BOUNDARY CONTROL OF THE TIMOSHENKO BEAM*
JONG UHN KIM† AND YURIKO RENARDY‡

Abstract. It is shown that the Timoshenko beam can be uniformly stabilized by means of a boundary control. A numerical study on the spectrum is also presented.

Key words. Timoshenko beam, uniform stabilization, exponential decay, boundary control, energy method, Co-semigroup, linear stability, eigenvalues, spectral method

AMS(MOS) subject classifications. 35B37, 35L15, 73K05, 93C20, 93D15, 65F15, 65N25

0. Introduction. The purpose of this paper is to investigate uniform stabilization of the Timoshenko beam with boundary control. The motion of a beam can be described by the Euler beam equation when the cross-sectional dimensions are small in comparison with the length of the beam. If the cross-sectional dimensions are not negligible, the effect of the rotatory inertia should be considered and the motion is better described by the Rayleigh beam equation. If the deflection due to shear is also taken into account in addition to the rotatory inertia, we arrive at a still more accurate model, which is called the Timoshenko beam. Its motion is described by the following system of equations:

\[ \frac{\partial^2 w}{\partial t^2} + K \frac{\partial^2 w}{\partial x^2} + K \frac{\partial \phi}{\partial x} = 0, \]

\[ \frac{I_p}{\rho} \frac{\partial^2 \phi}{\partial t^2} - EI \frac{\partial^2 \phi}{\partial x^2} + K \left( \phi - \frac{\partial w}{\partial x} \right) = 0. \]

Here, \( t \) is the time variable and \( x \) is the space coordinate along the beam in its equilibrium position. We denote by \( w(x, t) \) the deflection of the beam from the equilibrium line, which is described by \( w = 0 \), and by \( \phi(x, t) \) the slope of the deflection curve when the shearing force is neglected; for the precise meaning of \( \phi \), see Timoshenko [11] or Traill–Nash and Collar [12]. We assume that the motion occurs in the \( wx \)-plane and that \( 0 \leq x \leq L \). The coefficients \( \rho, I_p \), and \( E \) are the mass per unit length, the mass moment of inertia of the cross section, Young’s modulus and the moment of inertia of the cross section, respectively. The coefficient \( K \) is equal to \( kGA \), where \( G \) is the modulus of elasticity in shear, \( A \) is the cross sectional area and \( k \) is a numerical factor depending on the shape of the cross section. The boundary condition we employ at \( x = 0 \) is

\[ w(0, t) = 0, \quad \phi(0, t) = 0, \]

which is for the clamped end at \( x = 0 \), and the boundary control at \( x = L \) is of the form

\[ K \frac{\partial w}{\partial x} (L, t) - K \frac{\partial \phi}{\partial x} (L, t) = \alpha \frac{\partial w}{\partial t} (L, t), \]

\[ EI \frac{\partial \phi}{\partial x} (L, t) = -\beta \frac{\partial \phi}{\partial t} (L, t). \]

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‡ Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061. The work of this author was supported by National Science Foundation grant DMS-8615203 under the National Science Foundation Research Opportunities for Women Program.
where $\alpha$ and $\beta$ are positive constants depending on the control device. This boundary control corresponds to a control mechanism which monitors $\partial w/\partial t$ and $\partial \phi/\partial t$ at $x = L$ and transforms them into the lateral force and moment applied at $x = L$, respectively. Russell [10] and Washizu [13] derived (0.1) and (0.2) through the energy principle by using the natural energy of the beam given by

$$
(0.6) \quad e(t) = \frac{1}{2} \int_0^L \left\{ \rho \left(\frac{\partial w}{\partial t}\right)^2 + I_\rho \left(\frac{\partial \phi}{\partial t}\right)^2 + K \left(\phi - \frac{\partial w}{\partial x}\right)^2 + EI \left(\frac{\partial \phi}{\partial x}\right)^2 \right\} dx.
$$

One can also derive an equivalent fourth-order equation in terms of $w$; see Timoshenko [11] and Traill-Nash and Collar [12]. In particular, [12] discusses various boundary conditions.

