Stretch and hold: the dynamics of a filament governed by a viscoelastic constitutive model with thixotropic yield stress behavior

Y. Renardy$^{1, a)}$ and H. V. Grant$^1$

Department of Mathematics, 460 McBryde Hall, Virginia Tech, Blacksburg, VA 24061-0123

(Dated: 17 April 2016)

The transient behavior of filament stretching is studied for a viscoelastic constitutive model that combines a Partially Extending strand Convection model with a Newtonian solvent. The vertical filament is fixed at the bottom and the top is pulled up and held. Gravity and surface tension are also included in the model though they are not the primary mechanisms in this study. An axisymmetric circular slender jet approximation is applied. An asymptotic analysis for the initial stages of evolution is performed for large relaxation time, so that an interplay of fast and slow time scales emerges, and gives a criterion for whether the fluid yields immediately or whether slow dynamics ensues, depending on elastic stresses, gravity and capillary stress. The analysis guides the choice of parameters to exemplify thixotropy and yield stress behavior through numerical simulations of the full governing equations from start to finish of the filament evolution. Elastic effects promote a spring back of the filament toward its initial shape, while pulling at the top stretches the filament locally to promote yielding, with the lower portion of the filament remaining unyielded. In addition, a parameter regime that models extensional experiments in the literature for yield stress fluids sheds light on differences in filament shapes.

$^{a)}$renardy@vt.edu
I. INTRODUCTION

Recent developments have enabled detailed measurements of the properties of yield stress fluids that display thixotropic behavior\textsuperscript{1–4}. Not only do such fluids require a certain amount of stress to be applied before they begin to flow, but they have slow and fast time scales. Slow changes in the microstructure remain even when the flow has apparently stopped. Similarly, slow changes can lead to delayed yielding, which is difficult to anticipate. Familiar examples include suspensions such as make-up, foods and pastes\textsuperscript{5,6} and those with entangled microstructures such as wormlike micellar solutions. In this paper, we focus on a theoretical model of an idealized stretch-and-hold device for the PECN constitutive model that combines a viscoelastic component that is governed by the Partially Extending Strand Convection (PEC) model of [7] and a Newtonian solvent (N). A large relaxation time allows for a slow time scale, and the interplay of slow and yielded time scales is investigated in prior work\textsuperscript{8–10} for homogeneous shear flow driven by a prescribed shear stress. This approach has shown that the initial value problem from equilibrium proceeds in well-defined stages with closed-form dependence on time, thus contributing to a deeper understanding of how such a model can predict thixotropic yield stress behavior. In addition, the theoretical predictions are amenable to direct comparisons with experimental results in the literature for shear flow, such as immediate yielding, delayed yielding, hysteresis upon relieving the applied stress, and slow evolution to return to equilibrium. These phenomena are predicted with the PECN constitutive model in a natural way in contrast to phenomenological models that require the \textit{a priori} input of a yield stress value and evolutionary parameters.

While shear flows with prescribed stress have been extensively studied with the PECN model, elongational flows with prescribed tensile stress are less well understood. The analysis of elongational flows is simpler for unbounded rather than bounded geometries. The work of Grant \textit{et al.}\textsuperscript{11,12} concerns homogeneous uniaxial elongation and biaxial elongation with prescribed tensile stress. Experimental data for these geometries for thixotropic yield stress fluids are to our knowledge not available. On the other hand, the PECN model has a range of parameters that reflect the fluid properties of wormlike micellar solutions, analogous to more complex models such as the two-species model of Vasquez \textit{et al.}(author?)\textsuperscript{13,14}. The numerical simulation of the full equations in [12] compares with prior experimental work\textsuperscript{15,16} and theoretical work\textsuperscript{14}, which show extensional thickening for uniaxial extension but its
absence in biaxial extension. This paper is motivated by the need to understand how the PECN model behaves in a more realistic geometry than the prior unbounded studies, and to perform a theoretical analysis of an initial value problem with two time scales. We prescribe a fluid bounded laterally by a free surface, and by a fixed bottom boundary and a movable top boundary. The fluid is extended and held. We refer to this as stretch-and-hold.

Section II gives the derivation of the governing equations. Radial symmetry and the slender jet approximation are used. The initial extension may be instantaneous or a regularized displacement. The density of the fluid and effect of gravity are included in our model; thus, the fluid eventually drains away from the top boundary, and sags at the bottom boundary. Our derivation of the governing equations parallels those of Refs. [18, 19, 20] but with some differences; for example, the effect of the end boundary conditions on filament shape is investigated in [20] but is not in our analysis. The main focus is the role of the parameters associated with the constitutive model. Section III presents dimensionless parameters that have important roles. The relaxation time is assumed to be long compared to flow time, so that the ratio of retardation time to relaxation time is a small parameter $\epsilon$. An asymptotic analysis of the initial transient for small $\epsilon$ is given in section IV and shown to yield an energy equation that represents the elastic and capillary forces, which depend on the strain. This gives the condition for whether immediate yielding occurs or whether a slow time scale ensues. Beyond this stage, a direct numerical simulation of the full equations is conducted in section V. First, the particular case of yielding due to high strain and with the dominant mechanism of elastic force is given for the simplest initial shape, that of a cylindrical filament, and stretched at a constant rate for an interval of time. Secondly, we
investigate delayed yielding that is driven by small effects of gravity and surface tension. Thirdly, simulations are performed for yielding that is initiated by a dominant surface tension. This case is motivated by experimental work of [17] which gives profile data for several viscous fluids with apparent yield stress. Figure 1 reproduces two of the six photographs from their figure 1 that span a wide range of viscous fluids with apparent yield stress: a water/oil emulsion, and a shower gel. Also included in their work are water/oil and oil/water cremes, and an additional water/oil emulsion, and shampoo. There are two separate issues when it comes to the study of approach toward filament breakup; first, whether the fluid yields, and hence it evolves to breakup, and secondly, what the asymptotic nature of the breakup is. Since PECN becomes Newtonian after it yields, the breakup asymptotics is necessarily Newtonian$^{21–25}$. On the other hand, real yield stress fluids may show significant shear thinning after yielding and therefore have different breakup asymptotics$^{17,25,26}$.

II. GOVERNING EQUATIONS FOR FILAMENT DEFORMATION

A. Filament deformation

Figure 2 provides a schematic of an idealized stretch-and-hold experiment. A liquid bridge is placed in a device that stretches it vertically in a vacuum environment. Figure 2(a) shows the initial shape, in terms of the vertical coordinate denoted $Z$, $0 \leq Z \leq L$, and an initial prescribed radius $R_0(Z)$, with $R_0(Z) << L$. For example, an initially cylindrical filament satisfies $R_0(Z) = R_0$. The cross-sectional area is denoted $A(Z)$ initially.

Figure 2(b) shows the shape when the upper boundary moves for $0 < t < T$. The axial coordinate is denoted $z(Z,t)$, and the radius is $R(z(Z,t))$, $t \geq 0$, following a fluid particle which was at height $Z$ at $t = 0$. The cross-sectional area for $t > 0$ is denoted

$$a(z) = \pi R^2(z).$$

Figure 2(b) shows the top boundary moving upward and stretching the filament. At the start, the filament forms the shape shown here. Depending on the rate of pull, the relationship between $z$ and $R$ may continue to be roughly linear while the top thins out. The total length is prescribed by $\ell(t)$ and initially, $\ell(0) = L$. The top boundary stops moving at $t = T$, so that for $t \geq T$, the filament occupies $0 \leq z(Z,t) \leq \ell(T)$, unless it breaks. The aim is to calculate the shape $R(z(Z,t))$ for $t > 0$. 
FIG. 2. ‘Stretch and hold’. (a) The initial shape is a cylinder: \(0 \leq Z \leq L\), \(R_0(Z) = R_0\). (b) The top boundary location is prescribed as \(\ell(t)\). The cylinder occupies \(0 < z(Z,t) < \ell(t)\), the radius is \(R(z(Z,t))\), and the top boundary stops at \(t = T\).

