EFFECTIVENESS AND ROBUSTNESS WITH RESPECT TO TIME DELAYS
OF BOUNDARY FEEDBACK STABILIZATION IN
ONE-DIMENSIONAL VISCOELASTICITY*

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Abstract. The damping effect of a boundary feedback mechanism on torsional vibrations of a
homogeneous viscoelastic rod is examined. A characteristic equation is derived for the oscillatory modes,
and the solutions of this equation are studied by analytic and numerical methods, as functions of the
feedback gain parameter and of a parameter for feedback delay. Results are compared to recent studies of
elastic materials, where a feedback delay can cause exponential instability; here this phenomenon depends
significantly on the short-time behavior of the viscoelastic memory kernel. Finally, an existence result is
given, showing that the behavior of a weak solution corresponds in the expected way to the location of the
characteristic roots.

Key words. boundary stabilization, viscoelasticity, time delays

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1. Introduction. We examine the damping effect of a boundary feedback
mechanism on torsional vibrations of a homogeneous viscoelastic rod of uniform
circular cross section. In the purely elastic case it is known [15] that this mechanism
can force uniform exponential decay, but that time delays, which are inherent to
physical feedback mechanisms, can introduce exponentially growing vibrations. When
viscoelastic stress-strain laws are involved, exponential decay can occur even with
fixed or free ends, but the decay can be slow in the low-frequency oscillating modes.
We investigate whether boundary feedback can improve the rate of decay in these
modes and under what conditions delays in the feedback can lead to unbounded
vibrations. Analogous problems have been investigated for a variety of elastic bodies
and structures [9]-[12], [14], [23]-[26], [28]; for the viscoelastic case, we have chosen
a physical model that leads to the simplest equations of this type.

Specifically, we consider a homogeneous viscoelastic rod of length L and uniform
circular cross section of radius R. For the constitutive law relating cross-sectional shear
stress $\sigma$ and shear strain $\gamma$ at a point $p$ in the rod, we use a linear Boltzmann model
of the rate type [4] (see [29]):

$$
\sigma(p, t) = G\gamma(p, t) + \int_0^\infty g(\tau) \frac{\partial}{\partial t} \gamma(p, t-\tau) \, d\tau.
$$

The equilibrium stress modulus $G$ is a positive constant (that is, the material is a
solid), and $g(t)$ is completely monotonic and integrable on $(0, \infty)$ with $0 < g(0+) \leq \infty$.

We remark that complete monotonicity is a physically reasonable assumption that
is satisfied by our examples. As noted below, some of our results (e.g., on the presence
or absence of characteristic values in the right half-plane due to time delays) hold
when $g$ is merely positive, nonincreasing, and convex, with $-g'$ convex. By holding

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throughout to the stronger hypothesis, we lose little generality overall and avoid repetitious statements of various lemmas for slightly different hypotheses on $g$.

Observe that if the strain history $\gamma$ satisfies suitable decay conditions as $t \to -\infty$, then an integration by parts allows us to rewrite (1.1) as

$$\sigma(p, t) = G\gamma(p, t) + \frac{d}{dt} \int_0^t g(\tau)\gamma(p, t - \tau) d\tau + \int_0^\infty g'(\tau)\gamma(p, t - \tau) d\tau,$$

where the last integral on the right side of (1.2) is determined by the strain history for $t < 0$. Terms that arise from this contribution to the stress can be incorporated into a nonhomogeneous term. We do this throughout and make any required hypotheses directly of the resulting nonhomogeneous terms. As noted in §5, our hypotheses on those nonhomogeneous terms have reasonable interpretations in terms of the original history value problem.