This paper consists of two main parts. In the first part, we show that the energy $e(t)$ decays exponentially fast under (0.3), (0.4) and (0.5). For the one-dimensional wave equation with boundary control, Quinn and Russell [9] established the exponential decay of solutions. Later, Chen [1], [2] obtained the same result for a wave equation in any space dimension under some geometrical conditions on the domain. A very restrictive part of these conditions was eliminated by Lagnese [5] with the aid of a new energy estimate. Lagnese [6] also extended the result to linear elastodynamic systems. In contrast to the above works, Lasiecka and Triggiani [7] employed boundary feedback acting in the Dirichlet boundary condition to achieve exponential decay of solutions to the wave equation. More recently, Chen et al. [3] discussed the case of a chain of Euler beams and obtained a similar result. Our result for the Timoshenko beam is most closely related to [3]. We use the energy method combined with $C_0$-semigroup theory as in [1]-[3], [5] and [6]. The essence of the method is to construct a suitable energy functional associated with $e(t)$. Details are given in §2.

The second part of this paper is concerned with a numerical study. Since the nature of the spectrum is an important question in the investigation of the stability of a linear system, we carried out numerical experiments on the spectrum of (0.1) and (0.2) under (0.3)-(0.5). We express the temporal variation of the eigenfunction in normal modes of the form $e^{\alpha}$, transforming (0.1)-(0.5) to ordinary differential equations with boundary conditions. The Chebyshev-tau method [4], [8] is used to discretize the spatial variation of the eigenfunctions, thus yielding a matrix eigenvalue problem with discrete complex-valued eigenvalues $\lambda$. These are computed using a NAG routine in quadruple precision on a VAX 11/785. Results of numerical experiments are presented in §3.

1. Notation and preliminaries. We shall use the notation

$$
{f}_t = \frac{\partial f}{\partial t}, \quad {f}_x = \frac{\partial f}{\partial x}, \quad {f}_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad \text{etc.}
$$

$L^2$ always denotes $L^2(0, 1)$ and we write

$$
H^m = \left\{ f : f, \left[ \frac{d}{dx} \right]^k f \in L^2, \ k = 1, \ldots, m \right\}.
$$

Our basic function space $\mathcal{G}$ is the set of all quadruplets

$$
z = \begin{bmatrix}
{w}_1 \\
{w}_2 \\
{\phi}_1 \\
{\phi}_2
\end{bmatrix}
$$
satisfying
\[ w_1 \in H^1, \quad w_1(0) = 0, \quad w_2 \in L^2, \]
\[ \phi_1 \in H^1, \quad \phi_1(0) = 0, \quad \phi_2 \in L^2, \]
equipped with the inner product
\[
\langle z, \tilde{z} \rangle_{\mathcal{H}} = \int_{0}^{1} \left\{ \frac{K}{\rho} (\partial_x w_1)(\partial_x \tilde{w}_1) + w_2 \tilde{w}_2 + \frac{EI}{I_\rho} (\partial_x \phi_1)(\partial_x \tilde{\phi}_1) + \phi_2 \tilde{\phi}_2 \right\} dx.
\]
We shall also use the function space \( \mathcal{F} \) which is the set of all quadruplets
\[
z = \begin{bmatrix} w_1 \\ w_2 \\ \phi_1 \\ \phi_2 \end{bmatrix}
\]
satisfying
\[ w_1 \in H^2, \quad w_1(0) = 0, \quad w_2 \in H^1, \quad w_2(0) = 0, \]
\[ \phi_1 \in H^2, \quad \phi_1(0) = 0, \quad \phi_2 \in H^1, \quad \phi_2(0) = 0, \]
\[ K\phi_1(1) - K \partial_x w_1(1) = \omega w_2(1), \quad EI \partial_x \phi_1(1) = -\beta \phi_2(1), \]
equipped with the inner product induced by \( H^2 \times H^1 \times H^2 \times H^1 \). Here, \( K, \alpha, E, I \) and \( \beta \) are the same as in the previous section. It is easy to show that \( \mathcal{F} \) is dense in \( \mathcal{H} \).

