 \) and consider the eigenvalue problem for the case \( l = 0, k = 1 \). In this case the equations for \( \tilde{v} \) are decoupled from the rest of the problem, and it is easy to check that they only lead to negative eigenvalues. The remaining equations were solved numerically by the Chebyshev-tau method; we refer to [11] for details. Some guidance for choosing parameters for the numerical calculations is obtained from limiting situations where closed-form expressions for the interfacial eigenvalue are available. One such situation is the short-wave limit. In this case, the asymptotics of the interfacial eigenvalue can be obtained by focussing on a mode that decays rapidly away from the interface; the boundary conditions at the walls become irrelevant. It is found in eq. (32) of [13] that as \( \alpha \to \infty \) the interfacial eigenvalue is, at leading order, given by

\[ \sigma \sim \frac{1}{2 \alpha (1 + 1/m)} P \left( M_1 - \alpha^2 S \right), \]
where $M_1$ is given by (18). The dominant term is due to surface tension. Differences in density and the coefficient of cubical expansion enter at order $\alpha^{-1}$ (through $M_1$). If the density and coefficient of cubical expansion are equal and there is no surface tension, then the stratification in thermal conductivity is important. We show in the appendix (see eq. (A.16)) that thermal conductivity stratification results in short-wave instability.

A second situation where closed-form expressions can be obtained is the case where the two fluids have similar mechanical and thermal properties and the walls are replaced by stress-free slip surfaces [12, 13]. This problem is a perturbation of a one-fluid Bénard problem with a (neutrally stable) interface; the boundary conditions are chosen so that the one-fluid problem can be solved in closed form. We summarize the relevant results from [12] and [13] for similar liquids:

1. If the Rayleigh number is below criticality, then the interfacial eigenvalue $\sigma$ is real. In addition, we have:
   (a) Surface tension is stabilizing.
   (b) If the fluids differ only in thermal conductivity, then $\sigma$ is unstable if the thicker layer is the less conducting fluid and stable if the thicker layer is the more conducting fluid.
   (c) Density difference across the interface has the expected effect of being stabilizing if the lower fluid is heavier and destabilizing if the upper fluid is heavier. A dimensionless measure of the density difference across the interface is given by the quantity $M_1$ (see (18)), which involves the ratio $r$ of the densities (at the temperature of the upper plate) and the ratio $\beta$ of the thermal expansion coefficients. We note that, in applications where the Boussinesq approximation is justifiable, $G/RP$ is usually large, and therefore the first term in $M_1$ is dominant unless $r$ is very close to 1.
   (d) The stratification in viscosity and thermal diffusivity does not appear in $\sigma$ at leading order.

2. If the one-fluid problem is at criticality, then both the interfacial eigenvalue and one of the one-fluid eigenvalues are zero. When the problem is perturbed so that the two fluids differ slightly, this double eigenvalue zero may split up into real eigenvalues or into a complex conjugate pair. The following discussion applies to the case where the properties of fluid 1 are held fixed and fluid 2 is perturbed to have slightly different properties.
   (a) The introduction of surface tension leads to stable, complex conjugate eigenvalues.
   (b) When the fluids differ only in thermal conductivity, then real eigenvalues and instability result when the thicker layer is the less conducting fluid. When the thicker layer is more conducting, the eigenvalues become complex. Stability depends on the Prandtl number and depth ratio.
   (c) A density difference across the interface results in real eigenvalues and instability if the upper fluid is heavier. If the lower fluid is heavier, the eigenvalues become complex. If $r > 1$ and $G$ is large, the eigenvalues become stable. If the density difference at the interface is due to a difference in thermal expansion coefficients, the stability depends on the Prandtl number and depth ratio.
   (d) When the fluids differ only in viscosity or in thermal diffusivity, the eigenvalues are real. If the upper fluid (which is the one that is perturbed) is made less viscous or less diffusive, then instability results. This is expected because the introduction of a less viscous or less diffusive fluid increases the "effective" Rayleigh number.

From this discussion of similar liquids we conclude that three factors are important for obtaining a Hopf bifurcation: surface tension, density difference across the interface and difference in thermal conductivity. We have chosen to look at three situations where one of these factors dominates the problem. The density difference across the interface is generated by a difference in cubical expansion coefficients.
rather than a difference in densities at the temperature of the upper plate. This is because in applications $G$ is very large, and if $r$ were larger than 1, we would either have to choose $r$ extremely close to 1 or $R$ very large to get any instability. In all the following examples the Prandtl number is $P = 1$ and, since $r = 1$, $G$ does not enter into the problem.

Case (i): All fluid properties are equal but surface tension is non-zero

We choose $l_1 = 0.5$, $S = 0.01$. In this case, we find criticality at $R = 1707.94031$, $\alpha = 3.11612$ with a computed $\sigma = -0.18 \times 10^{-7} + 0.19i$. Eigenvalues for other values of $\alpha$ are stable. The least stable one-fluid eigenvalue at $\alpha = 0$ is $\sigma = -9.87$. It forms a complex conjugate pair with the interfacial eigenvalue for $\alpha$ between 2.8 and 3.7. As $\alpha \to \infty$, the one-fluid eigenvalue is proportional to $-\alpha^2$ and the interfacial eigenvalue is proportional to $-\alpha$. Fig. 1 presents $\text{Re} \, \alpha$ versus $\alpha$ for $\alpha$ between 0 and 10. Fig. 2 presents a

![Fig. 1](image1.png)

Fig. 1. Surface tension is $S = 0.01$; $R = 1707.94031$, $l_1 = 0.5$, $\beta = \gamma = r = \delta = m = P = 1$. Branches 1, 3 and 5 are associated with the interfacial mode. Branches 2, 3 and 4 are associated with a mode that has an analogue in the one-fluid Bénard problem. The eigenvalues are real except on branch 3 (dark line), where they are complex conjugates; at $\alpha = 3.11612$, they are at criticality. Branch 5 has a local maximum at about $\alpha = 7$ but this is below criticality (see fig. 2).

![Fig. 2](image2.png)

Fig. 2. This is a magnification of branch 5 in fig. 1, showing that the branch is below criticality.
Case (ii): *Stratification in coefficients of cubical expansion*

We choose \( l_1 = 0.4, \beta = 0.8 \) with all other fluid properties equal and zero surface tension. We find criticality at \( R = 1903.694, \alpha = 3.105, \) where \( \sigma = -0.17E-5 + 9.19i \). Eigenvalues at other values of \( \alpha \) are stable. Fig. 3 shows \( \text{Re} \sigma \) versus \( \alpha \) for this situation. As \( \alpha \to \infty \), the interfacial eigenvalue tends to 0 at order \( \alpha^{-1} \). This feature makes the nonlinear stability of short-wave disturbances a rather delicate problem. However, this problem can be circumvented by adding any positive surface tension.