For $0 \leq y \leq L$, $\theta(y, t)$ denotes the angular displacement of the cross section at $y$ from the equilibrium position. We assume that the angular displacement is known for $t \leq 0$, and we allow an initial jump in the angular velocity at $t = 0$, that is,

$$\theta(y, t) = \theta_0(y, t) \quad (t \leq 0), \quad \theta_1(y, 0^+) = \theta_1(y)$$

for $0 \leq y \leq L$. Then the constitutive law (1.2) together with a balance of angular momentum yields (compare [22, p. 42] for an elastic rod)

$$\theta_0(y, t) = G\theta_{yy}(y, t) + \frac{\partial}{\partial t} \int_0^t g(\tau)\theta_{yy}(y, t - \tau) d\tau + B(y, t),$$

$0 < y < L, \ t > 0$, where $\rho$ is the mass density of the rod, $\mathcal{R} = (\pi/2)R^4$, and $B(y, t)$ consists of a distributed applied torque together with the contribution from the initial history. We assume that the left end of the rod is fixed, i.e.,

$$\theta(0, t) = 0, \quad -\infty < t < \infty.$$

At the end $y = L$ we feed back a torque of the form

$$\tau(L, t) = -\kappa H(t - \varepsilon)\theta_1(L, t - \varepsilon), \quad t \geq 0,$$

where $\kappa \geq 0$ is the feedback gain, $\varepsilon \geq 0$ is a time delay in the feedback mechanism, and $H$ denotes the Heaviside step function. We consider both the case where there is a concentrated tip mass at $y = L$ with moment of inertia $I_m > 0$ about the axis of the rod, and the case where no such tip mass is present, i.e., $I_m = 0$. This leads to the balance law

$$I_m\theta_1(L, t) + \mathcal{R}\left\{G\theta_{yy}(L, t) + \frac{\partial}{\partial t} \int_0^t g(\tau)\theta_{yy}(L, t - \tau) d\tau\right\} = b(t) - \kappa H(t - \varepsilon)\theta_1(L, t - \varepsilon), \quad 0 \leq t < \infty,$$

where $b(t)$ consists of any other applied torque at $y = L$ together with the contribution that arises from the initial history.

We normalize the length and radius of the rod by introducing the variables

$$E = G\rho^{-1}L^{-2}, \quad a(t) = g(t)\rho^{-1}L^{-2},$$

$$I = I_m\mathcal{R}^{-1}\rho^{-1}L^{-1}, \quad k = \kappa\mathcal{R}^{-1}\rho^{-1}L^{-1},$$

$u(x, t) = \theta(Lx, t), \quad u_0(x, t) = \theta_0(Lx, t), \quad u_1(x) = \theta_1(Lx),$

$$F(x, t) = B(Lx, t)\mathcal{R}^{-1}\rho^{-1} \quad (0 \leq x \leq 1),$$

$$f(t) = b(t)\mathcal{R}^{-1}\rho^{-1}.$$
Then (1.3)-(1.5) and (1.7) become

\[ u_t(x, t) = E u_{xx}(x, t) + \frac{\partial}{\partial t} \int_0^t a(\tau) u_{xx}(x, t - \tau) \, d\tau + F(x, t), \quad 0 < x < 1, \quad t > 0, \]

(1.9)

\[ u(x, 0) = u_0(x), \quad u_t(x, 0^+) = u_1(x), \quad 0 \leq x \leq 1, \]

(1.10)

\[ u(0, t) = 0, \quad 0 \leq t < \infty, \]

(1.11)

\[ I u_t(1, t) + E u_x(1, t) + \frac{\partial}{\partial t} \int_0^t a(\tau) u_x(1, t - \tau) \, d\tau = f(t) - k H(t - \varepsilon) u_t(1, t - \varepsilon), \quad t > 0. \]

(1.12)

We remark that \( E \) and \( a(t) \) have dimension \( t^{-2} \), \( I \) is dimensionless, and \( k \) has dimension \( t^{-1} \).

Let \( A(t) \in \mathbb{R} + a(t) \) denote the (normalized) linear stress relaxation modulus. As mentioned above, we assume throughout that

\[ A(t) = E + a(t) \text{ with } E > 0 \text{ and } a(t) \text{ completely monotonic, } a \in L^1(0, \infty), \]

(1.13) and \( 0 < a(0^+) \).

If we expect to achieve exponential decay, we must assume that \( a(t) \) decays exponentially as \( t \to \infty \), so we also require that

\[ e^{\eta a(t)} \in L^1(0, \infty) \quad \text{for some } \eta > 0. \]

(1.14)

(In the analysis, the lack of hypothesis (1.14) manifests itself in a branch cut along the negative real axis that extends all the way up to the origin.)

Some important special cases follow.

