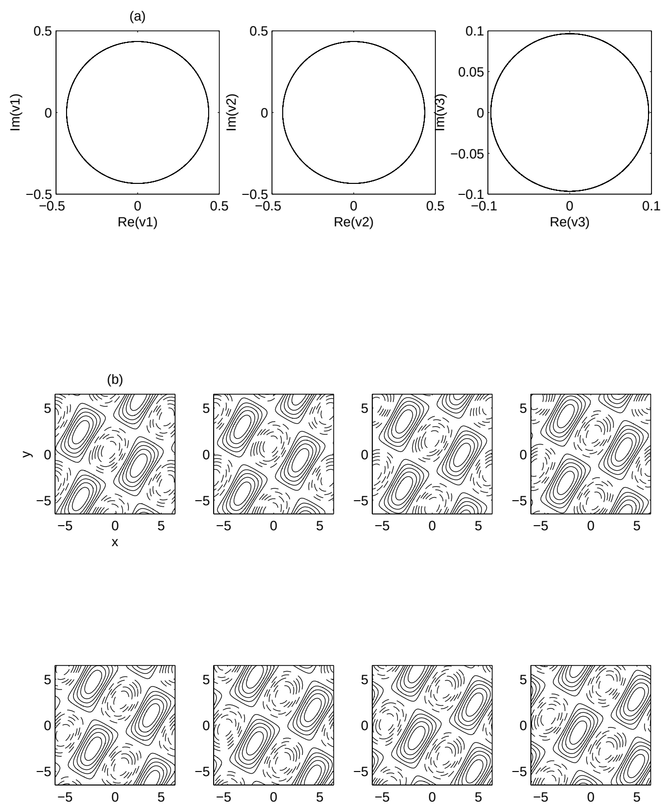

Figure 24: (a) Temporal evolution of v_1, v_2 and v_3 for Fluorinert/silicone oil, Case (b) of Table 5, and $\epsilon_2 = -3.2$. (b) Spatial evolution for Fluorinert/silicone oil, Case (b) of Table 5, and $\epsilon_2 = -3.2$.

7 Conclusion

We have derived the amplitude equations governing the evolution near a Takens-Bogdanov bifurcation with the symmetry of the hexagonal lattice. As in the classical Takens-Bogdanov problem, the leading order approximation is a Hamiltonian system, which is perturbed by dissipative terms at the next order of approximation. We have identified families of periodic solutions of the Hamiltonian system which bifurcate from one of the steady solutions. We have also proved the existence of heteroclinic connections between steady hexagons; these heteroclinic loops represent the limit of a family of periodic solutions ("oscillating triangles"). Numerical integration of the amplitude equations shows that steady solutions (rolls and hexagons) as well as periodic solutions (traveling rolls, wavy rolls (1), oscillating triangles, traveling patchwork quilt (1)) can be observed. The transition from steady to periodic regimes does not occur directly; instead chaotic and quasiperiodic solutions are observed in the intermediate region of parameter space.

Appendix. Coefficients in the amplitude equations

The coefficients in (7) and (11) - (12) are related as follows.The rest of the coefficients $b_i, c_i, \tilde{b}_i, \tilde{c}_i$ are not required in the analysis of Sec. 5.

$$\begin{aligned}
b_2 &= (-\beta_1 - \beta_2 - 2\beta_3 + \bar{\beta}_1 + \bar{\beta}_2 + 2\bar{\beta}_3)(\mu - \bar{\mu})/8, \\
b_3 &= (\beta_1 - \beta_2 + \bar{\beta}_1 - \bar{\beta}_2)/4, \\
c_3 &= (\gamma_1 + \gamma_3 - \gamma_4 - \gamma_6 + \bar{\gamma}_1 + \bar{\gamma}_3 - \bar{\gamma}_4 - \bar{\gamma}_6)/4, \\
c_4 &= (-\gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_6 + \bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3 + \bar{\gamma}_4 \\
&+ \bar{\gamma}_5 + \bar{\gamma}_6)(\mu - \bar{\mu})/16, \\
c_5 &= (-\gamma_1 + \gamma_2 + \gamma_3 - \gamma_4 - \gamma_5 + \gamma_6 - \bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3 - \bar{\gamma}_4 \\
&- \bar{\gamma}_5 + \bar{\gamma}_6)/8, \\
c_7 &= (-\gamma_{10} - \gamma_{11} - \gamma_{12} - \gamma_{13} - \gamma_{14} - \gamma_7 - \gamma_8 - \gamma_9 + \bar{\gamma}_{10} + \bar{\gamma}_{11} \\
&+ \bar{\gamma}_{12} + \bar{\gamma}_{13} + \bar{\gamma}_{14} + \bar{\gamma}_7 + \bar{\gamma}_8 + \bar{\gamma}_9)(\mu - \bar{\mu})/16, \\
c_8 &= (-\gamma_{10} + \gamma_{11} - \gamma_{12} + \gamma_{13} - \gamma_{14} + \gamma_7 - \gamma_8 + \gamma_9 - \bar{\gamma}_{10} + \bar{\gamma}_{11} \\
&- \bar{\gamma}_{12} + \bar{\gamma}_{13} - \bar{\gamma}_{14} + \bar{\gamma}_7 - \bar{\gamma}_8 + \bar{\gamma}_9)/8, \\
c_9 &= (\gamma_{10} + \gamma_{11} + \gamma_{12} - \gamma_{13} - \gamma_{14} - \gamma_7 - \gamma_8 + \gamma_9 + \bar{\gamma}_{10} + \bar{\gamma}_{11} \\
&+ \bar{\gamma}_{12} - \bar{\gamma}_{13} - \bar{\gamma}_{14} - \bar{\gamma}_7 - \bar{\gamma}_8 + \bar{\gamma}_9)/8, \\
c_{10} &= (-\gamma_{10} + \gamma_{11} + \gamma_{12} - \gamma_{13} - \gamma_{14} + \gamma_7 + \gamma_8 - \gamma_9 - \bar{\gamma}_{10} + \bar{\gamma}_{11} \\
&+ \bar{\gamma}_{12} - \bar{\gamma}_{13} - \bar{\gamma}_{14} + \bar{\gamma}_7 + \bar{\gamma}_8 - \bar{\gamma}_9)/8, \\
\tilde{b}_2 &= (\beta_1 + \beta_2 + 2\beta_3 + \bar{\beta}_1 + \bar{\beta}_2 + 2\bar{\beta}_3)/4, \\
\tilde{c}_4 &= (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3 + \bar{\gamma}_4 \\
&+ \bar{\gamma}_5 + \bar{\gamma}_6)/8, \\
\tilde{c}_7 &= (\gamma_{10} + \gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_7 + \gamma_8 + \gamma_9 + \bar{\gamma}_{10} + \bar{\gamma}_{11} \\
&+ \bar{\gamma}_{12} + \bar{\gamma}_{13} + \bar{\gamma}_{14} + \bar{\gamma}_7 + \bar{\gamma}_8 + \bar{\gamma}_9)/8.
\end{aligned} \tag{45}$$

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