Case (iii): *Stratification in thermal conductivity*

We choose \( l_1 = 0.3, \zeta = 0.8 \), again with all other fluid properties equal and zero surface tension. This yields \( \sigma = -0.64E-6 + 2.85i \) at \( R = 1692.881, \alpha = 3.081. \) However, due to the inherent short-wave instability of this stratification, the interfacial eigenvalue becomes unstable for \( \alpha \) larger than approximately 9. We use a small amount of surface tension to stabilize this. With a judicious amount of surface tension, it is possible to have two critical wavenumbers simultaneously. We add more surface tension so that the only criticality is the one around \( \alpha = 3.08. \) With \( S = 0.03, l_1 = 0.3, \zeta = 0.8 \), criticality occurs at \( R = 1693.33478, \alpha = 3.08 \) with \( \sigma = -0.1E-7 + 2.86i \). The addition of \( S = 0.03 \) raised the critical \( R \) by 0.03% over the case of \( S = 0 \) and the critical \( \alpha \) remains about the same. Fig. 4 shows \( \text{Re} \sigma \) versus \( \alpha \) for this situation. Fig. 5 is a magnification of fig. 4 showing a local maximum of \( \text{Re} \sigma \) near \( \alpha = 10.7. \)

We note that the critical Rayleigh number and \( \alpha \) in case (i) are very close to those of the one-fluid problem, cf. [4]. In case (ii) the critical Rayleigh number is higher and in case (iii) it is lower than in the one-fluid case. Although the results of [12] concern a different set of boundary conditions, and a direct comparison is therefore not possible, we remark that [12] would indeed predict a decrease in the critical Rayleigh number for case (iii), while case (ii) is close to marginal. Since the terms involving \( \beta \) are multiplied by the (large) factor \( R \) in the equations, a perturbation analysis based on nearly equal fluids is probably not applicable in case (ii).
Fig. 4. Stratification in thermal conductivity is $\xi = 0.8$; $R = 1692.881$, $t_1 = 0.3$, $S = 0.03$, $P = 1$, other fluid properties are equal. On branch 3 (dark line), the eigenvalues are complex conjugates; they are at criticality at $\alpha = 3.08$. On branch 5, there is a local maximum at about $\alpha = 10.7$ (see fig. 5).

Fig. 5. This is a magnification of branch 5 in fig. 4, showing a local maximum. This branch is below criticality.

4. The results of Roberts, Swift and Wagner

Roberts, Swift and Wagner [16] consider a reduced system of the form

$$\frac{du}{dt} + F(u, \lambda) = 0, \quad (26)$$

where $u \in \mathbb{R}^{12}$ and $\lambda$ is a real parameter. The reduction of the Bénard problem near criticality to such a system will be discussed in detail in subsequent sections. The parameter $\lambda$ corresponds to the difference between the Rayleigh number and its critical value, $R - R_c$, and, roughly speaking, the vector $u$ consists of amplitude factors that multiply the critical eigenfunctions. These critical eigenfunctions are waves propagating in the directions of $a_1, a_2, a_3 = -a_1 - a_2$ and $-a_1, -a_2, -a_3$. We denote the (complex) amplitude of the wave propagating in the direction of $a_i$ by $z_i$ and the amplitude of the wave propagating...
in the direction of $-a_i$ by $w_i$. We set $u = (z_1, z_2, z_3, w_1, w_2, w_3)$, and we rewrite (26) in the form

$$\frac{dz_i}{dt} + F_i(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = 0, \quad i = 1, 2, 3,$$

$$\frac{dw_i}{dt} + F_{i+3}(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = 0, \quad i = 1, 2, 3. \tag{27}$$

It follows from the hexagonal symmetry that

$$F_2(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = F_1(z_2, z_3, z_1, w_2, w_3, w_1, \lambda),$$

$$F_3(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = F_1(z_3, z_1, z_2, w_3, w_1, w_2, \lambda),$$

$$F_4(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = F_1(w_1, w_2, w_3, z_1, z_2, z_3, \lambda), \tag{28}$$

$$F_5(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = F_1(w_2, w_3, w_1, z_2, z_3, z_1, \lambda),$$

$$F_6(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = F_1(w_3, w_1, w_2, z_3, z_1, z_2, \lambda).$$

Roberts, Swift and Wagner show that the symmetries of the hexagonal lattice and the theory of normal forms can be used to reduce $F_1$ to the form

$$F_1(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = \mu(\lambda)z_1 + \alpha_1(\lambda)|z_1|^2z_1 + \alpha_2(\lambda)|w_1|^2z_1$$

$$+ \alpha_3(\lambda)(|z_2|^2 + |z_3|^2)z_1 + \alpha_4(\lambda)(|w_2|^2 + |w_3|^2)z_1 + \alpha_5(\lambda)(z_2w_2 + z_3w_3)\bar{w}_1 + \cdots, \tag{29}$$

where the dots denote terms of higher than third degree. It is assumed that criticality is at $\lambda = 0$ and that instability occurs for $\lambda > 0$, i.e. $\mu(0)$ is purely imaginary and $\Re\mu'(0) < 0$.

Roberts, Swift and Wagner find eleven qualitatively different classes of bifurcated solutions, each of which can be characterized by a certain relationship which gives $z_2$, $z_3$, $w_1$, $w_2$ and $w_3$ in terms of $z_1$. These solutions are as follows:

1. Standing rolls: $z = w_1$, $z_2 = z_3 = w_2 = w_3 = 0$.
2. Standing hexagons: $z_1 = z_2 = z_3 = w_1 = w_2 = w_3$.
3. Standing regular triangles: $z_1 = z_2 = z_3 = -w_1 = -w_2 = -w_3$.
4. Standing patchwork quilt: $z_1 = z_2 = w_1 = w_2$, $z_3 = w_3 = 0$.
5. Travelling rolls: $z_2 = z_3 = w_1 = w_2 = w_3 = 0$.
6. Travelling patchwork quilt (1): $z_1 = z_3$, $z_2 = w_1 = w_2 = w_3 = 0$.
7. Travelling patchwork quilt (2): $z_1 = w_3$, $z_2 = z_3 = w_1 = w_2 = 0$.
8. Oscillating triangles: $z_1 = z_2 = z_3$, $w_1 = w_2 = w_3 = 0$.
9. Wavy rolls (1): $z_1 = z_3 = w_1 = -w_3$, $z_2 = w_2 = w_3 = 0$.
10. Twisted patchwork quilt: $z_2 = e^{2\pi i/3}z_1$, $z_3 = e^{4\pi i/3}z_1$, $w_i = z_i$.
11. Wavy rolls (2): $z_2 = e^{2\pi i/3}z_1$, $z_3 = e^{4\pi i/3}z_1$, $w_i = -z_i$.

Of course other solutions can be obtained from these by symmetry, e.g. for the standing rolls we could also choose $z_2 = w_2$ and $z_1 = z_3 = w_1 = w_3 = 0$. For a visualization of the patterns associated with the solutions 1–11 we refer to fig. 3 of [16]. The first four solutions are standing waves, the next three are traveling waves and the remaining four have a more complicated time-dependence. There is, to our knowledge, no proof that the above list of bifurcated solutions is complete. However, Roberts, Swift and Wagner give a complete discussion of the stability of their solutions. The results are summarized in their
To determine whether solutions are subcritical or supercritical and whether supercritical solutions are stable, one has to evaluate certain combinations of the coefficients $\alpha_i$ through $\alpha_5$ in (29). Numerical values for the $\alpha_i$ are presented in section 7, together with the implications for stability. Without the numerical results, some conclusions can already be drawn from the table in [16]. The travelling patchwork quilt (1) (number 6) is never stable. Also there are some mutually exclusive cases. For example, if the travelling rolls are stable, then the standing rolls, oscillating triangles and travelling patchwork quilt (2) must all be unstable. The oscillating triangles and the twisted patchwork quilt or wavy rolls (2) cannot simultaneously be stable. Also, the wavy rolls (1) and travelling patchwork quilt (2) cannot both be stable.

5. Reduction to finite dimension

In order to obtain a system of the form (26), we must reduce the partial differential equations governing the Bénard problem to a finite set of ordinary differential equations. This is achieved by the center manifold approach. Roughly speaking, the center manifold theorem says that in the neighborhood of criticality the dynamics is governed by the (finitely many) critical modes. There is no version of the center manifold theorem in the literature which can be applied in a straightforward manner to our problem. A full justification of the center manifold approach would be quite tedious, due to the variable domain occupied by each fluid and the nonlinear nature of the interface conditions. We shall not attempt a proof here, although it can be done. The coercive PDE estimates required for this were derived in [10] for a problem involving parallel shear flow of two fluids, and they can easily be extended to the Bénard problem. In the following, we shall take the existence of a center manifold for granted and limit our scope to formal calculations.