(i) \( a(t) = 0 \) (excluded by (1.13)). This is the elastic case, and (1.9)-(1.12) (with \( I = 0 \)) reduces to the feedback stabilization problem for the one-dimensional wave equation which was included in [15].

(ii) \( A(0^+) < \infty \). Then

\[ E \gamma(p, t) + \frac{\partial}{\partial t} \int_0^t a(\tau) \gamma(p, t - \tau) \, d\tau = A(0^+) \gamma(p, t) + \int_0^t a'(\tau) \gamma(p, t - \tau) \, d\tau. \]

This is linear viscoelasticity of the Boltzmann type [4]. In what follows the cases \( A'(0^+) > -\infty \) and \( A'(0^+) = -\infty \) will be distinguished.

(iii) \( A(t) = E + \frac{\gamma}{\Gamma(1 - \alpha)} t^{-\alpha} e^{-\delta t}, \quad 0 < \alpha < 1, \quad \delta > 0 \)

(\( \Gamma \) = the gamma function.) This is a fractional derivative model modified by an exponential decay factor. Fractional derivative models have been used successfully to fit experimental complex modulus data for some real materials [1]-[3], [30].

In §2 we derive a characteristic equation for (1.9)-(1.12), solutions of which give the complex frequencies of asymptotic eigenvibrations. In §2 we also examine the possible destabilizing effect of time delays in (1.12). The effectiveness of (1.12), when \( \varepsilon = 0 \), in stabilizing high-frequency vibrations (possibly a consideration when \( A'(0^+) > -\infty \) is discussed in §3. In the case where there is no concentrated tip mass, i.e., \( I = 0 \) in (1.12), we present analytic and numerical results showing the following.

(i) When \( \varepsilon = 0 \), all roots of the characteristic equation for (1.9)-(1.12) are in the left half-plane and bounded away from the imaginary axis so that solutions should decay exponentially. (This is justified in §5.)
(ii) In the free-end and fixed-end cases ($k = 0$, $k = \infty$, respectively) with $\epsilon = 0$, low-frequency modes decay slowly, but rates of decay can be improved by choosing $k$ appropriately. Detailed numerical examples illustrating this are presented in § 4.

(iii) For sufficiently small positive $\epsilon$, the absence or presence of resonance (with unbounded solutions) depends on the growth of $A(t)$ as $t \to 0^+$.

We also analyze the effect of a concentrated tip mass and show the following:

(iv) When $I > 0$ in (1.12), variation of $k$ improves the decay rate for low-frequency modes, but does not affect the high-frequency modes. (In the case of regular kernels, i.e., $A'(0^+) > -\infty$, with $\epsilon = 0$, this result is in agreement with a recent result due to Desch and Miller [16] concerning exponential stabilization of abstract linear integrodifferential equations in Hilbert space. Namely, in this situation (1.9)–(1.12) can be formulated in a semigroup setting, and since $I > 0$, the operator (1.12) is a finite rank operator. Thus, by [16] the essential growth rate of the resolvent operator for the abstract integrodifferential equation cannot be changed by (1.12). See also Gibson [17].) Moreover, when $I > 0$, sufficiently small feedback delays ($\epsilon > 0$) do not lead to unbounded solutions.

As mentioned above, we present detailed numerical examples illustrating the effectiveness of (1.12) (when $\epsilon = 0$) in stabilizing low-frequency modes, both when $I > 0$ and $I = 0$. These numerical examples are discussed in § 4.

In § 5 we present theorems that guarantee the existence (in a weak sense) of the solutions to (1.9)–(1.12) discussed above. We also justify a Green's function representation which shows that the roots of the characteristic equation do determine the asymptotic decay of $u(x, t)$ as $t \to \infty$.

Finally, we postpone until § 6 the rather technical proofs of two lemmas.

2. The characteristic equation: robustness with respect to time delays. In this section we derive a Green's formula for the formal Laplace transform of the solution of (1.9)–(1.12), and we study how the location of the poles of this Green's formula is affected by time delays in (1.12). The interpretation of these results in terms of the existence of a solution and its asymptotic decay appears in § 5.

