STABILITY OF SHEAR FLOWS OF VISCOELASTIC FLUIDS UNDER PERTURBATIONS PERPENDICULAR TO THE PLANE OF FLOW

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Summary

We investigate the stability of plane Couette flow for two constitutive relations which interpolate between the upper and lower convected Maxwell models. Perturbations perpendicular to the plane of flow are considered. No instabilities are found in the range of parameters where the fluid climbs a rod and the shear stress increases with shear rate.

1. Introduction

Over the past twenty years there has been considerable interest in the question of whether shear flows of viscoelastic fluids can be unstable when the Reynolds number is low but the Weissenberg number is high. Two obvious sources of instability are when there is a maximum in the shear stress function or when the equations of motion lose evolutionarity [1]. These instabilities can be regarded as being due to pathological behavior of the constitutive relation in the following sense. They can be predicted from the constitutive relation alone, without solving the equations of motion. They are intrinsically different from the instabilities which are found from solving a dynamical problem, as in the case of Newtonian fluid mechanics. A more subtle question is whether instabilities that are more like those in Newtonian fluids occur for model equations which do not have these constitutive pathologies. Much research has focussed on the upper convected Maxwell model. Recent numerical studies [2–4] give strong indications that no instabilities at low Reynolds number exist. The numerical results concern two-dimensional disturbances, and indeed it is known that Squire's transfor-
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information applies to the upper convected Maxwell model [5], but not to other models of non-Newtonian fluids.

In [6], Akbay and Frischmann investigate the stability of plane Couette flow of a fluid that is an interpolation between the upper and lower convected Maxwell fluids (see Section 2 below). They consider only a special class of perturbations which are such that the perturbing velocity field is perpendicular to the plane of flow. The equations governing this restricted class of perturbations are considerably simpler than those for the general case. This class of perturbations leads to a degeneracy in Squire's transformation even in the Newtonian case, so that the transformation is not applicable: one needs to prove stability with respect to these perturbations in a different way. In the Newtonian case, the flow can be shown to be stable to this class of perturbations. Also, it was shown in [4] that plane Couette flow of the upper convected Maxwell fluid is stable with respect to this class of perturbations. This is in conflict with the results claimed in [6], which are in error. However, further research by Akbay et al. [7] has shown that there are instabilities for the lower convected Maxwell fluid.

The lower convected Maxwell fluid is not a very realistic model. It predicts that the ratio of the normal stresses is $N_2/N_1 = -1$, while it is well known that in the limit of zero shear rate one must have $-1/4 \leq N_2/N_1 \leq 0$ for every fluid which climbs a rod [8–10] and bulges in a tilted trough [11]. This raises the question of whether the instability discovered in [7] will persist when more realistic models, consistent with these requirements, are used. In this paper, we study this question for two families of models which interpolate between the upper and lower convected Maxwell models. The first family is the one considered in [6]; the stress is obtained by linear interpolation between the expressions for the two Maxwell models. For this case, we can show analytically that no instabilities occur in the parameter range which is consistent with rod climbing. Secondly, we present a numerical study of the Johnson–Segalman model [12]. This model contains a parameter $a$ which equals 1 for the upper convected Maxwell model and $-1$ for the lower convected Maxwell model. Rod climbing requires that $0.5 \leq a \leq 1$. Moreover, the shear stress as a function of shear rate reaches a maximum at a certain Weissenberg number. Our computations are confined to the range of $a$ consistent with rod climbing, and the range of Weissenberg numbers below the shear stress maximum. No instabilities are found.