In order to formulate the governing equations, a Lagrangian frame of reference is used, because of the advantage that the material derivative following a fluid parcel is replaced by a time derivative: \(\frac{D}{Dt} \equiv \frac{\partial}{\partial t}\). We define the stretch function \(s\) by

\[
s(Z,t) = \frac{\partial z}{\partial Z},
\]

and this is an important quantity to be determined from the governing equations. The physical coordinate value \(z\) is retrieved by

\[
z(Z,t) = \int_0^Z s(y,t)\,dy.
\]

The cross-sectional areas, initially \(A(Z)\) and later \(a(z(Z,t))\), are related because of volume conservation:

\[
\int_0^L A(Z)\,dZ = \int_0^{\ell(t)} a(z)\,dz.
\]

Substitution of (2) in (4) yields \(\int_0^L a(z(Z,t))s(Z,t)dZ\), or

\[
a(z)s = A(Z).
\]

The velocity \(\mathbf{v} = (u,v,w)\) represents homogeneous uniaxial extension for the bulk of the filament, ignoring the details of the motion at the boundaries. Thus, \(w = \frac{\partial z(Z,t)}{\partial t}\). Incompressibility is satisfied by \(\nabla \cdot \mathbf{v} = 0\), where \(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{1}{2} \frac{\partial w}{\partial z}\).

The extension rate is defined by \(\frac{\partial w}{\partial z}\), and the use of (2) in the form \(\frac{\partial}{\partial z} = \frac{1}{s} \frac{\partial}{\partial Z}\) gives the relationship between \(w\) and the stretch \(s\):

\[
\frac{\partial w}{\partial z} = \frac{1}{s} \frac{\partial w}{\partial Z} = \frac{1}{s} \frac{\partial^2 z}{\partial t \partial Z} = \frac{s_t}{s}.
\]
B. Balance of forces

Let $\eta$ denote the Newtonian component of viscosity, $\sigma$ the coefficient of surface tension, and $\rho$ the density. The total stress tensor is $\mathbf{T} + \mathbf{S} - p\mathbf{I}$, where the Newtonian contribution is given by $\mathbf{S} = \eta(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$, and $\mathbf{T}$ denotes the viscoelastic contribution. The component $T_{rr}$ is equal to $T_{xx}$ since each $x - y$ cross-section ($z = \text{constant}$) is axisymmetric, and if $y = 0$, then $r = x$.

The speeds are slow, thus justifying the neglect of inertia: $\rho \frac{\partial \mathbf{v}}{\partial t} = 0$. The consideration of forces acting on a small volume of the element yields a set of equations for describing the stretching filament\textsuperscript{18,20,27}. Figure 3 illustrates a small section of the filament, from $z$ to $z + dz$, with the notation used below. Gravity acts throughout the filament, and its effect is to thicken the lower part of the filament. Surface tension acts at the surface between the filament and surrounding vacuum. The motion of the top boundary upwards induces a decrease in the radius at the top, where eventually, the surface tension force induces pinch-off. In the case of an initially cylindrical configuration, this pinch-off is expected at the top, while if the initial configuration has a waist, the pinch-off at the waist may occur before that of the top.

In Figure 3, the force at the upper surface, $F(z + dz)$, is equal to the stress multiplied by surface area, $a(T_{zz} + S_{zz} - p)|_{z+dz}$, plus the surface tension force acting on the perimeter of the cross section, $2\pi R \sigma$ where $\sigma$ denotes the coefficient of surface tension. Similarly, the force at the lower surface is $F(z) = a(T_{zz} + S_{zz} - p)|_z + 2\pi R \sigma$. The difference $F(z + dz) - F(z)$ is equal to the weight, $-\rho g adz$. Therefore,

$$\frac{\partial}{\partial z} (a(T_{zz} + S_{zz} - p) + 2\pi R \sigma) = \rho g a.$$  \hspace{1cm} (7)

At the lateral free surface, the jump in the normal stress is balanced by surface tension force:

$$T_{rr}(z, t) + S_{rr}(z, t) - p(z, t) = -\sigma / R(z, t).$$  \hspace{1cm} (8)

This is used to eliminate the pressure in (7) to obtain

$$\frac{\partial}{\partial z} \left( a(T_{zz} + S_{zz} - T_{rr} - S_{rr}) + \frac{a\sigma}{R} \right) = \rho g a.$$  \hspace{1cm} (9)

The viscous stresses which are important are

$$S_{zz} = 2\eta \frac{\partial w}{\partial z}, \quad S_{rr} = -\eta \frac{\partial w}{\partial z}.$$  \hspace{1cm} (10)
Upon substitution of (6), we arrive at $S_{zz} = 2\eta \frac{a_t}{s}$, and $S_{rr} = -\eta \frac{a_t}{s}$. Moreover, their difference:

$$S_{zz} - S_{rr} = 3\eta \frac{s_t}{s}, \quad (11)$$

is important since it is the viscous stress difference after the fluid yields.

Next, a Lagrangian framework is adopted and all equations are re-written in terms of the initial configuration by using the mass conservation property (5) and substitution for $\frac{\partial}{\partial z}$ from (2):

$$\frac{\partial}{\partial Z} \left( \frac{A}{s} (T_{zz} + S_{zz} - T_{rr} - S_{rr}) + \sigma \sqrt{\frac{A\pi}{s}} \right) = \rho g A. \quad \quad (12)$$

Integration over $Z$ from 0 to a specific value $Z$ gives

$$\frac{A}{s} (T_{zz} + S_{zz} - T_{rr} - S_{rr}) + \sigma \int_{0}^{Z} A(\zeta) d\zeta + \lambda(t). \quad \quad (13)$$

The constant of integration is $\lambda(t)$, and there are essentially two cases to be considered. First, the filament is pulled up in a prescribed manner and held. In this case, a ‘stretch constraint’ determines $\lambda(t)$, and this is calculated in section III. Secondly, the filament may break at the value $Z = Z_{\text{break}}$, freeing the filament from the constraint. For the boundary condition of pulling at the top, the filament breaks at $Z_{\text{break}} = L$. If the filament is encouraged to break elsewhere with an initially cinched geometry, say in the middle, then the value of $Z_{\text{break}}$ is found as part of the solution to the initial value problem. At this point, pinch-off is driven by the surface tension force which gives rise to the term $\sigma \sqrt{\frac{A\pi}{s}}$ on the left hand side of (13). When the radius approaches 0, and the stretch function $s$ approaches infinity, this term is proportional to $1/\sqrt{s}$ and approaches 0. Note that the quantity $A$ is the initial value and does not depend on the later evolution of radius or $s$. The other stress terms on
the left hand side of (13) are responses to the primary surface tension force, and therefore they also decay to 0, and thus all terms on the left hand side vanish. On the right hand side, we have the gravity term (or ‘weight’) and therefore,

\[ \lambda(t) = -\rho g \int_0^{Z_{break}} A(\zeta) d\zeta. \]  

(14)

C. Constitutive equations

The partially extending strand convection (PEC) model is used for the microstructure, augmented with a Newtonian solvent (PECN) for yielded flow. The original intent of the PEC model, derived in [7], is to model polymeric melts consisting of polymers with long side branches that inhibit full extension and retraction. The polymers are represented by objects that are stretched by the flow and generate elastic forces for retraction. With such an object, we associate an orientation vector. The conformation tensor is a statistical average of the dyadic product of the orientation vector. The trace of the conformation tensor is measurable as microstructural deformation from equilibrium. The PECN constitutive model, in the limit of large relaxation time, is suggested in [8] as a natural way to predict thixotropy without phenomenological assumptions.