We consider an integrated version of (1.9)–(1.12), namely,

\begin{align}
(2.1) & \quad u_t(x, t) = \int_0^t A(t - \tau) u_{xx}(x, \tau) \, d\tau + F_0(x, t), \\
(2.2) & \quad u(x, 0) = u_0(x), \\
(2.3) & \quad u(0, t) = 0, \\
(2.4) & \quad I u_t(1, t) + \int_0^t A(t - \tau) u_x(1, \tau) \, d\tau = f_0(t) - k H(t - \epsilon)[u(1, t - \epsilon) - u_0(1)]
\end{align}

on $\{0 < x < 1, 0 \leq t < \infty\}$. Here $A$ satisfies (1.13) and (1.14), $I$, $k$, and $\epsilon$ are nonnegative constants, and

\begin{align}
F_0(x, t) &= \int_0^t F(x, \tau) \, d\tau + u_t(x), \\
f_0(t) &= \int_0^t f(\tau) \, d\tau + I u_t(1).
\end{align}

Assume that $F_0(x, t)$ and $f_0(t)$ are exponentially bounded as $t \to \infty$. Then taking formal Laplace transforms in (2.1)–(2.4) we obtain (for $\text{Re} \, s$ sufficiently large)

\begin{align}
(2.5) & \quad \hat{A}(s) U_{xx}(x, s) - s U(x, s) = -[\hat{F}_0(x, s) + u_0(x)], \\
(2.6) & \quad U(0, s) = 0, \\
(2.7) & \quad I[s U(1, s) - u_0(1)] + \hat{A}(s) U_x(1, s) = -k e^{-\epsilon s} \left[ U(1, s) - \frac{u_0(1)}{s} \right] + \hat{f}_0(s),
\end{align}
where \( U \) stands for the Laplace transform \( \hat{u} \). Proceeding formally, let

\[
(2.8) \quad \alpha(s) = (s\hat{A}(s))^{1/2},
\]

where we use the principal square root, i.e., \(|\arg z^{1/2}| < \pi/2\) when \(|\arg z| < \pi\) (see (2.15) below for justification), and define

\[
(2.9) \quad \beta(s) = \alpha(s)/\hat{A}(s),
\]

\[
(2.10) \quad \Delta(s) = [Is + k e^{-\varepsilon t}] \sinh \beta(s) + \alpha(s) \cosh \beta(s).
\]

Let \( s \) be such that \( \Delta(s) \) is defined and \( \Delta(s) \neq 0 \). Then the general solution of (2.5)–(2.6) is given by

\[
U(x, s)= C(s) \sinh (s)x - \alpha(s) \sinh \beta(s)(x-y)[u_0(y)+ F_0(s, y)] dy,
\]

and using the boundary condition (2.7) we see that the constant \( C(s) \) is determined by

\[
\Delta(s)C(s) = \left( \frac{ke^{-\varepsilon t}}{s} + I \right) u_0(1) + \int_0^1 \left\{ \alpha^{-1}(s)(ke^{-\varepsilon t} + Is) \sinh \beta(s)(1-y) + \cosh \beta(s)(1-y) \right\}
\]

\[
\cdot [u_0(y)+ F_0(y, s)] dy.
\]

Thus, for such \( s \) (2.5)–(2.7) has the unique solution

\[
(2.11) \quad U(x, s)= A(s) G(x, y, s)[u_0(y)+ F_0(y, s)] dy
\]

where the Green's function is defined by

\[
(2.12) \quad G(x, y, s)= -\sinh (s)x [\alpha^{-1}(s)(ke^{-\varepsilon t} + Is) \sinh \beta(s)(1-y)
\]

\[
+ \cosh \beta(s)(1-y)] \quad \text{for} \quad 0 \leq x < y \leq 1,
\]

\[
G(x, y, s) = G(y, x, s) \quad \text{for} \quad 0 \leq y < x \leq 1.
\]

We remark that \( sA(s) \rightarrow E \) as \( s \rightarrow 0 \) in \( \text{Re } s \geq 0 \) since \( a \in L^1(0, \infty) \). Thus, \( \Delta(0) = E^{1/2} > 0 \) and \( (\sinh \beta(s))/s \) is continuous at \( s = 0 \); hence, the expression in (2.11) is continuous at \( s = 0 \) whenever \( sF_0(y, s) \) and \( sF_0(s) \) are.