To describe the center manifold approach, we must first introduce some notations. We begin by reformulating the interface conditions according to (19) and restating the equations of motion and interface conditions (14)–(17) with terms ordered according to the degree of nonlinearity. Since we shall only compute terms up to third degree, we shall truncate after the cubic terms. The equations of motion for $0 \leq z \leq l_1$ are

\[
\begin{align*}
\dot{\theta} - A_1 w - \Delta \theta &= H_1 = -(v \cdot \nabla) \theta, \\
\dot{v} - P \Delta v + \nabla \tilde{p} - RP \theta e_z &= (H_2, H_3, H_4) = -(v \cdot \nabla) v, \\
\text{div } v &= 0.
\end{align*}
\]

For $l_1 \leq z \leq 1$ we have

\[
\begin{align*}
\dot{\theta} - A_2 w - \frac{1}{\gamma} \Delta \theta &= H_5 = -(v \cdot \nabla) \theta, \\
\dot{v} - \frac{r}{m} P \Delta v + r \nabla \tilde{p} - \frac{RP}{B} \theta e_z &= (H_6, H_7, H_8) = -(v \cdot \nabla) v, \\
\text{div } v &= 0.
\end{align*}
\]

The boundary conditions at the walls $z = 0$ and $z = 1$ are

\[
\begin{align*}
\begin{align*}
v &= 0, \\
\tilde{\theta} &= 0,
\end{align*}
\end{align*}
\]
and the interface conditions at $z = l_1$, truncated at third order, are as follows (subscripts 1 or 2 refer to the values for $z = l_1 -$ and $z = l_1 +$, respectively):

Continuity of velocity:

$$[\mathbf{v}] = (H_9, H_{10}, H_{11}) = -h[\mathbf{v}_z] - \frac{1}{2}h^2[\mathbf{v}_{zz}]. \quad (33)$$

Balance of shear stresses:

$$P\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_1}{\partial x}\right) - \frac{P}{m}\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right) = H_{12} \quad (34)_1$$

$$= -h\left[\frac{\partial \tilde{T}_{13}}{\partial z}\right] - \frac{\partial h}{\partial y} \left[\tilde{T}_{33} - \tilde{T}_{11}\right] + \frac{\partial^2 h}{\partial y^2} \left[\tilde{T}_{12}\right] + \left(\frac{\partial h}{\partial x}\right)^2 \left[\tilde{T}_{13}\right]$$

$$+ \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \left[\tilde{T}_{23}\right] + \frac{h^2}{4} \left[\frac{\partial^2 \tilde{T}_{13}}{\partial z^2}\right] - h \frac{\partial h}{\partial x} \left[\frac{\partial \tilde{T}_{33}}{\partial z} - \frac{\partial \tilde{T}_{11}}{\partial z}\right] + h \frac{\partial h}{\partial y} \left[\frac{\partial \tilde{T}_{12}}{\partial z}\right]. \quad (34)_2$$

Normal stress balance:

$$2P\frac{\partial w_1}{\partial z} - 2P\frac{\partial w_2}{\partial x} - \tilde{p}_1 + \tilde{p}_2 - M_1 h - S\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = H_{14}$$

$$= M_2 h^2 - h \left[\frac{\partial \tilde{T}_{33}}{\partial z}\right] + 2\frac{\partial h}{\partial x} \left[\tilde{T}_{13}\right] + 2\frac{\partial h}{\partial y} \left[\tilde{T}_{23}\right]$$

$$- \left(\frac{\partial h}{\partial x}\right)^2 \left[\tilde{T}_{11}\right] - \left(\frac{\partial h}{\partial y}\right)^2 \left[\tilde{T}_{22}\right] - 2\frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \left[\tilde{T}_{12}\right]$$

$$- \frac{h^2}{2} \left[\frac{\partial^2 \tilde{T}_{33}}{\partial z^2}\right] + 2h \frac{\partial h}{\partial x} \left[\frac{\partial \tilde{T}_{13}}{\partial z}\right] + 2h \frac{\partial h}{\partial y} \left[\frac{\partial \tilde{T}_{23}}{\partial z}\right]$$

$$+ S \frac{\partial^2 h}{\partial x^2} \left(\left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2\right) + S \frac{\partial^2 h}{\partial y^2} \left(\left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2\right)$$

$$- 2S \frac{\partial^2 h}{\partial x^2} \frac{\partial h}{\partial y} \frac{\partial h}{\partial y} + M_1 h \left(\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right). \quad (35)$$

Continuity of temperature:

$$[\mathbf{\tilde{\theta}}] = h (A_1 - A_2) = H_{15} = -h[\mathbf{\tilde{\theta}}_z] - \frac{1}{2}h^2[\mathbf{\tilde{\theta}}_{zz}]. \quad (36)$$
Continuity of heat flux:

\[
\frac{\partial \theta_1}{\partial z} - \frac{1}{\xi} \frac{\partial \theta_2}{\partial z} = H_{16} = -h \left( \frac{\partial^2 \theta_1}{\partial z^2} - \frac{1}{\xi} \frac{\partial^2 \theta_2}{\partial z^2} \right) + \frac{\partial h}{\partial x} \left( \frac{\partial \theta_1}{\partial x} - \frac{1}{\xi} \frac{\partial \theta_2}{\partial x} \right) + \frac{\partial h}{\partial y} \left( \frac{\partial \theta_1}{\partial y} - \frac{1}{\xi} \frac{\partial \theta_2}{\partial y} \right) \\
- \frac{1}{\xi} h^2 \left( \frac{\partial^3 \theta_1}{\partial z^3} - \frac{1}{\xi} \frac{\partial^3 \theta_2}{\partial z^3} \right) + h \frac{\partial h}{\partial x} \left( \frac{\partial^2 \theta_1}{\partial z \partial x} - \frac{1}{\xi} \frac{\partial^2 \theta_2}{\partial z \partial x} \right) + h \frac{\partial h}{\partial y} \left( \frac{\partial^2 \theta_1}{\partial z \partial y} - \frac{1}{\xi} \frac{\partial^2 \theta_2}{\partial z \partial y} \right).
\]

(37)

Kinematic free surface condition:

\[
\dot{h} - w_1 = H_{17} = h \frac{\partial w_1}{\partial z} - \frac{\partial h}{\partial x} u_1 - \frac{\partial h}{\partial y} v_1 + \frac{1}{\xi} h^2 \frac{\partial^2 w_1}{\partial z^2} - h \frac{\partial h}{\partial x} \frac{\partial u_1}{\partial z} - h \frac{\partial h}{\partial y} \frac{\partial v_1}{\partial z}.
\]

(38)

The unknowns in (30)–(38) are \(v, p, \theta\) and \(h\). We denote this set of unknowns by \(\Phi\). Eqs. (30)–(38) can be written in the schematic form

\[
L \Phi = N_2(\Phi, \Phi) + N_3(\Phi, \Phi, \Phi).
\]

(39)

Here the operator \(L\) incorporates the linear terms, while \(N_2\) and \(N_3\) stand for the quadratic and cubic terms. We can write \(L\) in the form \(A + B \frac{d}{dt}\). We also introduce the notation \(L(\sigma) = A + \sigma B\). The definitions of \(N_2\) and \(N_3\) are extended in a symmetric fashion to the case when their arguments are different:

\[
N_2(\Phi, \Psi) := \frac{1}{4} \left( N_2(\Phi + \Psi, \Phi + \Psi) - N_2(\Phi - \Psi, \Phi - \Psi) \right),
\]

(40)_1

\[
N_3(\Phi, \Psi, \Xi) := \frac{1}{4} \left( N_3(\Phi + \Psi + \Xi, \Phi + \Psi + \Xi, \Phi + \Psi + \Xi) + N_3(\Phi - \Psi - \Xi, \Phi - \Psi - \Xi, \Phi - \Psi - \Xi) \\
+ N_3(-\Phi + \Psi - \Xi, -\Phi + \Psi - \Xi, -\Phi + \Psi - \Xi) \\
+ N_3(-\Phi - \Psi + \Xi, -\Phi - \Psi + \Xi, -\Phi - \Psi + \Xi) \right).
\]

(40)_2

We look for solutions \(\Phi\) with the periodicity of the hexagonal lattice, where the period \(L\) is given by \(4\pi/\alpha\sqrt{3}\) and \(\alpha\) is the critical wavelength found in section 3. We regard all fluid properties as fixed and we view the Rayleigh number as a bifurcation parameter. We set \(\lambda = R - R_c\), where \(R_c\) is the critical value of the Rayleigh number.