The details of the PECN model is given in prior literature\textsuperscript{8,9} and is summarized next. Let \( T \) denote the extra stress tensor, \( \mathbf{C} \) denote the tube conformation tensor, \( c \) denote its trace \( \text{tr} \mathbf{C} \), and let \( \tau \) denote the relaxation time;

\[ \mathbf{C}^\nabla + \frac{1}{\tau} \phi(c)(\mathbf{C} - \mathbf{I}) = \mathbf{0}, \quad \phi(c) = c + \alpha, \]

\[ \mathbf{T} = \psi(c) \mathbf{C}, \quad \psi(c) = \frac{k_1}{c + \alpha}. \]  

(15)

The \( \mathbf{C}^\nabla \) denotes the upper convected derivative \( \frac{\partial \mathbf{C}}{\partial t} - (\nabla \mathbf{v}) \mathbf{C} - \mathbf{C}(\nabla \mathbf{v})^T \), and \( \frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \). The model functions \( \phi(c) \) and \( \psi(c) \) arise from a systematic derivation for entangled polymers, which are allowed to deform within tubelike regions that convect\textsuperscript{7}. The form of \( \psi(c) \) takes account of the fact that chains do not fully stretch with their tubes. For instance, \( \psi \) is constant for the upper convected Maxwell model\textsuperscript{28}, while here the fluid is less elastic and \( \psi \) decreases as \( \text{tr} \mathbf{C} \) increases. The form of \( \phi(c) \) reflects a relaxation time that shortens as the fluid is deformed and lengthens as it returns to equilibrium.

The model has a dimensionless material parameter \( \alpha \) and an elastic stress parameter \( k_1 \). In the original derivation for polymeric melts, \( \alpha > 0 \), with the limit \( \alpha \) large being the
upper convected Maxwell fluid which is solid-like, and the case $\alpha = 0$ being a Doi-Edwards model where polymers slip through entanglements. $\alpha$ is related to the ratio of yield stress to stress modulus$^{8,10}$. In order to relate these parameters to a real fluid, we need both the elastic modulus of the unyielded phase and the yield stress. Such data are currently very limited for elongational flow, but they are readily available for shear flow. In particular, homogeneous shear flow under a prescribed applied shear stress is investigated with the PECN model in $[9,10]$. A constant shear stress is applied initially and remains for the duration of the evolution. The initial value problem has the form $\frac{dC}{dt} = N_1(C) + \epsilon N_2(C)$, where $\epsilon = \text{retardation time/relaxation time}$, $N_1(C)$ and $N_2(C)$ are nonlinear interactions of the components of the conformation tensor. Numerical simulations of this system of equations are given, combined with asymptotic solutions for each stage of evolution, in terms of multiple time scales. It is found that an initial nonlinearly elastic evolution occurs in a solid-like state, followed by either unyielded microstructural evolution on a time scale of order $\frac{1}{\epsilon}$, or immediate yielding on a time scale of order 1. The results include yield stress hysteresis, with unyielding at much smaller shear stress than the apparent yield stress, and the dependence of apparent viscosity on the slow time scale: these are familiar properties of thixotropic yield stress fluids. The asymptotic analysis gives a formula for immediate yielding$^{10}$: the applied shear stress must exceed the maximum elastic shear stress denoted $T_{12}^{*}$, and together with the shear elastic modulus denoted $k_{1s}^{*}$, satisfy

$$\alpha = \left(\frac{2T_{12}^{*}}{k_{1s}^{*}}\right)^2 - 3, \quad k_1 = k_{1s}^{*}(3 + \alpha).$$

This formula determines $\alpha$ and $k_1$. For example, the experimental data of $[5]$ for ketchup is fitted in $[10]$ to be $\alpha \approx -2$. The range of negative $\alpha$, $-3$ to $-2$, is typically associated with suspensions and pastes. The wormlike micellar solution in the experimental work of $[15]$ on opposed jet flow is characterized in $[12]$ with $\alpha \approx 5$, and typically in the range 3 to 5.

**III. TRANSIENT DYNAMICS AND DIMENSIONLESS PARAMETERS**

In the stretch and hold problem, strain is an important dimensionless parameter, equal to the amount of extension divided by the initial length. The initial strain originates from the motion of the top boundary. From (2) evaluated at $Z = L$ and $t = T$, $s(L,T) =$
\( \frac{\partial z}{\partial Z} \bigg|_{Z=L,t=T} = \frac{L+\ell'(T)T}{L} = 1 + \frac{\ell'(T)T}{L} \). Hence,

\[
\text{strain} = s(L, T) - 1. \tag{17}
\]

The assumption of uniaxial extension simplifies the conformation tensor to a diagonal matrix with \( C_{xx} = C_{yy} \). Normalized with the average equilibrium length of the polymer molecules, we have \( \mathbf{C} = \mathbf{I} \) at equilibrium. This is the initial condition before the filament is pulled. The constitutive equation becomes

\[
\begin{align*}
\frac{dC_{xx}}{dt} &= -\frac{s}{s}C_{xx} + \frac{1}{\tau}(c + \alpha)[1 - C_{xx}], \\
\frac{dC_{zz}}{dt} &= 2\frac{s}{s}C_{zz} + \frac{1}{\tau}(c + \alpha)[1 - C_{zz}], \\
A \left[ \psi(c)(C_{zz} - C_{xx}) + 3\eta s t / s \right] + \sigma \sqrt{\frac{Ap}{s}} = \rho g \int_{0}^{Z} A(\zeta) d\zeta + \lambda(t),
\end{align*}
\tag{18}
\]

where \( c = C_{zz} + 2C_{xx} \). The constant \( \lambda(t) \) is implicitly determined by the condition that the total length of the filament is prescribed. The initial stretch is prescribed by (2):

\[
s(Z, 0) = \frac{dz}{dZ} \bigg|_{t=0} = \frac{dZ}{dZ} = 1, \quad 0 < Z < L. \tag{20}
\]

There are two time scales, a long relaxation time, and a short retardation time. This gives rise to a small parameter \( \epsilon \), equal to the ratio of retardation time \( \eta / k_1 \) to relaxation time \( \tau \),

\[
\epsilon = \frac{\eta}{k_1 \tau}, \quad \epsilon << 1. \tag{21}
\]

The yielding parameter \( \alpha \) has already been introduced. Next, the dimensionless space and time variables are defined with respect to the initial length \( L \) and the retardation time \( \eta / k_1 \):

\[
\tilde{z} = \frac{z}{L}, \quad \tilde{t} = \frac{tk_1}{\eta}. \tag{22}
\]

If \( R_0 \) is a representative size of the initial radius function \( R_0(Z) \), then the aspect ratio is

\[
\tilde{R}_0 = \frac{R_0}{L} << 1. \tag{23}
\]

The strain \( s(L, T) - 1 \) is

\[
s(L, T) - 1 = \tilde{\ell}(\tilde{T})\tilde{T}, \tag{24}
\]

where \( \tilde{\ell} = \ell(t)/L, \tilde{T} = Tk_1/\eta \).
There are four stresses which compete: an elastic stress represented by \( k_1 \), a gravitational stress \( \rho gL \), an initial capillary stress \( \sigma/R_0 \), and, after the fluid yields, a viscous stress difference given in (11), \( 3\eta s_t/s \). The competition between gravitational and elastic stresses gives a sagging number

\[
N_{sag} = \frac{\rho gL}{k_1}.
\]  

(25)

If \( N_{sag} \ll 1 \), the elastic stress causes the filament to spring back to its original shape in the initial fast dynamics. If \( N_{sag} \gg 1 \), the gravitational stress is dominant and the expectation is that the filament falls down. We focus on the new effects of elastic stress by choosing \( N_{sag} \) to be not large.