We assume that \( \hat{F}_0(s) \) and \( \hat{F}_0(x, s) \) are analytic functions of \( s \) in an appropriate right half-plane that satisfy suitable decay conditions as \( s \rightarrow \infty \) in this half-plane. Then \( U(x, s) \) is meromorphic in a half-plane with poles at the zeros of \( \Delta(s) \). In § 5, we use an argument based on Plancherel's theorem to obtain an exponentially weighted \( L^2 \)-estimate for the solution \( u(x, t) \) of (2.1)–(2.4) that depends on the location of these poles. Conversely, if \( \Delta(s_0) = 0 \) with \( \text{Re } s_0 > \mu \), then we can pick \( F_0, f_0, \) and \( u_0 \) so that (2.1)–(2.4) does not have a solution with \( e^{-\mu t}u(\cdot, t) \in L^2(\mathbb{R}^2; L^2(0, 1)) \). For the rest of this section as well as in §§ 3 and 4, we study the location of the zeros of the characteristic function \( \Delta(s) \).
Recall that if (1.13) holds, then Bernstein's theorem yields a nondecreasing function $\mu$ on $[0, \infty)$ with $0 = \mu(0) < \mu(0^+) = E < \mu(\infty) = A(0^+) \equiv \infty$ and $\mu(x) = \mu(x^-)$ for $0 < x < \infty$, such that

$$A(t) = \int_0^\infty e^{-xt} d\mu(x) \quad (t > 0).$$

Note that $\int_0^\infty d\mu(x)/x < \infty$ since $a \in L^1(0, \infty)$, and also observe that

$$(2.13) \int_0^\infty d\mu(x) = A(0^+), \quad \int_0^\infty x d\mu(x) = -A'(0^+),$$

where $A'(0^+) = -\infty$ and $A(0^+) = \infty$ are both allowed in (2.13). Since $a \in L^1(0, \infty)$ the Laplace transform $\hat{A}(s)$ exists for $s = \sigma + i\tau, \sigma \geq 0, s \neq 0$, and it follows from the theory of Laplace and Stieltjes transforms [32, Chap. 8] that $\hat{A}(s)$ can be continued analytically to the slit plane $\mathbb{C} \setminus (-\infty, 0]$ by the formula

$$(2.14) \hat{A}(s) = \int_0^\infty \frac{d\mu(x)}{s + x},$$

where the integral converges uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. (Since $\mu$ is real, $\hat{A}(s) = \hat{A}(s)$; this observation may be used without mention in the sequel.) Clearly $\hat{A}(\sigma) > 0$ when $\sigma > 0$, and by (2.14),

$$(2.15) \text{Im}(s\hat{A}(s)) = \frac{\tau x}{(\sigma + x)^2 + \tau^2} \int_0^\infty d\mu(x) \neq 0 \quad \text{when } \tau \neq 0.$$

If, in addition, (1.14) holds for some $\eta > 0$, then $\mu(x) \equiv E$ on $(0, \eta]$ and $\alpha(s)$ in (2.8) is defined and analytic in a slit plane $\mathbb{C} \setminus (-\infty, s^*]$, where $s^* < 0$ is defined by

$$(2.16) s^* \hat{A}(s^*) = E + s^* \hat{A}(s^*) = 0.$$

The transform $\hat{A}(s)$ has a simple pole at $s = 0$, and the functions $\beta(s)$ and $\Delta(s)$ defined in (2.9) and (2.10) are analytic on $\mathbb{C} \setminus (-\infty, s^*]$ and $\beta(0) = 0, \Delta(0) = E^{1/2}$.

In order to study the question of when the equation

$$(2.17) \Delta(s) = 0$$

has solutions in a right half-plane, it is necessary to have estimates that are uniform as $s \to \infty$ in a right half-plane. To obtain such estimates it is convenient to recall the following result due to Lindelöf (see [8, p. 2]). Fix $\sigma_0$ and suppose that there exists an $R > 0$ so that the function $h(s)$ is analytic in $G = \{\text{Re } s > \sigma_0, |s| > R\}$, and so that $h$ is bounded and continuous in the closure $\overline{G}$. If $h(\sigma_0 + i\tau) \to A$ as $|\tau| \to \infty$, then $h(s) \to A$ as $s \to \infty$ uniformly in $\{\text{Re } s \leq \sigma_0\}$.