We define the operator \( \Lambda \) in \( \mathcal{H} \):
\[
\Lambda = \begin{bmatrix}
0 & \text{id} & 0 & 0 \\
(K/\rho) \partial_{xx} & 0 & -(K/\rho) \partial_x & 0 \\
0 & 0 & 0 & \text{id} \\
(K/I_\rho) \partial_x & 0 & (EI/I_\rho) \partial_{xx} - (K/I_\rho) \text{id} & 0
\end{bmatrix}
\]
where \( \text{id} \) is the identity mapping and the domain of \( \Lambda \) is taken to be \( \mathcal{F} \). It is then easy to see that (0.1), (0.2) with (0.3), (0.4) and (0.5) can be put in the abstract form:
\[
\frac{dz}{dt} = \Lambda z \quad \text{where } z = \begin{bmatrix} w \\ w_2 \\ \phi_1 \\ \phi_2 \end{bmatrix} \quad \text{and } L \text{ is taken to be } 1.
\]

We also observe the following.

**Lemma 1.1.** \( \Lambda \) is an infinitesimal generator of a \( C_0 \)-semigroup in \( \mathcal{H} \).

**Proof.** Let us write \( \Lambda = \Lambda_0 + \Lambda_1 \), where
\[
\Lambda_0 = \begin{bmatrix}
0 & \text{id} & 0 & 0 \\
(K/\rho) \partial_{xx} & 0 & 0 & 0 \\
0 & 0 & 0 & \text{id} \\
0 & 0 & (EI/I_\rho) \partial_{xx} & 0
\end{bmatrix}
\]
with \( \mathcal{D}(\Lambda_0) = \mathcal{F} \) and
\[
\Lambda_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -(K/\rho) \partial_x & 0 \\
0 & 0 & 0 & 0 \\
(K/I_\rho) \partial_x & 0 & -(K/I_\rho) \text{id} & 0
\end{bmatrix}.
\]
It is apparent that $A_1$ is a bounded linear operator on $\mathcal{G}$. Thus, it is enough to show that $A_0$ is an infinitesimal generator of a $C_0$-semigroup. By virtue of the Lumer–Phillips theorem, it is enough to show that

$$\langle A_0 z, z \rangle_\mathcal{G} \leq c \|z\|^2_\mathcal{G} \quad \text{for all } z \in \mathcal{S},$$

where $c$ is a positive constant, and that

$$\text{Range of } (\lambda \text{ id} - A_0) = \mathcal{G} \quad \text{for some } \lambda > c.$$

By integration by parts using the boundary condition of $z \in \mathcal{S}$, we find that

$$\langle A_0 z, z \rangle_\mathcal{G} = \frac{K}{\rho} w_2(1) \partial_x w_1(1) + \frac{EI}{I_p} \phi_2(1) \partial_x \phi_1(1)$$

$$= \frac{1}{\rho} (K \phi_1(1) - \alpha w_2(1)) w_2(1) - \frac{\beta}{I_p} \phi_2(1)^2$$

$$\leq \frac{1}{4\rho \alpha} K^2 \phi_1(1)^2$$

$$\leq \frac{1}{4\rho \alpha} K^2 \int_0^1 (\partial_x \phi_1)^2 \, dx$$

$$\leq \frac{K^2}{4\rho \alpha E I} I_p \|z\|^2_\mathcal{G},$$

from which (1.6) follows. We next prove (1.7) for any $\lambda > 0$. Let $\lambda > 0$ and

$$\begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} \in \mathcal{G}.$$ 

Then, we have to find

$$\begin{bmatrix} w_1 \\ w_2 \\ \phi_1 \\ \phi_2 \end{bmatrix} \in \mathcal{S}$$

such that

$$\lambda w_1 - w_2 = f_1,$$  

$$\lambda w_2 - \frac{K}{\rho} \partial_{xx} w_1 = f_2,$$  

$$\lambda \phi_1 - \phi_2 = g_1,$$  

$$\lambda \phi_2 - \frac{EI}{I_p} \partial_{xx} \phi_1 = g_2.$$  

We can find $\phi_1 \in H^2$ such that

$$\left( \lambda^2 - \frac{EI}{I_p} \partial_{xx} \right) \phi_1 = g_2 + \lambda g_1,$$