During the stretch phase, viscous stress scales with \( \eta l''(t)/L \). Afterward, during elastic recoil, viscous and elastic stresses balance at roughly the same scales, meaning that \( \eta/\text{time} \sim \eta/(\text{retardation time}) \sim k_1 \). Since the viscous stress scales differently during the initial stretch phase and the subsequent hold phase, a capillary number for the initial stretching may be defined by

\[
Ca_{\text{stretch}} = \frac{\eta l''(t)R_0}{L\sigma}.
\]  

(26)

In the hold phase, the ratio of elastic and capillary stresses gives a capillary number associated with elastic recoil \( Ca_{\text{recoil}} \),

\[
Ca_{\text{recoil}} = \frac{k_1R_0}{\sigma}.
\]  

(27)

When a filament deforms, the two fundamental measurable quantities are the elastic modulus, which is relevant for small deformations, and the value of the yield stress, which is important when the stretch is large. This section is a discussion of capillary numbers with respect to these quantities. From (38),

\[
C_{rr}(t) = \frac{1}{s}, \quad C_{zz}(t) = s^2.
\]  

(28)

Hence, the elongational stress is \( T_{zz} - T_{rr} = \frac{k_1}{c+\alpha}(C_{zz} - C_{rr}) \), with \( c \approx 3 \). The strain is \( (s-1) \). Therefore, the elastic modulus \( k_1^* \) is \( \approx \frac{k_1}{3+\alpha}(s^2 - \frac{1}{s}) \). After the factor \( (s-1) \) is canceled out, we have \( k_1^* \approx \frac{k_1}{3+\alpha} \frac{s^2 + s + 1}{s} \). In the initial phase, we let \( s \approx 1 \), giving

\[
k_1^* \approx \frac{3k_1}{3+\alpha}.
\]  

(29)
A slightly different dimensionless capillary number for recoil (27) may be defined in terms of the elastic modulus and surface tension stress,

\[ Ca_{\text{recoil}}^* = \frac{k_1^* R_0}{\sigma}. \]  

When yielding occurs, \( s \) becomes large, and (28) gives a negligible \( C_{rr} \). This implies that \( c \approx C_{zz} = s^2 \), and the elongational stress is \( \frac{k_1}{c_{zz} + \alpha} C_{zz} \approx k_1 \). Thus, \( k_1 \) is the yield stress. Although the only difference between \( Ca_{\text{recoil}} \) and \( Ca_{\text{recoil}}^* \) is a factor of \( \frac{3}{3 + \alpha} \), \( Ca_{\text{recoil}}^* \) may be more convenient for experimental work because the elastic modulus is the measurable quantity, and \( k_1 \) is deduced from it. If the filament yields, pinch-off is inevitable. The time-scale for pinch-off of a viscous jet is \( \eta R/\sigma \). In contrast, if the Newtonian part were absent, pinch-off would occur immediately after yielding in the absence of inertia.

IV. INITIAL VALUE PROBLEM

In order to carry out the theoretical analysis and numerical computations, we introduce \( f_1(Z, t) = C_{xx} s \) and \( f_2(Z, t) = \frac{C_{zz}}{s^2} \) to decouple the time derivative term \( s_t \) in (18) to formulate a system of nonlinear ordinary differential equations of the form \( \dot{X}(t) = F(X, t) \), where \( X \equiv [f_1, f_2, s] \), and \( F \) denotes the nonlinear interactions of \( f_1, f_2 \) and \( s \).

\[
\frac{d}{dt} \ln f_1 = \frac{1}{\tau} (c + \alpha) [\frac{s}{f_1} - 1], \\
\frac{d}{dt} \ln f_2 = \frac{1}{\tau} (c + \alpha) [\frac{1}{f_2 s^2} - 1], \\
c = 2 \frac{f_1}{s} + f_2 s^2.
\]  

The equation for \( s_t \) follows from (19),

\[
s_t = \frac{1}{3\eta} \left[ -\psi(c)(s^3 f_2 - f_1) - \sigma \sqrt{\frac{\pi}{A}} s^{3/2} + \frac{\rho g}{A} s^2 \int_0^Z A(\zeta) d\zeta + \frac{s^2}{A} \lambda(t) \right].
\]  

From (2) evaluated at \( Z = L \), \( \int_0^s s(Z, t) \, dZ = \int_0^{\ell(t)} \, dz = [z(Z, t)]_{0}^{\ell(t)} = \ell(t) \), or

\[
\int_0^L s \, dZ = \ell(t).
\]  

This equation is the stretch constraint. The total length \( \ell(t) \) is prescribed for all \( t \); after the device stops, \( \ell(t) \) is obviously a constant, and \( \ell'(t) = 0 \). The derivative of (33) is

\[
\int_0^L s_t \, dZ = \frac{d\ell(t)}{dt}.
\]
Integration of (32) with respect to $Z$ gives the known quantity $\frac{dt(t)}{dt}$ in terms of $\lambda(t)$.

The initial condition from equilibrium becomes

$$f_1(Z,0) = s(Z,0) = 1, \quad f_2(Z,0) = \frac{1}{s^2(Z,0)} = 1, \quad 0 < Z < L.$$  \hfill (35)

This system depends on the continuous variable $Z$ as well as time. We use an in-house algorithm based on collocation in $Z$, with local refinement where stretch is large, and built upon a variable time step Runge-Kutta scheme within MATLAB.

A. Asymptotic analysis of the initial transient dynamics

The dimensionless form of (31) - (32) is

$$\frac{d}{dt}(\ln f_1) = \epsilon (c + \alpha) \left[ \frac{s}{f_1} - 1 \right],$$

$$\frac{d}{dt}(\ln f_2) = \epsilon (c + \alpha) \left[ \frac{1}{f_2 s^2} - 1 \right],$$

$$s\tilde{t} = \frac{1}{3} \left[ \frac{1}{c + \alpha} \left( s^3 f_2 - f_1 \right) - \frac{s^{3/2} \tilde{R}_0}{Ca_{recoil} \tilde{R}_0(\tilde{Z})} \right. + \frac{N_{sag}}{A(\tilde{Z})} s^2 \int_{\tilde{Z}_0}^{\tilde{Z}} \tilde{A}(\tilde{\zeta}) \, d\tilde{\zeta} + \frac{s^2}{A(\tilde{Z})} \hat{\lambda}(\tilde{t}) \right],$$

$$\tilde{A}(\tilde{Z}) = \pi \tilde{R}_0^2(\tilde{Z}),$$

$$s\tilde{t} = \frac{1}{3} \left[ \frac{1}{c + \alpha} \left( s^3 f_2 - f_1 \right) - \frac{s^{3/2} \tilde{R}_0}{Ca_{recoil} \tilde{R}_0(\tilde{Z})} \right. + \frac{N_{sag}}{A(\tilde{Z})} s^2 \int_{\tilde{Z}_0}^{\tilde{Z}} \tilde{A}(\tilde{\zeta}) \, d\tilde{\zeta} + \frac{s^2}{A(\tilde{Z})} \hat{\lambda}(\tilde{t}) \right],$$

$$\hat{\lambda}(\tilde{t}) = \left[ 3\tilde{p}(\tilde{t}) - N_{sag} \int_0^1 \frac{s^2}{\tilde{R}_0^2(\tilde{Z})} \left( \int_{\tilde{Z}_0}^{\tilde{Z}} \tilde{R}_0^2(\tilde{\zeta}) \, d\tilde{\zeta} \right) d\tilde{Z} \right. + \frac{\tilde{R}_0}{Ca_{recoil}} \int_0^1 \frac{s^{3/2}}{\tilde{R}_0(\tilde{Z})} \, d\tilde{Z} + \int_0^1 \frac{s(f_2 s^3 - f_1)}{(f_2 s^3 + 2f_1 + \alpha s)} \, d\tilde{Z} \right]$$

$$\left/ \int_0^1 \frac{s^2}{\pi \tilde{R}_0^2(\tilde{Z})} \, d\tilde{Z}. \right.$$