2. Linear interpolation model

We consider the linear stability of Couette flow between parallel plates. All our equations will be formulated in dimensionless variables. We assume the plates are at $y = \pm 1$ and are moving in the $x$-direction with velocities
For any constitutive relation, a possible velocity field is given by 
\[ v = ye_x. \] 
Following [6], we add to this basic flow a small perturbation which has a velocity only in the z-direction:
\[ v = ye_x + \epsilon w(y) e^{i\alpha x + \sigma t}e_z. \]

The equations of motion and the constitutive relation are linearized with respect to \( \epsilon \). It follows from a symmetry argument that contributions linear in \( \epsilon \) arise only in the stress components \( T_{13} \) and \( T_{23} \); we denote the linear perturbation to the stress by \( \epsilon S \). We also note that the velocity field (1) satisfies the incompressibility condition. The equation of motion then reduces to
\[ R(\sigma + i\alpha y)w e^{i\alpha x + \sigma t} = i\alpha S_{13} + S_{23}', \]
where the prime denotes the derivative with respect to \( y \). The no-slip boundary condition is assumed to hold at the plates, i.e.
\[ w = 0 \text{ at } y = \pm 1. \]

In this section, we consider the constitutive relation
\[ T(x, y, z, t) = \frac{1}{W^2(A + 1)} \int_0^\infty \left\{ A\left(C_t^{-1}(x, y, z, t-s) - I\right) \\
- \left(C_t(x, y, z, t-s) - I\right)\right\} e^{-s/W} ds. \]
Here \( C_t \) denotes the relative Cauchy strain, \( A \) is a nonnegative number and \( W \) is the Weissenberg number. The case \( A = \infty \) corresponds to the upper convected Maxwell model (called "Lodge-Modell" in [6]) and the case \( A = 0 \) corresponds to the lower convected Maxwell model (called "Maxwell-Modell" in [6]).

Let \( \chi_s(x, y, z, t) \) denote the position at time \( t-s \) of the fluid particle which occupies the position \( (x, y, z) \) at a fixed time \( t \), with \( s \) as the variable. The velocity field (1) has no component in the \( y \)-direction, so that the \( y \)-component of \( \chi_s \) remains \( y \) for all \( s \). Hence the \( x \)-component of the velocity at time \( t-s \) remains \( y \). On integration from time \( t \) to \( t-s \), we find that the \( x \)-component of the position at time \( t-s \) is \( x - ys \). Thus, the \( z \)-component of the velocity at time \( t-s \) is \( \epsilon w(y) \exp[i\alpha(x - ys) + \sigma(t-s)]. \) After integration, this yields
\[ \chi_s(x, y, z, t) = \left(x - sy, y, z + \frac{\epsilon w(y)}{\sigma + i\alpha y} e^{i\alpha x + \sigma t} [e^{-s(\sigma + i\alpha y)} - 1]\right). \]
This leads to the relative deformation gradient
\[ F_t(x, y, z, t-s) = \nabla \chi_s(x, y, z, t) = \begin{pmatrix} 1 & -s & 0 \\ 0 & 1 & 0 \\ ca & eb & 1 \end{pmatrix}, \]
where

\[ a = i\alpha e^{i\alpha x + \sigma t} \frac{w'(y)}{\sigma + i\alpha y} (e^{-s(\sigma + i\alpha y)} - 1), \]

\[ b = e^{i\alpha x + \sigma t} \left[ \left( \frac{w'}{(\sigma + i\alpha y)} - \frac{i\alpha w}{(\sigma + i\alpha y)^2} \right) e^{-s(\sigma + i\alpha y)} - 1 - \frac{i\alpha w}{\sigma + i\alpha y} e^{-s(\sigma + i\alpha y)} \right]. \]

Neglecting terms of quadratic order in \( \epsilon \), we find

\[ C_t(x, y, z, t-s) = F_t^T(x, y, z, t-s) F_t(x, y, z, t-s) \]

\[ = \begin{pmatrix} 1 & -s & \epsilon a \\ -s & 1 + s^2 & \epsilon b \\ \epsilon a & \epsilon b & 1 \end{pmatrix}. \]

and

\[ C_t^{-1}(x, y, z, t-s) \]

\[ = \begin{pmatrix} 1 + s^2 & s & \epsilon(-a-bs-as^2) \\ s & 1 & \epsilon(-b-as) \\ \epsilon(-a-bs-as^2) & \epsilon(-b-as) & 1 \end{pmatrix}. \]

By inserting into (4), we obtain the following stresses in the basic (\( \epsilon = 0 \)) Couette flow:

\[ T_{12} = 1, \quad N_1 = T_{11} - T_{22} = 2W, \quad N_2 = T_{22} + T_{33} = -\frac{2W}{A+1}. \] (10)