The reduction to the center manifold will involve the solution of problems of the form \(L(\sigma) \Phi = f\). We note that in (30)–(38) no nonlinear terms appear in the incompressibility condition and the boundary conditions on the walls, and we shall therefore always assume that the corresponding components of \(f\) are zero. The other components of \(f\) are labeled \(f_1\) through \(f_{17}\) in accordance with the labeling of the components of \(H\) in (30)–(38).

At criticality (\(\lambda = 0\)), we have a pair of complex conjugate eigenvalues \(\pm i\omega\) of the linearized problem, and each of them has six eigenfunctions. For \(\lambda\) near 0, we denote by \(-\mu(\lambda)\) the eigenvalue which arises from perturbing \(-i\omega\). We denote by \(\xi_k(\lambda), k = 1, 2, \ldots, 6\) the eigenfunctions belonging to \(-\mu(\lambda)\), i.e.

\[
L(-\mu(\lambda)) \xi_k(\lambda) = 0,
\]

(41)

and those belonging to \(-\bar{\mu}(\lambda)\) are their complex conjugates.
The eigenfunctions $\xi_k$ are related to each other by rotations. Let $P_\phi$ denote the transformation of rotation through angle $\phi$:

$$
P_\phi := \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}.
$$

With $\Phi = (u, v, w, \tilde{\phi}, h)$, we define

$$
Q_\phi \Phi(x, y, z) := (P_\phi(u, v), w, \tilde{\phi}, h)(P_{-\phi}(x, y), z).
$$

The eigenfunction computed in section 3 has the form $\xi_1 = \tilde{\xi}(z) \exp(ia_1 \cdot x)$, where $a_1 = (\alpha, 0, 0)$, $x = (x, y, z)$, and $\alpha$ is the critical wavenumber. The other eigenfunctions are obtained as follows:

$$
\xi_2 = Q_{2\pi/3}\xi_1, \quad \xi_3 = Q_{4\pi/3}\xi_1, \quad \xi_4 = Q_{5\pi/3}\xi_1, \quad \xi_5 = Q_{8\pi/3}\xi_1, \quad \xi_6 = Q_{10\pi/3}\xi_1.
$$

These correspond to the values of $(k, l)$ in (23) of (0, 1), (-1, -1), (-1, 0), (0, -1) and (1, 1).

Corresponding to the sixfold eigenvalue we have six solvability conditions for the equation $L(-\mu(\lambda))\Phi = f$: $(b_k, f) = 0$ for $k = 1, \ldots, 6$. The adjoint eigenfunctions $b_k$ are calculated from

$$
(b_k, L(-\mu(\lambda))\Phi) = 0, \quad k = 1, \ldots, 6,
$$

for every $\Phi$. Here $(\cdot, \cdot)$ denotes an appropriate inner product, e.g. the $L^2$ inner product. In the actual computations, we are dealing with a discretized problem where $L(-\mu(\lambda))$ becomes a finite-dimensional matrix. We can then choose $(\cdot, \cdot)$ to be the Euclidean inner product in the finite-dimensional space and the $b_k$ simply become the null vectors of the adjoint matrix. The adjoint eigenvector computed in section 3 has the form $b_1 = \tilde{b}(z) \exp(ia_1 \cdot x)$. In analogy with the $\xi_k$, the other $b_k$'s are obtained as follows:

$$
b_2 = R_{2\pi/3}b_1, \quad b_3 = R_{4\pi/3}b_1, \quad b_4 = R_{5\pi/3}b_1, \quad b_5 = R_{8\pi/3}b_1, \quad b_6 = R_{10\pi/3}b_1,
$$

where

$$
R_\phi f(x, y, z) = (f_1, P_\phi(f_2, f_3), f_4, f_5, P_\phi(f_6, f_7), f_8, P_\phi(f_9, f_{10}), f_{11}, P_\phi(f_{12}, f_{13}), f_{14}, f_{15}, f_{16}, f_{17})(P_{-\phi}(x, y), z).
$$

We normalize $\tilde{\xi}$ and $\tilde{b}$ such that

$$
(b_i, B_{ij}) = \delta_{ij}, \quad i, j = 1, \ldots, 6.
$$

We note that there are null spaces of $L(\sigma)$ associated with the $(0, 0)$ Fourier component ($k = l = 0$ in (23)). For any $\sigma$, the operator $L(\sigma)$ has a trivial one-dimensional null space ($u = v = w = \tilde{\phi} = h = 0, \tilde{\rho} = \text{const}$) due to the fact that we can add an arbitrary constant to the pressure; to make the pressure unique we can for example normalize it such that its (spatial) average is zero. In the following, we impose this restriction on $\Phi$. The corresponding solvability condition is found by noting that for the $(0, 0)$ Fourier component incompressibility yields $w' = 0$. Together with the boundary conditions (32), we find $w = 0$ in both fluids. Thus the solvability condition is that the $(0, 0)$ component of $f_{11}$ must be zero. In addition,
$L(\sigma)$ has an eigenvalue zero with an eigenfunction that has no $x$, $y$-dependence. This eigenfunction simply corresponds to a constant vertical shift of the interface and is given by $h = \text{const.}$, $u = v = w = 0$, and $\theta$ and $\bar{\theta}$ adjusted accordingly. We assume that the volume ratio of the two fluids is given, and we therefore have to rule out this eigenfunction; i.e. we require that the average of $h$ must be zero. The corresponding solvability condition is that the $(0,0)$ Fourier component of $f_{17}$ has to be zero. These solvability conditions are always satisfied below.

For convenience, we denote $w_1$, $w_2$ and $w_3$ in (27) by $z_4$, $z_5$ and $z_6$. We can decompose any real-valued function $\Phi$ in the form

$$\Phi = \sum_{i=1}^{6} z_i \xi_i + \sum_{i=1}^{6} \bar{z}_i \bar{\xi}_i + \Psi,$$

where $z_i$ are complex numbers and $\Psi$ represents a linear combination of eigenvectors belonging to stable eigenvalues. For example, suppose $\Psi$ represents an eigenvector belonging to an eigenvalue $\lambda$, not equal to $-\mu(\lambda)$. Then $L(s)\Psi = 0$ and hence $(b_j, (A + sB)\Psi) = 0$. But $A^*b_i = \bar{\mu}(\lambda)B^*b_i$ and thus $(b_j, (A + sB)\Psi) = (b_j, A\Psi) + s(b_j, B\Psi) = (A^*b_j, \Psi) + s(b_j, B\Psi) = (\mu(\lambda) + s)(b_j, B\Psi)$. Therefore,

$$(b_j, B\Psi) = 0, \quad i = 1, \ldots, 6. \quad (50)$$

Since $(b_j, (A - \mu(\lambda)B)\Psi) = 0$, we also have

$$(b_j, A\Psi) = 0, \quad i = 1, \ldots, 6. \quad (51)$$

Since $(A - \bar{\mu}(\lambda)B)\bar{\xi}_i = 0$, we have $(b_j, (A - \bar{\mu}(\lambda)B)\bar{\xi}_i) = 0$ and we conclude as before that $(b_j, B\bar{\xi}_j) = 0$ for $i, j = 1, \ldots, 6$. By taking the inner product of (49) with $b_i$, we therefore obtain

$$z_i = (b_i, B\Phi), \quad i = 1, \ldots, 6. \quad (52)$$

We define the projection operator $\Pi$ such that it picks out the components of $\Phi$ that consist of the critical modes and annihilates the stable modes:

$$\Pi \Phi = 2 \text{Re} \sum_{i=1}^{6} (b_i, B\Phi)\xi_i,$$

so that

$$\Psi = (I - \Pi) \Phi \quad (54)$$

in the above decomposition.