A cylinder of constant radius $\tilde{R}_0(\tilde{Z}) = \tilde{R}_0$ is a simple initial shape that has an advantage of further decoupling the governing equations. In the initial transient stage, when the filament is starting to be pulled up, the initial conditions (20)-(35) are used to estimate that $f_1 \sim f_2 \sim 1$ and $c \sim \frac{2}{s} + s^2$. In comparison, the right hand side of (36) is $O(\epsilon)$ and is negligible: $\frac{d}{dt}(\ln f_1) = 0, \frac{d}{dt}(\ln f_2) = 0$. Integration, together with $f_1(0) = 1, f_2(0) = 1$ gives

$$f_1(\tilde{t}) = 1, \quad f_2(\tilde{t}) = 1.$$  \hfill (38)
This motion is called ‘fast dynamics’, since the deformation occurs on the time scale \( 1 \). This stage represents a nonlinear elastic deformation. The cross-sectional area of the filament is (5) for \( \tilde{t} > 0 \), and substitution of (38) in (32) reduces the governing equations to a single equation for \( s_\tilde{t} \),

\[
\begin{align*}
\frac{s_\tilde{t}}{3} = & -s + 2 + sa \frac{1}{Ca_{recoil}} - s^{3/2} + N_{sag} \tilde{Z} s^2 + \frac{s^2 \tilde{\lambda}(\tilde{t})}{\pi R_0^2}, \\
\frac{\tilde{\lambda}(\tilde{t})}{\pi R_0^2} = & -\int \frac{s(s - 1)}{s + \alpha s} d\tilde{Z}^2 \int 1 \\
+ & \int \frac{s(s - 1)}{s + \alpha s} d\tilde{Z}^2 \int 1 \\
(39)
\end{align*}
\]

B. Slow dynamics

The steady state solutions of fast dynamics, (39), are unyielded states. These states appear to be steady on the fast time scale, but evolve on a slow time scale \( \tilde{t} \sim \epsilon^{-1} \). Such solutions form the ‘slow manifold’. If \( s_\tilde{t} = 0 \), then \( s(\tilde{Z}, \tilde{t}) = s(\tilde{Z}) \), and \( \lambda(\tilde{t}) = \lambda \). Substitution in (39) gives the equation for the slow manifold to be

\[
h(s) = N_{sag} \tilde{Z} + \frac{\tilde{\lambda}}{\pi R_0^2}, \tag{40}
\]

where \( h(s) \) represents the force in the filament, denoted

\[
h(s) = \frac{s^2 - \frac{1}{2}}{s^3 + 2 + \alpha s} + \frac{1}{Ca_{recoil}} s^{-1/2}. \tag{41}
\]

This expression represents the elastic and capillary forces that depend on stretch. The first term is the elastic force, which must increase with stretch \( s \). The mass conservation equation (5) gives \( \pi R^2 s = \pi R_0^2 \), or \( s \sim 1/R^2 \), so that the second term, which is capillary force, decreases as \( s \) increases (equivalently, the capillary stress is the force divided by area, and is proportional to \( \sqrt{\pi Ca_{recoil}} \), increasing with stretch as the radius thins). Therefore, only the values of \( s \) for which \( h(s) \) has positive slope at the beginning are attainable. Once the stretch surpasses the value for the maximum of \( h(s) \), the filament yields, flows and collapses.

The value of \( \tilde{\lambda} \) in (40) is not known \emph{a priori}, so that the values of \( s \) for the slow manifold can not be solved from this equation. However, the inverse function \( h^{-1} \) is defined where it is increasing. Thus, the solution of (40) is

\[
s = h^{-1}(N_{sag} \tilde{Z} + \frac{\tilde{\lambda}}{\pi R_0^2}), \tag{42}
\]
and this is substituted into the volume constraint to find
\[
\int_0^1 h^{-1} \left(N_{\text{sag}} \tilde{Z} + \frac{\tilde{\lambda}}{\pi R_0^2} \right) d\tilde{Z} = \tilde{l}(\tilde{t}).
\] (43)

Since \(N_{\text{sag}} \tilde{Z} + \frac{\tilde{\lambda}}{\pi R_0^2} \) is a monotonically increasing function of \(\tilde{\lambda} \), (43) can be solved for \(\tilde{\lambda} \) as a function of \(\tilde{Z} \). This is substituted in (42) to express \(s \) as a function of \(\tilde{Z} \).

If gravity is absent, (43) simplifies to \(s = \tilde{l}(\tilde{t}) \), and \(\tilde{\lambda} = \pi R_0^2 h(s) \). An initially uniform filament will stay uniform.

1. **Zero surface tension and zero gravity**

If surface tension and gravity are neglected, then the study of \(h(s)\) focuses on the effects of the fluid rheology and stretch. A simplification is that \(\tilde{\lambda} \) in (39) is independent of \(\tilde{Z} \), and thus a constant for the entire length of the filament. From (40), \(h(s)\) is a constant for the filament. For any given \(\alpha \), this means that the stretch \(s\) is constant along the entire filament.

Figure 4(a) shows \(h(s)\) versus \(s\) for zero surface tension, or \(Ca_{\text{recoil}} = Ca^*_{\text{recoil}} = \infty \), and \(\alpha = -2.95 \) (—), \(-2.8 \) (- -), \(-2.5 \) (.-.), \(0 \) (…), and \(5 \) (- -). For each \(\alpha \), \(h(s)\) increases from \(s = 1\) to \(s = s_{\text{max}}\). For example, at \(\alpha = -2.95\), \(max[h] \approx 3.6\) at \(s_{\text{max}} = 1.2\) and at the slightly larger value of \(\alpha = -2.8\), \(max[h] \approx 1.7\) at \(s_{\text{max}} = 1.3\). If the initial pull produces a stretch that exceeds \(s_{\text{max}}\) locally, then the filament yields there. The figure shows two important features. First, \(s_{\text{max}}\) increases as \(\alpha\) increases from \(-3\). Secondly, the maximum of \(h(s)\) decreases as \(\alpha\) increases (reduction to half its value from \(-2.95\) to \(-2.8\)), and that \(max[h]\) decreases as \(\alpha\) increases.

If \(\alpha\) is close to \(-3\), the initial stretch must be quite small for the solution to arrive at the slow manifold. If the initial stretch exceeds \(s_{\text{max}}\), the solution arrives at the slow manifold only for a moment, and yields. In practical terms, an experiment to determine yield stress must find \(s_{\text{max}}\) easily, and this becomes easier as \(\alpha\) increases because the initial pull can access both below and above \(s_{\text{max}}\) without resorting to minute increments in \(s\).

2. **Non-zero surface tension**

With non-zero surface tension, a parametric study like figure 4(a) for a fixed capillary number requires a decision on whether \(Ca_{\text{recoil}}\) or \(Ca^*_{\text{recoil}}\) is fixed. The former is essentially
FIG. 4. $h(s)$ defined in (41) versus $s$ for stretching from equilibrium. In (a)-(c), $\alpha = -2.95$ (—), $-2.8$ (- -), $-2.5$ (-.-), $0$ (…), $5$ (- -) in order of decreasing maximum. (a) $Ca_{recoil} = \infty$. (b) $Ca_{recoil} = 1$. (c) $Ca_{recoil} = 0.01$. (d) Comparison of predictions for the two capillary numbers, $Ca_{recoil} = 1$ (—) and $Ca_{recoil}^* = 1$ (- -). $1 < s < 5$, at $\alpha = 5$.

yield stress divided by capillary stress, and the latter is the elastic modulus divided by capillary stress. These are alternative capillary number definitions which are defined in (27) and (30).