$$\phi_1(0) = 0, \quad \beta \lambda \phi_1(1) + EI \partial_x \phi_1(1) = \beta g_1(1).$$
In fact, $\phi_1$ is given by

$$
\phi_1(x) = c \sinh \mu x - \frac{I_p}{\mu EI} \int_0^x (g_2(s) + \lambda g_1(s)) \sinh \mu(x-s) \, ds
$$

where $\mu = \lambda (I_p/EI)^{1/2}$ and $c$ is uniquely determined from $\beta \lambda \phi_1(1) + EI \partial_x \phi_1(1) = \beta g_1(1)$.

Then, $\phi_2$ is determined by (1.13). It is obvious that $\phi_2 \in H^1$, $\phi_2(0) = 0$ and $EI \partial_x \phi_1(1) = -\beta \phi_2(1)$. Similarly, we can find $w_1 \in H^2$ such that

$$
(\lambda^2 - \frac{K}{\rho} \partial_{xx}) w_1 = f_2 + \lambda f_1,
$$

$$
w_1(0) = 0, \quad \alpha \lambda w_1(1) + K \partial_x w_1(1) = K\phi_1(1) + \alpha f_1(1).
$$

Then, $w_2$ is determined from (1.19). It is easy to see that

$$
\begin{bmatrix}
w_1 \\
w_2 \\
\phi_1 \\
\phi_2
\end{bmatrix} \in \mathcal{G}
$$

and (1.9)-(1.12) hold.

We shall use the following elementary inequality later on:

$$
\int_0^1 w_x^2 \, dx \leq 2 \int_0^1 (\phi - w_x)^2 \, dx + 2 \int_0^1 \phi_x^2 \, dx
$$

for all $\phi \in H^1$, $w \in H^1$ satisfying $\phi(0) = 0$.

2. Statement and proof of the main result. In this section, we take $L = 1$ without loss of generality. Let $S(t)$ be the $C_0$-semigroup in $\mathcal{G}$ generated by $\Lambda$ in the previous section. We assert the following.

THEOREM 2.1. The operator norm of $S(t)$ satisfies

$$
\|S(t)\| \leq M e^{-rt} \quad \text{for all } t \geq 0
$$

where $M$ and $r$ are positive constants.

Before giving details of the proof, we shall outline our arguments. Let us fix any $z_0 \in \mathcal{G}$. Using $\varepsilon(t)$ associated with $S(t) z_0$, we define

$$
F(t) = \mu \varepsilon(t) + G(S(t) z_0)
$$

where $\mu$ is a positive constant depending only on the coefficients of (0.1), (0.2), and $G(\cdot)$ is a suitable functional on $\mathcal{G}$ such that

$$
|G(z)| \leq C \|z\|_{\mathcal{G}}^2 \quad \text{for all } z \in \mathcal{G}.
$$

With the aid of (1.18), we can derive that

$$
d_1 \|S(t) z_0\|_{\mathcal{G}}^2 \leq \varepsilon(t) \leq d_2 \|S(t) z_0\|_{\mathcal{G}}^2
$$

holds for all $t \geq 0$, where $d_1$ and $d_2$ are positive constants depending only on the coefficients of (0.1) and (0.2). We then show that

$$
F(t) \leq M_1 \|z_0\|_{\mathcal{G}}^2
$$
holds for all \( t \geq 0 \), where \( M_1 \) is a positive constant depending only on \( \alpha, \beta \) and the coefficients of (0.1) and (0.2). Formulae (2.3) and (2.4) imply

\[
\| S(t)z_0 \|_\mathcal{F}^2 \leq \frac{1}{t} M_2 \| z_0 \|_\mathcal{F}^2 \quad \text{for all } t > 0
\]

where \( M_2 \) is a positive constant depending only on \( \alpha, \beta \) and the coefficients of (0.1) and (0.2). Since \( \mathcal{F} \) is dense in \( \mathcal{G} \), (2.6) implies

\[
\| S(t)z \|_\mathcal{G} \leq \left( \frac{M_2}{t} \right)^{1/2} \| z_0 \|_\mathcal{G} \quad \text{for all } z \in \mathcal{G}.
\]

Finally, we use the semigroup property of \( S(t) \) to arrive at (2.1).