The decomposition of a real-valued function $f$ motivated by the normalization conditions (48) and the decomposition (49) is

$$f = \sum_{i=1}^{6} (b_i, f) B\xi_i + \sum_{i=1}^{6} (b_i, f) B\bar{\xi}_i + g. \quad (55)$$

Here

$$(b_i, g) = 0, \quad i = 1, \ldots, 6. \quad (56)$$
With the projection operator $\hat{\Pi}$ defined as

$$\hat{\Pi}f = 2 \text{Re} \sum_{i=1}^{6} (b_i, f) B_{i}$$

we have

$$g = (I - \hat{\Pi})f.$$  

We form the inner product of (39) with $b_i$, where $L\Phi = (A + B \frac{d}{dt})\Phi$. Using $(b_i, A\Phi) = \mu(\lambda)(b_i, B\Phi)$, together with (52), we obtain

$$\frac{dz_i}{dt} + \mu(\lambda) z_i = (b_i, N_2(\Phi, \Phi)) + (b_i, N_3(\Phi, \Phi, \Phi)), \quad i = 1, \ldots, 6.$$  

Next, we apply the projection $I - \hat{\Pi}$ to (39). Since $A_{i} = \mu(\lambda) B_{i}$, we have

$$L\Phi = \sum_{i=1}^{6} \left( \frac{d}{dt} + \mu \right) z_i B_{i} + \sum_{i=1}^{6} \left( \frac{d}{dt} + \bar{\mu} \right) \bar{z}_i \bar{B}_{i} + L\Psi.$$  

The application of $I - \hat{\Pi}$ to the terms under summations yields zero. Hence $(I - \hat{\Pi})L\Phi = (I - \hat{\Pi})L\Psi$. Using (50), (51), we obtain $\hat{\Pi}L\Psi = 0$ and thus

$$\left( A + B \frac{d}{dt} \right) \Psi = N_2(\Phi, \Phi) + N_3(\Phi, \Phi, \Phi)$$

$$- 2 \text{Re} \left( \sum_{i=1}^{6} (b_i, N_2(\Phi, \Phi) + N_3(\Phi, \Phi, \Phi)) B_{i} \right).$$  

The center manifold theorem states that, in a neighborhood of $\Phi = 0$, there is a manifold $\Gamma$ (called the center manifold) of the form $\Psi = \tau(z_1, z_2, z_3, z_4, z_5, z_6)$ with the following properties:

1. All solutions with initial data on the center manifold remain on the center manifold as long as they remain small.
2. All small periodic solutions lie on the center manifold.
3. The stability of a small periodic solution is determined by its stability within the center manifold, in other words, all Floquet multipliers corresponding to directions outside the center manifold are stable.

The center manifold is usually not uniquely determined but, even if there are many center manifolds, their asymptotic expansions agree to all orders. The above properties hold for every center manifold.

For our present purposes, we require only quadratic terms in the asymptotic approximation to the center manifold. We thus make the ansatz

$$\Psi = 2 \text{Re} \left( \sum_{i,j=1}^{6} z_i z_j \psi_{ij} + z_i \bar{z}_j \chi_{ij} \right) + \cdots,$$  

where the dots indicate terms of higher than quadratic order. W.l.o.g. we may assume the symmetry.
conditions
\[ \psi_{ij} = \psi_{ji}, \quad \chi_{ij} = \chi_{ji}. \]

We insert (62) into (61) and use (59) to express the time derivatives \( \frac{dz_j}{dt} \). By comparing quadratic terms, we obtain

\[ (A - 2\mu(\lambda)B)\psi_{ij} = N_2(\xi_i, \xi_j) - \sum_{k=1}^{6} (b_k, N_2(\xi_i, \xi_j)) B\xi_k - \sum_{k=1}^{6} (\bar{b}_k, N_2(\xi_i, \xi_j)) B\bar{\xi}_k, \]

\[ (A - (\mu(\lambda) + \bar{\mu}(\lambda))B)\chi_{ij} = N_2(\xi_i, \xi_j) - \sum_{k=1}^{6} (b_k, N_2(\xi_i, \xi_j)) B\xi_k - \sum_{k=1}^{6} (\bar{b}_k, N_2(\xi_i, \xi_j)) B\bar{\xi}_k. \]

In our actual computations, the inner product of two functions \( g_1 \) and \( g_2 \), defined in fluid \( j \) and expressed in terms of Chebyshev polynomials as follows,

\[ g_1 = \sum_{i=0}^{N} g_{1i} T_i(z_j) \exp(\alpha x + i\beta_j y), \]

\[ g_2 = \sum_{i=0}^{N} g_{2i} T_i(z_j) \exp(\alpha_2 x + i\beta_2 y), \]

\[ z_1 = \frac{2}{l_1} z - 1, \quad z_2 = \frac{2}{l_2} (z - 1) + 1, \]

is the following Euclidean inner product:

\[ (g_1, g_2) = \int_A \int_A \exp\left[i(-\alpha_1 + \alpha_2)x + i(-\beta_1 + \beta_2)y\right] dx \, dy \times \sum_{i=0}^{N} \bar{g}_{1i} g_{2i}. \]

Here \( A \) represents a cell of the hexagonal lattice, i.e. the parallelogram spanned by the two basis vectors \( x_1 \) and \( x_2 \) of (20). Thus, unless \(-\alpha_1 + \alpha_2 = -\beta_1 + \beta_2 = 0\), the inner product vanishes. We note that many of the inner products in (64) vanish. For example, in the equation for \( \psi_{11} \), \( N_2(\xi_1, \xi_1) \) is proportional to \( \exp(2\alpha x) \) and none of the \( b_k \) have this \((x, y)\)-dependence. Moreover, we only need to evaluate \( \psi_{11} \) at \( \lambda = 0 \), and hence we compute it from \((A - 2\mu(0)B)\psi_{11} = N_2(\xi_1, \xi_1)\). Similarly, we find \( \chi_{11} \) from \( A\chi_{11} = N_2(\xi_1, \xi_1) \). Here, \( N_2(\xi_1, \xi_1) \) is independent of \( x \) and \( y \) and the comments in the paragraph following (48) are relevant for the calculation of \( \chi_{11} \). Finally, we note that our goal is to compute the coefficients given by (75) below (at \( \lambda = 0 \)) and only a subset of the \( \psi \)'s and \( \chi \)'s is actually required for that purpose.
With the notations
\[ \Phi_1 = 2 \text{Re} \sum_{k=1}^{6} z_k \xi_k, \]
\[ \Psi_2 = 2 \text{Re} \left( \sum_{i,j=1}^{6} z_i z_j \Psi_{ij} + z_i \tilde{z}_j \chi_{ij} \right), \]
we obtain the following reduced system, which is accurate to third order, from (59)
\[ \frac{dz_i}{dt} + \mu(\lambda) z_i = (b_i, N_2(\Phi_1, \Phi_1)) + 2(b_i, N_2(\Phi_1, \Psi_2)) + (b_i, N_3(\Phi_1, \Phi_1, \Phi_1)). \]