This is in disagreement with the expressions (3b), (3c) in [6] which contain an erroneous sign. Equations (10) are consistent with rod climbing only if \( A \geq 3 \). For the perturbed stresses we obtain

\[ S_{13} = \frac{1}{W^2(A+1)} \int_0^\infty e^{-s/W} (-A+1) a - Abs - Aas^2) \, ds \]

\[ = e^{i\alpha x + \sigma t} \left[ \frac{i\alpha w}{W(\sigma + i\alpha y)} \left(1 - \frac{1}{S}\right) + \frac{A}{A+1} \left( \frac{w'}{\sigma + i\alpha y} - \frac{i\alpha w}{(\sigma + i\alpha y)^2} \right) \right] \]

\[ \times \left(1 - \frac{1}{S^2}\right) + 2 \frac{A}{A+1} W \frac{i\alpha w}{\sigma + i\alpha y}, \] (11)
and

\[ S_{23} = \frac{1}{W^2(A+1)} \int_0^\infty e^{-s/W} \left(- (A + 1)b - Aas\right) ds \]

\[ = e^{i\alpha x + \sigma t} \left[ \left( \frac{w'}{W(\sigma + i\alpha y)} - \frac{i\alpha w}{W(\sigma + i\alpha y)^2} \right) \left(1 - \frac{1}{S}\right) \right.\]

\[ + \frac{i\alpha w}{S^2(\sigma + i\alpha y)} + \frac{A}{A + 1} \frac{i\alpha w}{\sigma + i\alpha y} \left(1 - \frac{1}{S^2}\right) \right]. \tag{12} \]

Here we have set

\[ S := 1 + W(\alpha + i\alpha y). \tag{13} \]

Further algebraic simplification yields

\[ S_{13} = e^{i\alpha x + \sigma t} \left[ \frac{i\alpha w}{S} + \frac{A}{A + 1} \frac{Ww'(S + 1)}{S^2} + \frac{A}{A + 1} \frac{W^2 i\alpha w (2S + 1)}{S^2} \right], \tag{14} \]

\[ S_{23} = e^{i\alpha x + \sigma t} \left[ \frac{w'}{S} - \frac{W i\alpha w}{S^2} + \frac{A}{A + 1} \frac{W i\alpha w (S + 1)}{S^2} \right]. \tag{15} \]

After inserting (14) and (15) into (2), we obtain

\[ R(\sigma + i\alpha y)w = \frac{w''}{S} + 2 \frac{A}{A + 1} i\alpha Ww' \frac{S + 1}{S^2} - \frac{2i\alpha Ww'}{S^2} \]

\[ - \frac{\alpha^2 w}{S} - \frac{2\alpha^2 W^2 w}{S^3} + 2 \frac{A}{A + 1} \frac{\alpha^2 W^2 w}{S^3} \left(1 - \frac{S^2}{S^3}\right). \tag{16} \]

We make the substitution \( w = S^{1/(A+1)} \phi \) (in [6] the exponent is \( A/(A + 1) \) instead of \( 1/(A + 1) \); this must be a misprint). Equation (16) now assumes the form

\[ RS(\sigma + i\alpha y)\phi = \phi'' + 2i\alpha W \frac{A}{A + 1} \phi' - \alpha^2 \phi - 2 \frac{A}{A + 1} \alpha^2 W^2 \phi \]

\[ - \frac{\alpha^2 W^2}{(A + 1)^2} \frac{2S + 1}{S^2} \phi. \tag{17} \]

The further substitution \( \phi = \exp(-i\alpha W(\alpha/A + 1)\psi) \) leads to the equation

\[ RS(\sigma + i\alpha y)\psi = \psi'' - \alpha^2 \psi - \alpha^2 W^2 A(A + 2) \frac{(A + 1)^2}{(A + 1)^2} \psi - \alpha^2 W^2 A \frac{2S + 1}{S^2} \psi. \tag{18} \]

Akbay and Frischmann [6] obtain \( S + 2 \) instead of \( 2S + 1 \) in the last term; the discrepancy must be due to algebraic error. In any case, the argument that follows is not affected by this discrepancy.
Equation (18), together with the boundary conditions
\[ \psi(-1) = \psi(1) = 0 \] (19)
forms an eigenvalue problem for \( \sigma \). Akbay et al. [7] show that for \( A = 0 \) there are unstable eigenvalues if \( RW > 1 \) and \( W \) is sufficiently large. We shall show here that there is a maximal value of \( A \) beyond which unstable eigenvalues cannot exist. This maximal value of \( A \) turns out to be less than the "rod-climbing threshold" \( A = 3 \).