**Proof of Theorem 2.1.** Fix any \( z_0 \in \mathcal{G} \). Then,

\[
S(t)z_0 \in C([0, \infty); \mathcal{G}) \cap C^1([0, \infty); \mathcal{G})
\]

and

\[
\frac{d}{dt} S(t)z_0 = \Lambda S(t)z_0 \quad \text{for every } t \geq 0.
\]

Hence, we can write

\[
S(t)z_0 = \begin{bmatrix}
w(x, t) \\
\partial_t w(x, t) \\
\phi(x, t) \\
\partial_t \phi(x, t)
\end{bmatrix}
\]

where \( w(x, t) \) and \( \phi(x, t) \) satisfy (0.1)-(0.5).

We now construct \( F(t) \):

\[
F(t) = \frac{\mu t}{2} \int_0^1 \left\{ \rho w^2_t + I_\rho \phi_t^2 + K (\phi - w_x)^2 + EI \phi_x^2 \right\} dx
\]

\[
+ \rho \int_0^1 x w w_x dx + I_\rho \int_0^1 x \phi \phi_x dx + \frac{1}{2 + \eta} I_\rho \int_0^1 \phi \phi_t dx
\]

\[
- \frac{1}{2 + \eta} \rho \int_0^1 w w_t dx
\]

where \( \mu \) and \( \eta \) are positive constants which will be determined later on. By virtue of (2.8) and (2.10), we can differentiate (2.11) to obtain

\[
\frac{dF}{dt} = \mu J_1 + \mu t J_2 + \sum_{n=3}^9 J_n
\]

where

\[
J_1 = \varepsilon(t) = \frac{1}{2} \int_0^1 \left\{ \rho w^2_t + I_\rho \phi_t^2 + K (\phi - w_x)^2 + EI \phi_x^2 \right\} dx,
\]

\[
J_2 = \int_0^1 \left\{ \rho w_t w_x + I_\rho \phi_t \phi_x + K (\phi - w_x)(\phi_t - w_{xt}) + EI \phi_x \phi_{xt} \right\} dx,
\]
Using (0.1)-(0.5), we can integrate by parts to arrive at

\[
J_2 = -\alpha w_t(1, t)^2 - \beta \phi_t(1, t)^2, \\
J_3 = \frac{1}{2} \rho w_t(1, t)^2 - \frac{1}{2} \rho \int_0^1 w_t^2 \, dx, \\
J_4 = \int_0^1 \rho w_t(\delta) \, dx, \\
J_5 = \frac{1}{2} \rho \phi_t(1, t)^2 - \frac{1}{2} \rho \int_0^1 \phi_t^2 \, dx, \\
J_6 = \int_0^1 \phi_t(\delta) \, dx.
\]