Specifically, how does the force in the filament, $h(s)$, depend on $\alpha$ if $Ca_{recoil}^*$ is used in place of $Ca_{recoil}$? From (41),

$$h(s) = \frac{s^2 - \frac{1}{s}}{s^3 + 2 + s\alpha} + \frac{3s^{-1/2}}{(3 + \alpha)Ca_{recoil}^*}. \quad (44)$$

Consider the limit $\alpha \to \infty$. Stress is non-dimensionalized with respect to $k_1$ in section III and the limit $\alpha \to \infty$ is taken with $k_1 = O(\alpha)$. This keeps the dimensional stress of $O(1)$, and the dimensional yield stress is infinite in this limit.

First, if $Ca_{recoil}$ is fixed, then capillary stress is also $O(\alpha)$ and dominates $h(s)$. The stretch function grows the most at the top where the filament is pulled, and eventually, $s$ is large enough that we can approximate $h(s) \sim \frac{s^{-1/2}}{Ca_{recoil}}$. Thus, capillary force dominates over elastic force to balance out (40). Figures 4(a)-(c) shows the behavior with fixed $Ca_{recoil} = \infty$ to .01, for $\alpha = -2.95$ (—), $-2.8$ (- -), $-2.5$ (-.-), $0$ (…), $5$ (- -). Initially, $s = 1$ and $h(1) = \frac{1}{Ca_{recoil}}$; therefore, the vertical scale is magnified from figure (a) to (c). As $Ca_{recoil}$ decreases from $\infty$ to 10, the essential features remain the same: $h(s)$ starts out increasing,
and reaches a maximum, beyond which the system is unstable. The maximum is lower and takes place at ever larger $s_{\text{max}}$ as $\alpha$ increases. Eventually, when $\alpha$ is greater than a critical value $\alpha_{\text{switch}}$, $h(s)$ becomes a decreasing function from the start, and the fluid yields right away. For example, Fig. 4(b) shows $Ca_{\text{recoil}} = 1$, where the $h(s)$ is a decreasing function for $\alpha_{\text{switch}} \approx 5$. If $-3 < \alpha < \alpha_{\text{switch}}$, then it is possible for slow evolution or delayed yielding to ensue if the stretch remains less than the critical value $s_{\text{max}}$.

The behavior for large capillary stress becomes noticeable at small $Ca_{\text{recoil}}$ and this is illustrated in figure 4(c) at $Ca_{\text{recoil}} = 0.01$; $h(s)$ increases initially only for a narrow window of $\alpha$ close to $-3$, and the rest of $\alpha$ exhibit monotonically decreasing $h(s)$. $h(s)$ is already decreasing at $\alpha = -2.8$. For $\alpha > -2.8$, immediate yielding takes place. For $Ca_{\text{recoil}} << 0.01$, the initial $h(1)$ is proportionately larger, and the curves for the $\alpha$'s begin to overlap.

Secondly, if $Ca_{\text{recoil}}^*$ is fixed and $\alpha \to \infty$, then (44) becomes $\approx \left( \frac{s^{1/2} - 1}{s^2} + \frac{s^{-1/2}}{(1/3)Ca_{\text{recoil}}^*} \right)/\alpha$, which retrieves the expression for the force balance for the upper convected Maxwell model case (cf. (5.2) of [28]).

Next, consider the limit in the opposite direction of $\alpha \to -3$. First, if $Ca_{\text{recoil}}$ is fixed, then the elastic and capillary effects compete. Secondly, if $Ca_{\text{recoil}}^*$ is fixed, then capillary stress dominates the force in the filament.

Figure 4(d) illustrates a major difference in keeping one or the other capillary number fixed at a large $\alpha = 5$. The case $Ca_{\text{recoil}} = 1$ (—) is a decreasing function, and hence exhibits immediate yielding. On the other hand, $Ca_{\text{recoil}}^* = 1$ ( - -) does not yield for $1 < s < 2$, and yields immediately if the initial stretch is beyond this.

V. NUMERICAL RESULTS

We present numerical computations of stretching and holding a filament, and tie the results to the asymptotic analysis of section IV. We begin with the initially cylindrical filament illustrated in figure 2, with initial radius $\tilde{R}_0$. The top of the filament is given by

$$\tilde{z}(1, \tilde{t}) = \tilde{\ell}(\tilde{t}) = 1 + \tilde{\ell}'(\tilde{t})\tilde{t}, \quad \tilde{\ell}'(\tilde{t}) = \begin{cases} \text{constant,} & 0 < \tilde{t} < \tilde{T} \\ 0, & \tilde{t} > \tilde{T}. \end{cases}$$

(45)

For $\tilde{t} \geq \tilde{T}$, the top boundary is fixed at $\tilde{\ell}(\tilde{T}) = 1 + \tilde{\ell}(\tilde{T}^-)\tilde{T}$.

We shall ignore the uninteresting case of large gravitational stress with the predictable
fall of the mass. Similarly, a large surface tension stress leads to filament breakup and is not pursued here.

We present four cases. The first singles out the elastic effects on stretching that contribute to yielding. The second illustrates an evolution on a slow time scale which eventually leads to yielding. The third shows effects of increasing elastic stress. The fourth case is a model for filament shape evolution in a capillary breakup extensional rheometer.

A. Yielding due to stretching and elastic effects

We focus on the effects of stretching and elastic stress on the onset of filament breakup. For this purpose, we choose parameters such that capillary and gravitational effects are not significant; \( N_{\text{sag}} \) is small and \( Ca_{\text{recoil}} \) is large. The ratio of retardation time to relaxation time is kept small at \( \epsilon = 10^{-4} \), and a constant initial radius of \( \tilde{R}_0 = 0.1 \) is used, so that the asymptotic analysis of the initial transient in section IV, leading to \( h(s) \) in (41), predicts what parameter values we need for the numerical computation of the filament evolution to yield due to elastic effects. At zero surface tension, the discussion of figure 4(a) for the elastic force \( h(s) \) versus stretch, concludes that \( \alpha \) needs to be close to \(-3\) to see interesting effects of yielding versus slow microstructural evolution. Thus, we choose \( \alpha = -2.9 \).

Figures 5(a) and (b) show the case of zero surface tension, or \( Ca_{\text{recoil}} = \infty \), and \( \tilde{T} = 1, N_{\text{sag}} = 0.01; t \) in the figures correspond to \( \tilde{t} \) in the text. In (a), the strain is 0.1, and in (b), the strain is 0.2. Specifically, if the filament is pulled out a small amount, to where the stretch \( s \) is below the first maximum of \( h(s) \), as in the case of strain 0.1 in Fig. 5(a), the initial dynamics is on the slow manifold and the filament does not yield. Fig. 5(a) plots up to \( \tilde{t} = 990 \), and there is no evidence of elastic effects in the form of springing back to the initial shape.

On the other hand, Fig. 5(b) with strain 0.2 is pulled to where \( h(s) \) starts to decrease with \( s \) in Fig. 4(a). Thus, this situation must yield immediately due to elastic effects. We see that this is indeed the case in the numerical computation. The filament breaks at \( \tilde{t} \approx 12 \), and since just the top experiences the large \( s \) locally, this is where the breakup takes place. In comparison, the stretch is not large below the breakage, so that much of the filament springs back to the initial shape. This is an example of the case when breakup is due solely to the elastic effects. In summary, if \( \alpha \) is close to -3, and only elastic effects are acting, then
FIG. 5. $\tilde{z}(\tilde{Z}, \tilde{t})$ versus $\tilde{R}(\tilde{t})$ for zero surface tension, $\epsilon = 10^{-4}$, $\alpha = -2.9$, $\tilde{T} = 1, N_{sag} = 0.01$, $Ca_{recoil} = \infty$, $\tilde{R}_0 = 0.1$. $t$ in the figure corresponds to $\tilde{t}$ in the text. (a) Strain 0.1 without breaking, (b) strain 0.2 reached in the same time interval using a higher pull rate, resulting in breaking at $\tilde{t}_{break} \approx 12$. Initially, the free surface is a straight vertical line for $0 < \tilde{z} < 1$, with constant radius.

yielding depends on the strain.