As in [4], we multiply eqn. (18) by \( \bar{S} \psi \) and integrate from \(-1\) to \(1\). In the resulting equation we take the real part. This yields after an integration by parts (note that \( S' = iA W \))
\[
R \text{ Re } \sigma \int_{-1}^{1} |S|^2 |\psi|^2 \, dy
= -(1 + W \text{ Re } \sigma) \int_{-1}^{1} |\psi'|^2 + \alpha^2 \left(1 + W^2 \frac{A(A + 2)}{(A + 1)^2}\right) |\psi|^2 \, dy
- \alpha W \text{ Im } \int_{-1}^{1} \psi' \bar{\psi} \, dy - \alpha^2 W^2 \frac{A}{(A + 1)^2} \int_{-1}^{1} \text{ Re } \left( \frac{2S + 1}{S^2} \right) |\psi|^2 \, dy.
\] (20)

If \( \sigma \) is an unstable eigenvalue, i.e. an eigenvalue with positive real part, then the left hand side of (20) is positive and therefore the right hand side must be positive. We now find a range of \( A \) for which the right hand side is negative when \( \text{ Re } \sigma > 0 \). If \( \sigma \) has positive real part, then \( |S| > 1 \) and hence
\[
\left| \frac{(2S + 1)\bar{S}}{S^2} \right| < 2 + \left| \frac{1}{S} \right| < 3.
\] (21)
Moreover, the inequality \( 2ab \leq a^2 + b^2 \) with \( a = |\psi'| \) and \( b = \alpha W |\psi|/2 \) yields
\[
\alpha W |\psi'| \leq |\psi'|^2 + \frac{1}{4} \alpha^2 W^2 |\psi|^2.
\] (22)
Hence the right hand side of (22) is bounded from above by
\[
-(1 + W \text{ Re } \sigma) \int_{-1}^{1} |\psi'|^2 + \alpha^2 \left(1 + W^2 \frac{A(A + 2)}{(A + 1)^2}\right) |\psi|^2 \, dy
+ \int_{-1}^{1} |\psi'|^2 + \frac{\alpha^2 W^2}{4} |\psi|^2 \, dy + 3\alpha^2 W^2 \frac{A}{(A + 1)^2} \int_{-1}^{1} |\psi|^2 \, dy.
\] (23)
This expression is negative if \( \text{Re} \sigma > 0 \) and the coefficient of \( \alpha^2 W^2 |\psi|^2 \) is negative:

\[
\frac{A(A + 2)}{(A + 1)^2} > \frac{1}{4} + \frac{3A}{(A + 1)^2},
\]

i.e. if \( A > 1 + \frac{1}{3}\sqrt{12} \). We conclude that, for this range of \( A \), \( \text{Re} \sigma \) cannot be positive.

3. The Johnson–Segalman model

We consider the special case of one relaxation mode, which can also be regarded as a special case of the Oldroyd 6-constant model. The stress is related to the velocity gradient by the following system of differential equations [12]:

\[
W \left( \dot{T} + (\nu \cdot \nabla)T - \frac{a + 1}{2} ((\nabla \nu)T + T(\nabla \nu)^T) \right) + \frac{1 - a}{2} (T(\nabla \nu) + (\nabla \nu)^T T) + T = \nabla \nu + (\nabla \nu)^T. \tag{25}
\]