Let us choose \( \eta > 0 \) such that

\[
K \left( \frac{1}{2} - \frac{1}{2 + \eta} \right) \leq \frac{1}{4} EI.
\]
and then, choose \( \mu > 0 \) such that
\[
\mu (2K + EI) \leq \left( \frac{1}{4} + \frac{1}{2 + \eta} \right) EI,
\]
\[(2.20)\]
\[
2 \mu \leq \frac{1}{2} - \frac{1}{2 + \eta}.
\]
Then, we find that
\[
\frac{dF}{dt} \leq -\frac{1}{2} \left( \frac{1}{4} + \frac{1}{2 + \eta} \right)EI \int_0^1 \phi_x^2 dx - \frac{K}{2} \left( \frac{1}{2} - \frac{1}{2 + \eta} \right) \int_0^1 w_x^2 dx
\]
\[\quad - \alpha \mu w_t(1, t)^2 - \beta \mu t \phi_t(1, t)^2 + \frac{1}{2} \rho \phi_t(1, t)^2
\]
\[\quad + \frac{1}{2} K \phi_x(1, t)^2 + \frac{1}{2} L \phi_x(1, t)^2 + \frac{1}{2} EI \phi_x(1, t)^2
\]
\[\quad - \frac{1}{2} K \phi(1, t)^2 + \frac{1}{2 + \eta} EI \phi(1, t) \phi_x(1, t)
\]
\[\quad + \frac{1}{2 + \eta} K \phi(1, t) w(1, t) - \frac{1}{2 + \eta} K w_x(1, t) w(1, t).
\]
By means of (0.4) and (0.5) and the inequalities
\[
\phi(1, t)^2 \leq \int_0^1 \phi_x(x, t)^2 dx,
\]
\[(2.23)\]
\[
w(1, t)^2 \leq \int_0^1 w_x(x, t)^2 dx
\]
\[(2.24)\]
we deduce from (2.22) that for all \( t \geq T \),
\[
\frac{dF}{dt} \leq 0
\]
\[(2.25)\]
where \( T \) is a positive constant depending only on \( \alpha, \beta \) and the coefficients of (0.1) and (0.2). Consequently, we arrive at (2.5). The argument following (2.5) completes the proof.

Remark 2.2. Finally, we remark that our arguments with the same energy functional also yield exponential stabilization for a hinged boundary condition at \( x = 0 \):
\[
w(0, t) = 0, \quad \frac{\partial \phi}{\partial x}(0, t) = 0.
\]
However, in this case it seems necessary to impose the zero mean condition \( \int_0^1 \phi dx = 0 \) in order to avoid some technical difficulties (see [1]).

3. Numerical study of the spectrum. We present numerical results on the linear stability of our system. We use normal mode analysis and set
\[
w(x, t) = e^{\lambda t} P(x),
\]
\[(3.1)\]
\[
\phi(x, t) = e^{\lambda t} Q(x).
\]
\[(3.2)\]
Thus, (0.1)-(0.5) become the following system of ordinary differential equations with boundary conditions:

\[(3.3) \quad -KP_{xx} + KQ_x + \lambda^2 \rho P = 0,\]
\[(3.4) \quad -EIQ_{xx} + K(Q - P_x) + \lambda^2 I_\rho Q = 0,\]
\[(3.5) \quad P = Q = 0 \quad \text{at} \quad x = 0,\]
\[(3.6) \quad EI\!Q_x + \lambda \beta Q = 0 \quad \text{at} \quad x = L,\]
\[(3.7) \quad K(Q - P_x) - \lambda \alpha P = 0 \quad \text{at} \quad x = L.\]

This system is of the form \(A_0 + A_1 \lambda + A_2 \lambda^2 = 0\), where

\[(3.8) \quad A_0 = \begin{bmatrix} -KP_{xx} + KQ_x \\ -EIQ_{xx} + K(Q - P_x) \\ P(0) \\ Q(0) \\ EIQ_x(L) \\ KQ(L) - KP_x(L) \end{bmatrix},\]
\[(3.9) \quad A_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \beta Q(L) \\ -\alpha P(L) \end{bmatrix},\]
\[(3.10) \quad A_2 = \begin{bmatrix} \rho P \\ I_\rho Q \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.\]

We rewrite this in the form

\[(3.11) \quad \det \begin{bmatrix} \lambda & -A_0 \\ 1 & A_1 + \lambda A_2 \end{bmatrix} = 0\]

so that our system takes on the customary form \(A = \lambda B\), where

\[(3.12) \quad A = \begin{bmatrix} 0 & -A_0 \\ 1 & A_1 \end{bmatrix},\]
\[(3.13) \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -A_2 \end{bmatrix}.\]