Obviously this system is of the form (27) and it satisfies the symmetry conditions (28). To determine the form of the function \( F_1 \), we note that \( b_1 \) is of the form \( \tilde{b}(z) \exp(\alpha x) \) and hence we need to consider only those terms in \( N_2(\Phi_1, \Phi_1) + 2N_2(\Phi_1, \Psi_2) + N_3(\Phi_1, \Phi_1, \Phi_1) \) which have an \((x, y)\)-dependence which is also proportional to \( \exp(\alpha x) \). We obtain the following form for \( F_1 \):
\[ F_1(z_1, z_2, z_3, w_1, w_2, w_3, \lambda) = \mu(\lambda) z_1 + \beta_1(\lambda) w_2 w_3 + \beta_2(\lambda) \tilde{z}_2 \tilde{z}_3 + \beta_3(\lambda)(w_2 \tilde{z}_3 + w_3 \tilde{z}_2) \]
\[ + \gamma_1(\lambda)|(z_1|^2 z_1 + \gamma_2(\lambda)|w_1|^2 z_1 + \gamma_3(\lambda)|z_2|^2 w_1 + \gamma_4(\lambda)\tilde{w}_1^2 \tilde{z}_1 + \gamma_5(\lambda)|z_1|^2 \tilde{w}_1 + \gamma_6(\lambda)|w_1|^2 \tilde{w}_1 \]
\[ + \gamma_7(\lambda)(|z_2|^2 + |z_3|^2) z_1 + \gamma_8(\lambda)(|z_2|^2 + |z_3|^2) \tilde{w}_1 + \gamma_9(\lambda)(|w_2|^2 + |w_3|^2) z_1 \]
\[ + \gamma_{10}(\lambda)(|w_2|^2 + |w_3|^2) \tilde{w}_1 + \gamma_{11}(\lambda)(z_2 w_2 + z_3 w_3) z_1 + \gamma_{12}(\lambda)(z_2 w_2 + z_3 w_3) \tilde{w}_1 \]
\[ + \gamma_{13}(\lambda)(\tilde{z}_2 \tilde{w}_2 + \tilde{z}_3 \tilde{w}_3) z_1 + \gamma_{14}(\lambda)(\tilde{z}_2 \tilde{w}_2 + \tilde{z}_3 \tilde{w}_3) \tilde{w}_1. \]

The coefficients \( \beta_i \) and \( \gamma_i \) are as follows:
\[ \beta_1(\lambda) = -2(b_1, N_2(\xi_5, \xi_5)), \]
\[ \beta_2(\lambda) = -2(b_1, N_2(\xi_5, \xi_4)), \]
\[ \beta_3(\lambda) = -2(b_1, N_2(\xi_5, \xi_3)), \]
\[ \gamma_1(\lambda) = -2(b_1, N_2(\xi_5, \xi_1)) - 4(b_1, N_2(\xi_1, \xi_1)) - 3(b_1, N_3(\xi_1, \xi_1, \xi_1)), \]
\[ \gamma_2(\lambda) = -4(b_1, N_2(\xi_5, \xi_4)) - 4(b_1, N_2(\xi_4, \xi_4)) - 4(b_1, N_2(\xi_5, \xi_4)) \]
\[ - 6(b_1, N_3(\xi_1, \xi_4, \xi_4)), \]
\[ \gamma_3(\lambda) = -2(b_1, N_2(\xi_4, \xi_1)) - 4(b_1, N_2(\xi_1, \xi_4)) - 3(b_1, N_3(\xi_1, \xi_4, \xi_4)), \]
\[ \gamma_4(\lambda) = -2(b_1, N_2(\xi_5, \xi_4)) - 4(b_1, N_2(\xi_4, \xi_4)) - 3(b_1, N_3(\xi_1, \xi_4, \xi_4)), \]
\[ \gamma_5(\lambda) = -4(b_1, N_2(\xi_5, \xi_4)) - 4(b_1, N_2(\xi_5, \xi_4)) \]
\[ - 6(b_1, N_3(\xi_1, \xi_4, \xi_4)), \]
\[ \gamma_6(\lambda) = -2(b_1, N_2(\xi_4, \xi_4)) - 4(b_1, N_2(\xi_4, \xi_4)) - 3(b_1, N_3(\xi_4, \xi_4, \xi_4)), \]
\[ \gamma_7(\lambda) = -4(b_1, N_2(\xi_5, \xi_4)) - 4(b_1, N_2(\xi_5, \xi_4)) \]
\[ - 6(b_1, N_3(\xi_1, \xi_4, \xi_4)), \]
\[ \gamma_8(\lambda) = -4(b_1, N_2(\xi_5, \xi_4)) - 4(b_1, N_2(\xi_5, \xi_4)) - 4(b_1, N_2(\xi_5, \xi_4)) \]
\[ - 6(b_1, N_3(\xi_1, \xi_4, \xi_4)), \]
\[ \gamma_9(\lambda) = -4(b_1, N_2(\xi_1, X_{55})) - 4(b_1, N_2(\xi_5, X_{15})) - 4(b_1, N_2(\bar{\xi}_5, \psi_{15})) - 6(b_1, N_3(\xi_1, \xi_5, \bar{\xi}_5)), \]

\[ \gamma_{10}(\lambda) = -4(b_1, N_2(\bar{\xi}_4, X_{55})) - 4(b_1, N_2(\bar{\psi}_{45})) - 4(b_1, N_2(\bar{\xi}_5, X_{54})) - 6(b_1, N_3(\bar{\xi}_4, \xi_5, \bar{\xi}_5)), \]

\[ \gamma_{11}(\lambda) = -4(b_1, N_2(\xi_1, \psi_{25})) - 4(b_1, N_2(\xi_2, \psi_{15})) - 4(b_1, N_2(\xi_5, \psi_{12})) - 6(b_1, N_3(\xi_1, \xi_2, \xi_5)), \]

\[ \gamma_{12}(\lambda) = -4(b_1, N_2(\bar{\xi}_4, \psi_{25})) - 4(b_1, N_2(\bar{\xi}_2, X_{54})) - 4(b_1, N_2(\bar{\xi}_5, X_{24})) - 6(b_1, N_3(\bar{\xi}_4, \bar{\xi}_2, \bar{\xi}_5)), \]

\[ \gamma_{13}(\lambda) = -4(b_1, N_2(\bar{\xi}_1, \bar{\psi}_{25})) - 4(b_1, N_2(\bar{\xi}_2, X_{15})) - 4(b_1, N_2(\bar{\xi}_5, X_{12})) - 6(b_1, N_3(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_5)), \]

\[ \gamma_{14}(\lambda) = -4(b_1, N_2(\bar{\xi}_4, \bar{\psi}_{25})) - 4(b_1, N_2(\bar{\xi}_2, \bar{\psi}_{45})) - 4(b_1, N_2(\bar{\xi}_5, \bar{\psi}_{24})) - 6(b_1, N_3(\bar{\xi}_4, \bar{\xi}_2, \bar{\xi}_5)). \]

6. Transformation to Birkhoff normal form

The form (29) represents a normal form (i.e., as many terms as possible are transformed out of the differential equation at leading order) into which (68) can be put after a suitable coordinate transformation. We refer to [3] for a general discussion of the theory of normal forms and to [5] for applications to Hopf bifurcation with symmetry.