The parameter \( a \) ranges between \(-1\) and 1; for \( a = -1 \) we have the lower convected Maxwell model, and for \( a = 1 \) we have the upper convected Maxwell model. In the basic Couette flow (\( \nu = (y, 0, 0) \)), eqn. (25) yields the following stresses:

\[
T_{12} = \frac{1}{1 + (1 - a^2)W^2}, \quad T_{11} = \frac{(a + 1)W}{1 + (1 - a^2)W^2},
\]

\[
T_{22} = -\frac{(1 - a)W}{1 + (1 - a^2)W^2}.
\]

This is consistent with rod climbing only if \( a \geq \frac{1}{2} \). Moreover, the shear stress as a function of the shear rate has a maximum for \( W = 1/\sqrt{1 - a^2} \). Therefore we are looking for instabilities only in the range \( a \geq \frac{1}{2}, W \leq 1/\sqrt{1 - a^2} \).

We now consider the perturbed velocity field (1), and we denote the linearized perturbation to the stress by \( e^\sigma e^{i\alpha x + \sigma t} \). After some algebra, we obtain the following linearized system of equations:

\[
R(\alpha \dot{s} + i\alpha y) = i\alpha s_{13} + s_{13}',
\]

\[
S_{13} - \frac{a + 1}{2} W s_{23} = i\alpha w + \frac{(a + 1)^2 W^2}{2(1 + (1 - a^2)W^2)} i\alpha w
\]

\[+
\frac{(a + 1)W}{2(1 + (1 - a^2)W^2)} w', \tag{27}
\]
\[ S_{23} + \frac{1 - a}{2} \, W_{s13} = \frac{(a + 1)W}{2(1 + (1 - a^2)W^2)} \, i\alpha w \]
\[ + \frac{(a^2 - 1)W^2}{2(1 + (1 - a^2)W^2)} w' + w''. \]
\[ w(-1) = w(1) = 0. \]

Here \( S = 1 + W(\sigma + i\gamma) \) has the same meaning as before.

The eigenvalue problem for \( \sigma \) given by eqns. (27) was solved numerically using the Chebyshev-tau method (see [13]). By combining (27) into a single equation for \( w \) and setting the coefficient of \( w'' \) equal to zero, it can be determined analytically that there is a continuous spectrum given by

\[
\text{Re} \, \sigma = \frac{1}{W} \left( \frac{(1 - a^2)W^2}{2(2 + (1 - a^2)W^2)} - 1 \right), \quad -\alpha \leq \text{Im} \, \sigma \leq \alpha. \tag{28}
\]

The numerical results were checked against this. In addition to this continuous spectrum there is an infinite number of discrete eigenvalues which are essentially lined up along a parallel to the imaginary axis.

For \( a = -1 \) and low values of \( RW \), the discrete eigenvalues lie exactly on \( \text{Re} \, \sigma = -1/2W \), and they form complex conjugate pairs. As observed in [7], if \( RW > 1 \), it is possible for one of these pairs to reach the real axis and then split into two real eigenvalues. If \( W \) is large enough, one of these real eigenvalues can cross through 0 and become unstable. Figure 1 shows a neutral stability curve in the \( \alpha \) versus \( RW \) plane for \( W = 20 \). This graph is in qualitative but not in quantitative agreement with Fig. 3 in [7]. We

Fig. 1. Neutral stability curve for \( W = 20 \).
suspect that this discrepancy has some trivial reason such as mislabeling of the graph in [7]. Figure 3 of [7] cannot be correct as it is. To demonstrate this, we show that for the wave number $\alpha = 1$ and arbitrary $W$, an eigenvalue $\sigma = 0$ cannot occur for $RW \leq 6$. If we set $A = \sigma = 0$ and $\alpha = 1$ in (18), we obtain

$$Ri y \psi = \psi'' - \psi + RWy^2 \psi.$$  \hspace{1cm} (29)

We multiply by $\bar{\psi}$, integrate and take the real part, which leads us to

$$\int_{-1}^{1} |\psi'|^2 + |\psi|^2 - RWy^2 |\psi|^2 \, dy = 0.$$  \hspace{1cm} (30)