We discretize \(P(x)\) and \(Q(x)\) by the Chebyshev-tau method [4], [8]. This is a spectral method where the expansion functions are the Chebyshev polynomials \(T_n(z)\) defined by \(T_n(\cos \theta) = \cos n\theta\) when \(z = \cos \theta\). This method approximates discrete eigenvalues belonging to \(C^\infty\) eigenfunctions with infinite-order accuracy.
We rescale the spatial variable to $z = (2x)/L - 1$, so that $-1 \leq z \leq 1$. We set

$$P(z) = \sum_{n=0}^{N} p_n T_n(z),$$

$$Q(z) = \sum_{n=0}^{N} q_n T_n(z)$$

and substitute into (3.3)–(3.7). Thus, there are $2N + 2$ unknowns. In the differential equations, we equate coefficients of like powers of the Chebyshev polynomials. Since the first equation (3.3) contains $P_{xx}$, it yields equations for the coefficients up to degree $N - 2$ in the polynomials, thus giving $N - 1$ equations. Similarly, (3.4) yields $N - 1$ equations. In the tau approximation, the expansion functions $T_n(z)$ are not required to satisfy the boundary conditions individually. The four boundary conditions are imposed as part of the conditions determining the coefficients $p_n$ and $q_n$. The total number of equations is $2N + 2$. The size of the final matrix equation $A = \lambda B$ is $4N + 4$ square. Our computer program uses the NAG routine F02GJF to compute the eigenvalues in complex quadruple precision on a VAX 11/785.

The accuracy of our numerical results was established in the following way. The eigenvalues must satisfy the characteristic equation:

$$\det \begin{bmatrix} m_{11}(\lambda) & m_{12}(\lambda) & m_{13}(\lambda) & m_{14}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) & m_{23}(\lambda) & m_{24}(\lambda) \\ m_{31}(\lambda) & m_{32}(\lambda) & m_{33}(\lambda) & m_{34}(\lambda) \\ m_{41}(\lambda) & m_{42}(\lambda) & m_{43}(\lambda) & m_{44}(\lambda) \end{bmatrix} = 0$$

where, for $j = 1, 2, 3, 4$

$$m_{1j} = 1, \quad m_{2j} = I_{\rho} \lambda^2 \eta_j - EI \eta_j^3, \quad m_{3j} = (EI \eta_j + \beta \lambda) e^{\eta_j \rho},$$

$$m_{4j} = \left( EI \lambda \eta_j^2 - I_{\rho} \lambda^3 - \frac{\alpha}{\rho} I_{\rho} \lambda^2 \eta_j + \frac{\alpha}{\rho} EI \eta_j^2 \right) e^{\eta_j \rho},$$

and $\eta_j$ are the roots of

$$\eta^2 = \frac{1}{2EI} \left[ \lambda^2 \left( I_{\rho} + \frac{\rho EI}{K} \right) \pm \left( \lambda^4 \left( I_{\rho} - \frac{\rho EI}{K} \right)^2 - 4\rho E I \lambda^2 \right)^{1/2} \right].$$

In order to check that our computed eigenvalues satisfy the determinant equation (3.16), we have chosen moderate-sized parameters so that the evaluation of the determinant avoids cancellation between large numbers. We choose $\rho = 1, K = 1.5, I_{\rho} = 2, E = 2.5, I = 3, L = 0.1, \alpha = 3.5$ and $\beta = 4.1$. Computations at $N = 15, 20, 25$ and 30 showed that a few eigenvalues are already converged to about 15 digits at $N = 15$. About 12 eigenvalues at $N = 15$ are converged to at least 5 digits, and satisfy (3.16) to that accuracy. All converged eigenvalues have negative real parts and are either real or complex conjugates. The number of digits to which each pair is a complex conjugate is an indication of the amount of roundoff error present. The eigenvalues consist of two groups. One group is lined up approximately along the line $-4.47$ and the imaginary parts are almost multiples, starting with $\lambda = -4.4709$ and then $\lambda = -4.4769 \pm 38.475i, -4.4770 \pm 76.951i$ and so on. The other group is lined up approximately along $-34.45$, and the imaginary parts are almost multiples, starting with $\lambda = -34.505$ and then $\lambda = -34.446 \pm 60.829i, -34.452 \pm 121.68i$ and so on.
Computations were done at the following set of parameters to model a solid aluminum bar: \( \rho = 400 \text{ g/cm} \), \( K = 2.8 \times 10^{13} \text{ g} \cdot \text{cm/sec}^2 \), \( I_p = 3,332 \text{ g} \cdot \text{cm} \), \( E = 7.6 \times 10^{11} \text{ g/cm/sec}^2 \), \( I = 833 \text{ cm}^4 \) and \( L = 200 \text{ cm} \). We allow \( \alpha \) and \( \beta \) to be