B. Breakup on slow time scales

A signature of a thixotropic yield stress fluid is delayed yielding. To illustrate this phenomenon for filament stretching, we are guided by the asymptotic analysis of the initial transient motion in section IV and figure 4. Here, we explore yielding that occurs ultimately from small effects of gravity and surface tension. Thus a small $N_{sag} = 0.098$, and high $Ca_{recoil} = 10, 14.3$ are chosen. This situation is similar to that of fig. 4(a). We seek a longer interval where $h(s)$ is increasing and choose the material parameter $\alpha$ accordingly. This consideration leads to larger $\alpha$ and $\alpha = -1$ is a reasonable choice. This case lies between those of $-2.5$ (...) and 0 (...). The interval for increasing $h(s)$ is $1 \leq s < 1.65$ at $\alpha = -1$ and if the initial stretch is in this interval, motion proceeds at a slow time scale.
FIG. 6. Evolution of filament shape, $\tilde{z}(\tilde{Z}, \tilde{t})$ versus $\tilde{R}(\tilde{t})$, for $\epsilon = 10^{-4}$, $\alpha = -1$, $N_{sag} = 0.098$, $\tilde{R}_0 = 0.1$. $t$ in the figure corresponds to $\tilde{t}$ in the text. (a) strain = 0.4. $Ca_{recoil} = 14.3$. The top is pulled according to (45) to length 1.4 at time 1s and held. The pull phase is shown at $\tilde{t} = 0, 0.24, 0.73, 0.98$. After full extension, the evolution is slow, and eventually sags down. (b) Strain 0.5, slightly higher than (a), $Ca_{recoil} = 10$. The top radius shrinks and at roughly $\tilde{t} = 42$, capillary pinching takes place, followed by sagging due to gravity.

The numerical simulations for $\tilde{z}(\tilde{t})$ versus $\tilde{R}(\tilde{z}, \tilde{t})$ are shown in Figure 6(a) for $Ca_{recoil} = 14.3$ and strain = 0.4. When the top boundary reaches the final position, $s$ moves from 1 to 1.4, and according to our asymptotic analysis in section IV, the elastic energy $h(s)$ is increasing at $s = 1.4$. Thus, the subsequent pinching is not due to an elastic instability which would occur only if $h(s)$ is decreasing. After $\tilde{t} = 1$, the top is fixed at $\tilde{z} = 1.4$, and the filament simply sags under gravity, increasing the radius at the bottom. We see that the strain is not high enough to make the filament yield immediately. The evolution occurs on a slow time scale during this phase with $\tilde{t} \sim \epsilon^{-1}$, so that the shapes at $\tilde{t} = 48740$ and at $\tilde{t} = 78000$ are approaching a linear profile. The stretch function eventually becomes large locally at the top. As soon as the top radius is small enough, capillary breakup would occur on a fast time scale.
Figure 6(b) is at a slightly larger strain $= 0.5$, and shows an elastically driven snap back toward its original radius that is not seen in the case of strain $= 0.4$ ($t$ in the figure corresponds to $\tilde{t}$). The top of the filament is pulled up and held at $\tilde{z} = 1.5$ after $\tilde{t} = 1$. From the asymptotic analysis of section IV, $h(s)$ is still increasing at strain 0.5 (or equivalently, $s = 1.5$ from (17)), we see that the eventual pinching is not due to elasticity but due to the sagging at the top after stretching, and eventually, the radius shrinks enough to undergo a capillary pinch-off.

The numerical simulations in Fig. 6(b) show that while being stretched upwards, the radius shrinks all along the filament to conserve volume. At the top, the radius shrinks the fastest and $s$ increases the fastest. By $\tilde{t} \approx 50$ ($t$ in the figure is $\tilde{t}$), this necessitates the use of local mesh refinement. We designed an in-house algorithm for this calculation. The details can be obtained from the authors.

At the same time that the radius decreases at the top, the bulk of the filament springs back toward its initial radius. At $\tilde{t} = 1000$, the filament has fallen to roughly $\tilde{z} \approx 1.2$. The rest of the filament is sloped toward the original radius, forming a step, with gravity pulling the volume down. The slow settling of the yield surface downward is due to gravity and takes place at $\tilde{t} \sim \epsilon^{-1}$. Whether this would be observed in reality is debatable because a pinch-off of this type would result in the immediate loss of axisymmetry, and thus our model would not apply beyond this point. In summary, the strain here is large enough for the fluid to yield; yielding takes place at the top, and is followed by sagging.

C. Effect of the elastic stress

We illustrate the effect of increasing the elastic stress while gravitational stress is fixed. This situation highlights an elastic spring back of the lower part of the filament which remains unyielded, while the top yields and thins due to gravity. The asymptotics of initial transients in section IV is used to guide the choice of parameters for a high elastic stress. Figures 4(a)-(b) clearly show that $\alpha$ should be close to -3 to achieve a higher maximum in $h(s)$, and therefore a higher elastic stress. For example, let $\alpha = -2.9$ and $Ca_{recoil} = O(1)$, and we see from Fig. 4(b) that if the initial $s$ very close to 1 at $z$, then that location of the filament settles to the slow manifold. If the initial stretch rate is high enough, the lower part of the filament experiences low stretch and the top experiences very high stretch. At
FIG. 7. Evolution of filament shape $\tilde{z}(\tilde{Z}, \tilde{t})$ versus $\tilde{R}(\tilde{t})$ illustrating elastic spring back for the unyielded lower portion of the filament. $\alpha = -2.9$. $t$ in the figure represents $\tilde{t}$. (a) $N_{sag} = 0.33$, $Ca_{recoil} = 4.3$, $\epsilon = 3.3 \times 10^{-4}$, (b) $N_{sag} = 0.05$, $Ca_{recoil} = 28.6$, $\epsilon = 5.0 \times 10^{-5}$. The initial radius is $\tilde{R}_0 = 0.1$. The strain is the same for (a)-(b), at 1. The top is pulled according to (45) to $\tilde{z} = 2$.

an intermediate point of the filament, therefore, there is a jump from unyielded to yielded flow. The effect of increasing the elastic stress while the gravitational stress is fixed is shown in Fig. 7. In both (a) and (b), elastic stress makes the lower half of the filament spring back to its original shape, and the stretching at the top leads to yielding. The difference between (a) and (b) is that gravitational stress is relatively lower in (b), and thus the filament takes longer to sag to the same level as in (a).

D. Yielding initiated by a dominant surface tension effect

A capillary breakup extensional rheometer (CaBER) is used in the work of [17] to make observations of yielding initiated by a dominant surface tension force for highly viscous fluids with apparent yield stress. The device pulls the fluid into a filament at prescribed speed until it pinches off. Several products are studied in [17], such as a water-oil emulsion, composed of densely packed small water droplets in a continuous paraffin oil phase, and a high viscosity thickener solution, which is a dispersion of swollen microgel particles. Their photographs
of the filament shape are centered around the cinching area, and may not be of the entire filament. We prescribe our initial condition to be slightly cinched midway between the ends as shown in Fig. 8. The initial radius at the top and bottom are assumed equal, denoted $R_0$, and cinched to a fraction $\beta R_0$ at $Z = L/2$ with $0 < \beta < 1$. We choose $\beta = 0.8$ as an example. The surface shape is initially set to be linear functions as follows:

$$R_0(Z) = \begin{cases} R_0 \left( \frac{(\beta - 1)Z}{L/2} + 1 \right) & \text{if } 0 < Z \leq L/2 \\ R_0 \left( \frac{(1-\beta)Z}{L/2} + 2\beta - 1 \right) & \text{if } L/2 < Z \leq L. \end{cases}$$

(46)

This initial condition modifies the calculation of prior sections that initialize the filament as a cylinder. The modifications include the calculation of $\lambda(t)$ because of the evaluation of the integrals over $Z$: these are now performed separately over $0 < Z < L/2$ and $L/2 < Z < L$, and summed. The cross sectional area in the initial configuration now depends on $Z$, where $A(Z) = \pi R_0^2(Z)$ is given as polynomials in $Z$ for the two intervals of $Z$. This modifies the evolution equations, for instance, through the integrals $\int_0^Z A(\zeta) d\zeta$, which must be evaluated to derive $s_t$ in (32).