We first transform away the quadratic terms in (69) by setting

\[ \tilde{z}_1 = z_1 - \frac{\beta_1(\lambda)}{\mu_2(\lambda)} w_2 w_3 + \frac{\beta_2(\lambda)}{\mu(\lambda) - 2\hat{\mu}(\lambda)} \tilde{z}_2 \tilde{z}_3 - \frac{\beta_3(\lambda)}{\hat{\mu}(\lambda)} (w_2 \tilde{z}_3 + w_3 \tilde{z}_2). \]  

(71)

The variables \( \tilde{z}_2, \tilde{z}_3, \tilde{w}_1, \tilde{w}_2 \) and \( \tilde{w}_3 \) are defined in such a way that the hexagonal symmetry is preserved, i.e. using the same permutations of the arguments that appear in (28). We obtain a new transformed system, which is again of the form (27) and satisfies

\[ \begin{align*}
\tilde{F}_1(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \lambda) &= \mu(\lambda) \tilde{z}_1 + \tilde{\gamma}_1(\lambda) |\tilde{z}_1|^2 \tilde{z}_1 + \tilde{\gamma}_2(\lambda) |\tilde{w}_1|^2 \tilde{z}_1 + \tilde{\gamma}_3(\lambda) \tilde{z}_1^2 \tilde{w}_1 \\
&+ \tilde{\gamma}_4(\lambda) \tilde{w}_1^2 \tilde{z}_1 + \tilde{\gamma}_5(\lambda) |\tilde{z}_1|^2 \tilde{w}_1 + \tilde{\gamma}_6(\lambda) |\tilde{w}_1|^2 \tilde{w}_1 \\
&+ \tilde{\gamma}_7(\lambda) (|\tilde{z}_1|^2 + |\tilde{z}_3|^2) \tilde{z}_1 + \tilde{\gamma}_8(\lambda) (|\tilde{z}_1|^2 + |\tilde{z}_3|^2) \tilde{w}_1 \\
&+ \tilde{\gamma}_9(\lambda) (|\tilde{w}_1|^2 + |\tilde{w}_3|^2) \tilde{z}_1 + \tilde{\gamma}_{10}(\lambda) (|\tilde{w}_2|^2 + |\tilde{w}_3|^2) \tilde{w}_1 \\
&+ \tilde{\gamma}_{11}(\lambda) (\tilde{z}_2 \tilde{w}_2 + \tilde{z}_3 \tilde{w}_3) \tilde{z}_1 + \tilde{\gamma}_{12}(\lambda) (\tilde{z}_2 \tilde{w}_2 + \tilde{z}_3 \tilde{w}_3) \tilde{w}_1 \\
&+ \tilde{\gamma}_{13}(\lambda) (\tilde{z}_2 \tilde{w}_2 + \tilde{z}_3 \tilde{w}_3) \tilde{w}_1 + \tilde{\gamma}_{14}(\lambda) (\tilde{z}_2 \tilde{w}_2 + \tilde{z}_3 \tilde{w}_3) \tilde{w}_1 + \cdots,
\end{align*} \]  

(72)
Here the $\tilde{\gamma}_i$ are as follows:

$$
\tilde{\gamma}_i = \gamma_i, \quad i = 1, 2, 3, 4, 5, 6, \quad \tilde{\gamma}_7 = \gamma_7 + \frac{|\beta_2|^2}{\mu - 2\mu} - \frac{\beta_1\beta_3}{\mu} \tilde{\gamma}_9 = \gamma_9 + \frac{|\beta_3|^2}{\mu - 2\mu} - \frac{\beta_2\beta_3}{\mu},
$$

$$
\tilde{\gamma}_9 = \gamma_9 - \frac{\beta_1\beta_3}{\mu} - \frac{|\beta_3|^2}{\mu - 2\mu}, \quad \tilde{\gamma}_{10} = \gamma_{10} - \frac{\beta_1\beta_2}{\mu - 2\mu} - \frac{\beta_2^2}{\mu}, \quad \tilde{\gamma}_{11} = \gamma_{11} - \frac{\beta_1^2}{\mu - 2\mu} - \frac{\beta_2\beta_3}{\mu},
$$

Finally, we can achieve the form (29) by using the further transformation

$$
\tilde{z}_1 = \frac{\tilde{\gamma}_1(\lambda)}{2\mu(\lambda)} z_1^2 \tilde{w}_1 + \frac{\tilde{\gamma}_4(\lambda)}{\mu(\lambda) - 3\tilde{\mu}(\lambda)} \tilde{w}_1 z_1 - \frac{\tilde{\gamma}_5(\lambda)}{2\mu(\lambda)} |z_1|^2 \tilde{w}_1 - \frac{\tilde{\gamma}_6(\lambda)}{2\mu(\lambda)} |\tilde{w}_1|^2 \tilde{w}_1
$$

$$
- \frac{\tilde{\gamma}_8(\lambda)}{2\mu(\lambda)} (|z_2|^2 + |z_3|^2) \tilde{w}_1 - \frac{\tilde{\gamma}_{10}(\lambda)}{2\mu(\lambda)} (|\tilde{w}_2|^2 + |\tilde{w}_3|^2) \tilde{w}_1
$$

$$
- \frac{\tilde{\gamma}_{11}(\lambda)}{2\mu(\lambda)} (z_2 \tilde{w}_2 + z_3 \tilde{w}_3) z_1 - \frac{\tilde{\gamma}_{13}(\lambda)}{2\mu(\lambda)} (z_2 \tilde{w}_2 + z_3 \tilde{w}_3) \tilde{z}_1
$$

$$
+ \frac{\tilde{\gamma}_{14}(\lambda)}{\mu(\lambda) - 3\tilde{\mu}(\lambda)} (\tilde{z}_2 \tilde{w}_2 + \tilde{z}_3 \tilde{w}_3) \tilde{w}_1.
$$

The coefficients in the new system are

$$
\alpha_1(\lambda) = \tilde{\gamma}_1(\lambda), \quad \alpha_2(\lambda) = \tilde{\gamma}_2(\lambda), \quad \alpha_3(\lambda) = \tilde{\gamma}_3(\lambda), \quad \alpha_4(\lambda) = \tilde{\gamma}_4(\lambda), \quad \alpha_5(\lambda) = \tilde{\gamma}_5(\lambda).
$$

### 7. Results and discussion

We investigated the three situations discussed for the linear problem in section 3. For the first case (non-zero surface tension), all branches turned out to be subcritical. In the second case (density difference across the interface), the standing rolls, travelling patchwork quilt (1) and oscillating triangles are

<table>
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<td>Case 1</td>
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<tr>
<td>$\alpha_1$</td>
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<tr>
<td>$\alpha_2$</td>
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<td>$\alpha_4$</td>
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<td>$\alpha_5$</td>
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supercritical, but unstable; the remaining branches are subcritical. In the third case (thermal conductivity stratification) only the travelling rolls, travelling patchwork quilt (2) and wavy rolls (1) are subcritical; the remaining branches are supercritical, but unstable. Hence, in all three cases, all branches are unstable near the bifurcation point. It remains an open question whether any of the branches may be stable at a larger amplitude. The numbers $a_i$, $i=1,\ldots,5$ for the three cases are listed in table I. It should be noted that these values are not independent of the normalization of the eigenfunctions $\xi_i$ and are thus determined only up to a common positive factor.