Our first lemma is an elementary result.

**Lemma 2.1.** Assume that (1.13) holds and let $0 \leq \sigma_0 > -\infty$ be given. Then

(i) $s\hat{A}(s) \to A(0^+) \quad \text{as } s \to \infty$, 

(ii) $\alpha(s)/s \to 0 \quad \text{as } s \to \infty$, 

and the convergence in (2.18) and (2.19) is uniform in $\{\text{Re } s \geq \sigma_0\}$. (The right-hand side of (2.18) is infinity when $A(0^+) = \infty$.)

If $A'(0^+) > -\infty$, then

$$s^2 \hat{A}(s) - sA(0^+ - A'(0^+) \to 0 \quad \text{as } s \to \infty$$

uniformly in $\{\text{Re } s \geq \sigma_0\}$. 

Proof. Assume that $A(0^+)<\infty$. Then

$$(\sigma_0 + i\tau)\hat{A}(\sigma_0 + i\tau) = \int_0^\infty \frac{(\sigma_0 + i\tau)}{(\sigma_0 + i\tau) + x} \, d\mu(x) \to A(0^+)$$

as $|\tau| \to \infty$ by Lebesgue's dominated convergence theorem and (2.13). Since $s\hat{A}(s)$ is analytic and bounded in $\{\text{Re} \ s \geq \sigma_0, |s| \geq 2|\sigma_0|\}$, Lindelöf’s theorem yields (2.18) when $A(0^+) < \infty$. Similarly, if $A'(0^+) > -\infty$, then

$$s^2 \hat{A}(s) - sA(0^+) = -\int_0^\infty \frac{s x}{s + x} \, d\mu(x)$$

is analytic and bounded in $\{\text{Re} \ s \geq \sigma_0, |s| \geq 2|\sigma_0|\}$. Moreover, setting $s = \sigma_0 + i\tau$ in this expression and letting $|\tau| \to \infty$, we see by dominated convergence and (2.13) that $(\sigma_0 + i\tau)^2 \hat{A}(\sigma_0 + i\tau) - (\sigma_0 + i\tau)A(0^+) \to A'(0^+)$ as $|\tau| \to \infty$; hence, (2.20) follows by Lindelöf’s theorem.

To prove (2.18) when $A(0^+) = \infty$, let $s = \sigma_0 + i\tau$ and note that

$$|s\hat{A}(s)| \leq |\tau| ||\hat{A}(s)|| = \int_0^\infty \frac{\tau^2}{(\sigma_0 + x)^2 + \tau^2} \, d\mu(x)$$

$$\leq \int_0^\infty \frac{\tau^2}{\tau^2 + \tau^2} \, d\mu(x) \leq \frac{1}{2} \int_0^\infty \, d\mu(x) \to \infty$$

as $|\tau| \to \infty$. Thus, since $1/s\hat{A}(s)$ is analytic and bounded in $\{\text{Re} \ s \geq \sigma_0, |s| \geq 2|\sigma_0|\}$, (2.18) is proved when $A(0^+) = \infty$.

Finally, rewriting (2.14) as

$$\hat{A}(s) = \int_0^\infty \frac{1 + x}{s + x} \frac{d\mu(x)}{1 + x},$$

and recalling that $\int_0^\infty (d\mu(x)/x) < \infty$, we see that $\hat{A}(s) \to 0$ as $s \to \infty$ uniformly in $\text{Re} \ s \geq \sigma_0$ by the dominated convergence theorem, and, in particular, (2.19) holds by the definition of $\alpha$. □

As an easy consequence of Lemma 2.1 we get the following lemma.