(i) \( \alpha = 10^3 \text{ g/sec}, \beta = 10^2 \text{ g} \cdot \text{cm}^2/\text{sec}; \)
(ii) \( \alpha = 2 \times 10^3 \text{ g/sec}, \beta = 2 \times 10^2 \text{ g} \cdot \text{cm}^2/\text{sec}; \)
(iii) \( \alpha = 5 \times 10^3 \text{ g/sec}, \beta = 5 \times 10^2 \text{ g} \cdot \text{cm}^2/\text{sec}. \)

By comparing the results of \( N = 40 \) and \( N = 45 \), we conclude that about 20 complex conjugate pairs have converged to 5 digits at \( N = 40 \), and there are no real-valued eigenvalues. The results for case (i) are plotted in Figs. 1 and 2. Figure 2 is a magnification of Fig. 1 close to the origin. All eigenvalues have negative real parts. Figure 1 does not indicate that the ratio \( \text{Im}(\lambda)/\text{Re}(\lambda) \) approaches a constant for large \(|\lambda|\). Results of cases (i)-(iii) are displayed in Table 1. Essentially, the real parts of \( \lambda \) are approximately proportional to \( \alpha \) or \( \beta \) and the imaginary parts of the three cases...
This table displays to 5 digits the first 22 to 23 complex conjugate pairs for cases (i)-(iii) described in § 3.

<table>
<thead>
<tr>
<th>Case (i)</th>
<th>Case (ii)</th>
<th>Case (iii)</th>
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<tr>
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are approximately equal. This indicates that when $\alpha$ and $\beta$ are zero, the real parts are zero and the eigenvalues are purely imaginary. The relative importance of the damping terms in our computations is seen from the dimensionless equations. There are 6 dimensionless parameters: $C_1 = \frac{LQ}{\bar{P}}$, where $\bar{Q}$ is the scale of $Q$ and $\bar{P}$ is the scale of $P$, $C_2 = \frac{\lambda^2 \rho L^2}{K}$, $C_3 = \frac{KL^3}{EI}$, $C_4 = \frac{\lambda^2 L^2}{EI}$, $C_5 = \frac{\lambda \beta L}{EI}$ and $C_6 = \lambda \alpha L/K$. Our values for case (i) are: $C_2 = 2.6 \times 10^{-7}$, $C_3 = 1.8 \times 10^3$, $C_4 = 2 \times 10^{-7}$, $C_5 = \lambda \times 3 \times 10^{-6}$ and $C_6 = \lambda \times 7 \times 10^{-9}$. The dimensionless equations are

\begin{align}
(3.19) \quad -P_{xx} + C_1 Q_x + C_2 P = 0, \\
(3.20) \quad -Q_{xx} + C_3 Q - \frac{C_3}{C_1} P + C_4 Q = 0, \\
(3.21) \quad P = Q = 0 \quad \text{at} \quad x = 0, \\
(3.22) \quad Q_x + C_5 Q = 0 \quad \text{at} \quad x = 1, \\
(3.23) \quad C_1 Q - P_x - C_6 P = 0 \quad \text{at} \quad x = 1
\end{align}

where $P$, $Q$ and $x$ have been made dimensionless. The largest eigenvalues in Fig. 1 are $O(10^3)$ so that $C_5$ is $O(10^{-1})$ and $C_6$ is $O(10^{-3})$, indicating that the damping terms are not large. For the smallest eigenvalues, the damping terms are small so that the property of proportionality of $Re(\lambda)$ to $\alpha$ or $\beta$ may be an asymptotic behavior for small damping.

We note that the qualitative features obtained for the computation with moderate data are very different from those of the model of an aluminium bar. This is reminiscent of the qualitative differences in Figs. 5 and 7 of Chen et al. [3].
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REFERENCES