For $RW \leq 6$, we conclude

$$\int_{-1}^{1} |\psi'|^2 + |\psi|^2 - 6y^2 |\psi|^2 \, dy \leq 0,$$  \hspace{1cm} (31)

and we shall now show that this is impossible for non-zero $\psi$. Obviously the integrand is non-negative for $1 - 6y^2 \geq 0$, or approximately $|y| \leq 0.4$, and we must therefore have

$$\int_{0.4}^{1} |\psi'|^2 + |\psi|^2 - 6y^2 |\psi|^2 \, dy \leq 0,$$  \hspace{1cm} (32_1)

and/or

$$\int_{-1}^{-0.4} |\psi'|^2 + |\psi|^2 - 6y^2 |\psi|^2 \, dy \leq 0.$$  \hspace{1cm} (32_2)

Let us assume without loss of generality that the first case applies. Using $y < 1$ in the third term of the integrand, we conclude

$$\int_{0.4}^{1} |\psi'|^2 - 5 |\psi|^2 \, dy \leq 0.$$  \hspace{1cm} (33)

If $\psi$ is any function in $H^1(0.4, 1)$ which vanishes at $y = 1$ (as the case here, see (19)), then $\psi$ can be continued by reflection across $y = 0.4$ to a function in $H^1(-0.2, 1)$ which vanishes at both endpoints. It is known that the ratio of $\int |\psi'|^2 \, dy$ to $\int |\psi|^2 \, dy$ is at least equal to the first eigenvalue of the Dirichlet problem on $(-0.2, 1)$: $-\psi'' = \lambda \psi, \, \psi(-0.2) = \psi(1) = 0$. Thus $\int |\psi'|^2 \, dy / \int |\psi|^2 \, dy \geq \pi^2 / (1.2)^2 \sim 6.85 > 5$. This contradicts (33), unless $\psi$ vanishes.

There are four parameters involved in our problem: $R$, $W$, $\alpha$ and $a$, and exhaustive computations would therefore be quite difficult. We undertook computations for $\alpha = 2, 4, 6, 8, 10$ and $R = 1, 3, 5, 7, 10$ at the following combinations of $a$ and $W$:

\[
\begin{array}{cccccccc}
  a = & 0.8 & 0.85 & 0.9 & 0.95 & 0.97 & 0.99 \\
  W = & 1.66 & 1.89 & 2.29 & 3.2 & 4.11 & 7.08
\end{array}
\]
A number of calculations were also done for $a = 0.6$ and $W = 1.2$. The Weissenberg numbers were chosen just barely below $1/\sqrt{1 - a^2}$, the value for the shear stress maximum, because we felt that instabilities would be more likely there than at lower Weissenberg numbers. Note that the Weissenberg number of the shear stress maximum is quite low unless $a$ is very close to 1. This feature may severely limit the prospects of finding instabilities; we note that instabilities at $a = -1$ only occur for $W > 4$. In none of the cases we computed were any instabilities found. We recall that the onset of instability in [7] is preceded by a pair of complex conjugate eigenvalues merging together and splitting off into two real eigenvalues (one of which may then become unstable). We therefore paid attention to real eigenvalues or eigenvalues with small imaginary part. For $\alpha = 0$, the system (27) can be reduced to

$$R\sigma \left( (1 + \sigma W)^2 + \frac{1 - a^2}{4} W^2 \right) w$$

$$= \left[ \frac{2\sigma W(2 + (1 - a^2)W^2) + 4 + (1 - a^2)W^2}{4(1 + (1 - a^2)W^2)} \right] w'' ,$$

$w(-1) = w(1) = 0.$  \hspace{1cm} (34)

The eigenfunctions are $\sin(n(\pi/2)(x + 1))$, and for each value of $n$ we find a third degree polynomial for $\sigma$, which has a real negative root. Hence for $\alpha = 0$, there is an infinite sequence of negative eigenvalues. Indeed we found such stable real eigenvalues for small values of $\sigma$. For larger values of $\alpha$, there can be eigenvalues with very small imaginary parts; for instance at $a = 0.99$, $W = 7.08$, $R = 10$ and $\alpha = 4$, we found the pair $\sigma = -0.64171E - 1 \pm 0.15045E - 2i$. However, a number of computations undertaken for parameters in the vicinity of this set do not show this eigenvalue becoming unstable.

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