The rheological data for the water-oil emulsion in [17] is adopted within our framework. All quantities are in CGS units unless otherwise stated; $\rho g = 980 \text{ cm s}^{-2}$, the surface tension coefficient is $\approx \sigma = 50 \text{ dyne cm}^{-1}$. Data in fig. 2 of [17] is used to estimate our Newtonian constant viscosity model to be $\eta = 100 \text{ P}$. Their data does, however, display shear thinning.
after yielding, and we comment on this later. The yield stress from their shear data is on the order of 100 dyn cm$^{-2}$. In order to calculate our model parameters $\alpha$ and $k_1$, as described in section II, we require both the elastic modulus of the unyielded phase and the yield stress. However, the models addressed in [17] are of Bingham type, and there would be no need to pay attention to the elastic modulus at zero shear, and it is not provided. Thus, we have some leeway in using (16), because we match the asymptotic formula for immediate yielding with yield stress exceeding the maximum elastic shear stress in the initial deformation, $rac{k_1}{2\sqrt{3+\alpha}}$, which depends on both $k_1$ and $\alpha$. Thus, a specific yield stress is achieved by pairs of $\alpha$ and $k_1$. The choice of $\alpha$ and $k_1$ affects the elastic modulus $\frac{k_1}{2(3+\alpha)}$ which is not provided. Numerical simulations for $\alpha$ from $-2$ to 5 are conducted, and found to be qualitatively similar; the case $\alpha = 4$ is chosen for illustration.

Figure 8 of [17] does not show the full filament but the mid-section away from the ends. We adopt their definition of $t = 0$ as the time of full elongation. The rate of pull influences the balance of viscous versus capillary force, and affects the evolution as shown in figure 9(a). We use their initial length $L = 0.3$ cm, and their extension at the final length $\ell(t) \approx 1.664$ cm, which gives the rate of pull $\frac{d\ell}{dt}$ to be 34 to 45 cm s$^{-1}$. This gives our pulling time $T = 0.03$ to 0.04 s. The radius at full elongation is roughly 0.045 cm at the middle in fig. 8 of [17], and already noticeably cinched. Finally, we use $\frac{d\ell}{dt} = 35$ cm s$^{-1}$, and the time to full extension $T = 39$ ms. Pinching occurs at 0.35 to 0.4s in their data.

Figure 9(a) shows the numerical simulation (—, blue online) for the minimum diameter (mm) versus time (s) superposed on their fig. 9 data. Our simulation (—) leads to pinch-off at $t_{\text{break}} \approx 0.3$ s after full extension, which is in the ballpark of their data. Figure 9(b) shows our simulation for the shape of the filament from full extension (−) to just before pinch-off 0.3s later (−−). The dimensionless parameters from (21), (25), (26), and (24) are the strain $= \frac{d\ell}{dt} \frac{T}{L} = 4.55$, dimensionless radius before pull $\tilde{R}_0 = 0.5$, $\epsilon = 0.02$, and $Ca_{\text{recoil}} = 1.7$. The effect of gravity is not negligible based on the value of $N_{sag} = 0.56$, and this accounts for the build-up of material below the middle, and promotes thinning above. This results in the smaller conical volume at the top compared with the bottom in Fig. 9(b), and evident also in the photographs of Fig. 1. The final filament shape in our simulation has a thin long middle, which is similar to the photograph for a shower gel. We conjecture that the filament shapes in fig. 1 are a reflection of viscosity after yielding. If the post-yield rheology is Newtonian,
FIG. 9. (a) Evolution of diameter (mm) with time (s). Our simulation (− blue online) at $\alpha = 4$, $k_1 = 530$, $L = 0.3$ cm, $T = 39$ ms, $d\ell/dt = 35$ cm s$^{-1}$, $\eta = 100$, $\sigma = 50$ CGS units, superposed on a reproduction from Figure 9 of [17]. (b) Our numerical simulation at full extension (−) and 0.3 s later just before pinch-off (−−).

then the literature on pinching of filaments driven by surface tension provides similarity solutions of one-dimensional models in Stokes flow$^{21}$, with inertia$^{22}$, and a viscous outer fluid$^{23}$. Further, the similarity solutions for a power law fluid with a shear rate dependent viscosity shows that the curvature at the waist and the jet profile changes markedly with the power law exponent$^{26}$. For highly shear thinning fluids, the profile is pointed, while less shear thinning fluids have elongated concave profiles. Figure 1(a) is similar to the jet profile for a highly shear thinning fluid shown in figure 5 of [26] The elongated waist in figure 1(b) is reminiscent of jet profiles of less shear thinning fluids. The behavior observed in fig. 9(a) for our PECN simulation reflects the model assumption of a constant Newtonian viscosity.

VI. CONCLUSIONS

Filament breakup of thixotropic yield stress fluids is one way to characterize the rheology of complex materials such as gels and suspensions. On the one hand, experimental work of this nature faces challenges that are currently starting to be addressed. On the other, our work is the first for a simplified but more realistic geometry in extension, taking advantage
of a constitutive model that does not rely on purely phenomenological derivations but, at the same time, has success in in predicting thixotropic yield stress behavior for shear flows such as hysteresis upon unyielding. The novelty of this work also lies in the use of multi scale asymptotics with two time scales, in order to obtain analytical information on whether breakup occurs immediately or later.

The stretch-and-hold problem consists of a filament which is pulled up from equilibrium in a prescribed manner and held. The PECN constitutive model is used to combine a viscoelastic model for an entangled microstructure (Partially Extending strand Convection model) and a Newtonian solvent. We focus on the small parameter \( \epsilon = \text{retardation time/relaxation time} \), since this regime gives rise to thixotropy. A slender filament approximation is used with uniaxial extension and axisymmetry. The dimensionless material parameter \( \alpha \) is related to the maximum elastic shear stress and the elastic modulus, and \( \alpha \rightarrow \infty \) relates to the upper convected Maxwell model. Pastes have been modeled in prior work in the range \(-3 < \alpha < 0\), and wormlike micellar solutions in the range \( \alpha > 0^{9,12} \). A capillary number and a sagging number for the effect of gravity are included.

An asymptotic analysis with a slow time scale on the order of \( 1/\epsilon \) and a fast time scale of order 1 yields a simplified force balance for the initial dynamics. There is a critical value of strain above which immediate yielding occurs. The critical value increases as \( \alpha \) increases, allowing for more stretching before the fluid yields. The analysis provides guidance for numerical simulations based on the full governing equations without the small \( \epsilon \) assumption. Results of filament shape evolution from beginning to pinch-off illustrate yielding due to strain and dominated by elastic effects, delayed yielding and immediate yielding. Delayed yielding can be due to small effects of gravity and surface tension, and achieved by a slowly growing strain, and a slow sag, and is sensitive to the magnitude of the initial pull. Simulations that model extensional experiments of [17] for fluids with apparent yield stress and notable surface tension predict yielding on a fast time scale to approach pinching. In comparison, we attribute the importance of the PEC model parameters in predicting the time to yielding. The evolution of the thinning filament shape is attributed to the post-yield viscosity model.
ACKNOWLEDGMENTS

This research was partly funded by the National Science Foundation under NSF-DMS-1311707. We thank Michael Renardy for discussions.

REFERENCES

13. P. A. Vasquez, L. P. Cook, and G. H. McKinley. A network scission model for wormlike mi-


27 M. Renardy. Some comments on the surface-tension driven break-up (or the lack of it) of