These results might at first not appear believable. It is well known [6] that all bifurcating branches are supercritical in the Bénard problem for one fluid, and we are here looking at situations where the two fluids differ only slightly; case 1 in particular is close to the one-fluid problem. Hence one may expect that the bifurcation should also be supercritical. In the following, we show a simple model problem which demonstrates that such an intuitive expectation is not justified. The equation

$$\dot{x} = \lambda x - x^3 \quad (76)$$

(with $\lambda$ considered the bifurcation parameter) undergoes supercritical bifurcation at $\lambda = 0$. In going from the one-fluid to the two-fluid Bénard problem, a new variable, the interface position, is introduced. In analogy, let us add a new variable $y$ to eq. (76). A small parameter $\epsilon$ is analogous to the difference in physical properties of the fluids. Thus, for $\epsilon = 0$, the dynamics of $x$ is not influenced by $y$. The full system might for example look like this:

$$\dot{x} = \lambda x - x^3 - \epsilon y, \quad \dot{y} = x + K\epsilon y^3. \quad (77)$$

By linearizing (77), we can easily see that there is a Hopf bifurcation from the trivial solution $x = y = 0$ at $\lambda = 0$. In (77) we introduce the scaling $x = \epsilon^{1/4} x', \quad y = \epsilon^{-1/4} y', \quad \lambda = \epsilon^{1/2} \lambda', \quad t = \epsilon^{-1/2} t'$. In this way we obtain the new system

$$\dot{x}' = \lambda' x' - y' - x'^3, \quad \dot{y}' = x' + K\epsilon y'^3. \quad (78)$$

This system no longer involves $\epsilon$. Hence whether the Hopf bifurcation in (77) is supercritical or subcritical depends only on $K$ and not on $\epsilon$. Further analysis shows that the bifurcation is subcritical if $K > 1$. We conclude from this example that the supercriticality of the bifurcation in the one-fluid problem need not rule out subcritical bifurcation in the two-fluid problem, even if the two fluids are similar.

We emphasize that the steady solutions of (76) do not correspond to steady solutions of (77), even for $\epsilon = 0$. Instead, the steady solutions of (77) for $\epsilon = 0$ are given by $x = 0$ and $y$ arbitrary. This mimics the features of the two-fluid Bénard problem. Steady flows of the one-fluid problem do not remain steady solutions when the interface position is introduced as an extra variable. Instead, the “steady” solutions of the problem with equal fluids are given by arbitrary interface deformations and no flow.

Acknowledgements

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Appendix

Short-wave asymptotics for the interfacial mode

We consider the special case where only the thermal conductivities of the two fluids differ with all other fluid properties equal and surface tension equal to zero. We introduce a stream function $\psi$ by setting $w = i\alpha \psi$, $u = -\psi'$. After some algebra, the linear stability problem is reduced to

$$
\begin{align*}
\left\{ -P(D^2 - \alpha^2)^3 + (P + 1)\sigma(D^2 - \alpha^2)^2 - \sigma^2(D^2 - \alpha^2) \right\} \psi = \alpha^2 A_1 RP\psi, & \quad 0 \leq z \leq l_1, \\
\left\{ -P(D^2 - \alpha^2)^3 + (P + 1)\sigma(D^2 - \alpha^2)^2 - \sigma^2(D^2 - \alpha^2) \right\} \psi = \alpha^2 A_2 RP\psi, & \quad l_1 \leq z \leq 1,
\end{align*}
$$

(A.1)-(A.2)

and the interface conditions become

$$
\begin{align*}
[\psi] = [\psi'] = [\psi''] = [\psi'''] = 0, \\
[\psi^{iv}] = -i\alpha Rh[A], \quad \left[ k\left( (D^2 - \alpha^2)^2 \psi' - \frac{\sigma}{P}(D^2 - \alpha^2) \psi' \right) \right] = 0, \quad \sigma h = i\alpha \psi.
\end{align*}
$$

(A.3)

For large $\alpha$, we look for eigenfunctions which decay rapidly away from the interface. We ignore the influence of the boundary conditions at the walls, which would only lead to correction terms which are exponentially small in $\alpha$. We use the rescaled coordinate $\xi = \alpha(z - l_1)$. In the rescaled coordinate, (A.1)–(A.3) read as follows:

$$
\begin{align*}
-P\alpha^6(D^2 - 1)^3 \psi + (P + 1)\alpha^4\sigma(D^2 - 1)^2 \psi - \alpha^2\sigma^2(D^2 - 1) \psi = \alpha^2 A_1 RP\psi, & \quad 0 \leq \xi < l_1, \\
-P(D^2 - 1)^3 \psi = \alpha^2 A_2 RP\psi, & \quad l_1 \leq \xi \leq 1,
\end{align*}
$$

(A.4)

in fluid $i$, and the interface conditions become

$$
\begin{align*}
[\psi] = [\psi'] = [\psi''] = [\psi'''] = 0, \\
[\psi^{iv}] = -i\frac{R}{\alpha^3} h[A], \quad \left[ k\left( \alpha^4(D^2 - 1)^2 \psi' - \frac{\sigma\alpha^2}{P}(D^2 - 1) \psi' \right) \right] = 0, \quad \sigma h = i\alpha \psi.
\end{align*}
$$

(A.5)

For large $\alpha$, (A.4) and (A.5) are satisfied at leading order if we set $\psi = 0$, $\sigma = 0$ and $h = 1$. From the interface conditions, it is clear that the leading term in $\psi$ is of order $\alpha^{-3}$ and the leading term in $\sigma$ is of order $\alpha^{-2}$. To find the leading term in $\psi$, we must solve the equation

$$
(D^2 - 1)^3 \psi = 0,
$$

(A.6)

and we use $h = 1$ in the conditions at the interface. Let $\chi = (D^2 - 1)\psi$, $\eta = (D^2 - 1)^2\psi$. From (A.6) and the condition of decay away from the interface, we find

$$
\begin{align*}
\eta(\xi) = a e^\xi, & \quad \xi < 0, \\
\eta(\xi) = b e^{-\xi}, & \quad \xi > 0.
\end{align*}
$$

(A.7)

The interface conditions yield

$$
[\eta] = a - b = -\frac{iR}{\alpha^3}[A], \quad [k\eta'] = k_1 a + k_2 b = 0.
$$

(A.8)
This leads to
\[ a = -\frac{i R k_2}{\alpha^2 (k_1 + k_2)} [A], \quad b = \frac{i R k_1}{\alpha^2 (k_1 + k_2)} [A]. \quad (A.9) \]

Observing that \((D^2 - 1)\chi = \eta\), we find
\[ \chi(\xi) = \frac{a}{2} \xi e^\xi + c e^\xi, \quad \xi < 0, \]
\[ \chi(\xi) = -\frac{b}{2} \xi e^{-\xi} + d e^{-\xi}, \quad \xi > 0. \quad (A.10) \]

The interface conditions yield
\[ [\chi] = c - d = 0, \quad [\chi'] = \frac{a}{2} + c + \frac{b}{2} + d = 0, \quad (A.11) \]
which results in
\[ c = d = -\frac{a + b}{4} = \frac{i R (k_2 - k_1)}{4 \alpha^2 (k_1 + k_2)} [A]. \quad (A.12) \]

The equation \((D^2 - 1)\psi = \chi\) now yields
\[ \psi(\xi) = \frac{a}{8} \xi^2 e^\xi + \left(\frac{c}{2} - \frac{a}{8}\right) \xi e^\xi + f e^\xi, \quad \xi < 0, \]
\[ \psi(\xi) = \frac{b}{8} \xi^2 e^{-\xi} + \left(\frac{b}{2} - \frac{d}{2}\right) \xi e^{-\xi} + g e^{-\xi}, \quad \xi > 0. \quad (A.13) \]

From the interface conditions, we find
\[ [\psi] = f - g = 0, \quad [\psi'] = f + g + \frac{c + d}{2} - \frac{a + b}{8} = 0, \quad (A.14) \]
and hence
\[ f = g = \frac{a + b}{16} - \frac{c + d}{4} = \frac{3}{16} (a + b) = \frac{3i R (k_1 - k_2)}{16 \alpha^2 (k_1 + k_2)} [A]. \quad (A.15) \]

The last of the interface conditions now gives the leading contribution to \(\sigma\) as
\[ \sigma \sim i af = \frac{3 R (k_2 - k_1)}{16 \alpha^2 (k_1 + k_2)} [A] = \frac{3 R (k_2 - k_1)^2}{16 \alpha^2 (k_1 + k_2) (k_2 l_1 + k_1 l_2)} > 0. \quad (A.16) \]

Hence short waves are unstable if the two fluids differ only in thermal conductivities.
References