Lemma 2.2. Assume that (1.13) holds and that $A'(0^+) > -\infty$. Then for any $\sigma_0 > -\infty$,

$$(2.21) \quad \exp(-2\beta(s))/C \exp(-2sA(0^{+1/2})^{-1}) \to 1$$

as $s \to \infty$ uniformly in $\text{Re} \ s \geq \sigma_0$, where

$$(2.22) \quad C = \exp(A'(0^+)/A(0^{+1/2})).$$

Proof. By (2.20) and a Taylor approximation to the square root,

$$s\alpha(s)A(0^+)^{-1} - sA(0^+)^{-1/2} - A'(0^+)A(0^+)^{-2} \to 0$$

as $s \to \infty$ uniformly in $\text{Re} \ s \geq \sigma_0$. Also, using (2.18) and (2.20) we easily see that

$$\hat{A}(s)^{-1} - sA(0^+)^{-1} + A'(0^+)A(0^+)^{-2} \to 0$$

as $s \to \infty$ uniformly in $\text{Re} \ s \geq \sigma_0$. Thus, since $\beta(s) = \alpha(s)/\hat{A}(s)$,

$$\beta(s) - sA(0^+)^{-1/2} + A'(0^+)A(0^+)^{-1} \to 0$$

as $s \to \infty$ uniformly in $\text{Re} \ s \geq \sigma_0$, and (2.21) is proved. □

The proof of the next lemma is somewhat technical and it is postponed until § 6.

Lemma 2.3. Assume that (1.13) holds and that $A'(0^+) = -\infty$. Then for any $\sigma_0 > -\infty$,

$$(2.23) \quad \exp(-2\beta(s)) \to 0 \quad \text{as} \quad s \to \infty$$

uniformly in $\text{Re} \ s \geq \sigma_0$. 
We remark that Lemmas 2.1-2.3 all hold when $\sigma_0 = 0$ with complete monotonicity in (1.13) weakened to $a$ is positive, nonincreasing, and convex, with $-a'$ convex on $(0, \infty)$. (The proofs are sketched in [20].) In particular, as we noted in the Introduction, our results on the presence or absence of solutions of (2.17) in $\{\text{Re } s \geq 0\}$ due to time delays in (2.4) all hold, with complete monotonicity replaced by this weaker assumption on $a$.

For the rest of this section we study robustness with respect to time delays, and investigate the question of whether (2.17) has solutions in $\{\text{Re } s \geq 0\}$. We will see that the behavior of the stress relaxation modulus $A(t)$ near $t = 0$, and the presence (I $>$ 0) or absence (I $= 0$) of a concentrated tip mass, are of primary importance in this regard.

From (2.14) we write
\[
\hat{A}(s) = \varphi_\sigma(\tau) - i\psi_\sigma(\tau) = \int_0^\infty \frac{\sigma + \chi}{(\sigma + \chi)^2 + \tau^2} d\mu(x) - i \int_0^\infty \frac{\tau}{(\sigma + \chi)^2 + \tau^2} d\mu(x),
\]
and, in particular, we obtain
\[
\varphi_\sigma(\tau) > 0(\sigma \geq 0, \tau \in \mathbb{R}, s \neq 0), \quad \tau\psi_\sigma(\tau) > 0(\sigma \geq 0, \tau \neq 0).
\]

By these two inequalities we see that
\[
|\arg \alpha(s)| < \pi/4 \quad \text{when } \text{Re } s \geq 0, \quad s \neq 0,
\]
and that $\beta(s)$ defined by (2.9) is the same as $\beta(s) = (s/\hat{A}(s))^{1/2}$ (principal branch) when $\text{Re } s \geq 0$. (We caution the reader that this formula for $\beta(s)$ need not be valid when $\text{Re } s < 0$.) In particular,
\[
|\arg \beta(s)| < \pi/2 \quad \text{when } \text{Re } s \geq 0, \quad s \neq 0.
\]

Finally, recall that $s = 0$ never satisfies (2.17) since $\Delta(0) = E^{1/2}$.

As an elementary consequence of (2.25) and (2.26) we obtain the following proposition.

\begin{proposition}
Let (1.13) hold.
\begin{enumerate}
\item If $k = 0$, then (2.17) has no solution in $\{\text{Re } s \geq 0\}$.
\item If $k > 0$ and $I, \epsilon$ are nonnegative, then (2.17) has no solution in $\{\sigma \geq 0, 0 \leq |\tau| < \pi/2\epsilon\}$. In particular, (2.17) has no solution in $\{\text{Re } s \geq 0\}$ when $\epsilon = 0$.
\end{enumerate}
\end{proposition}

\begin{proof}
Rewrite (2.17) as
\[
\exp(-2\beta(s)) = \frac{I s + k e^{-\epsilon s} + \alpha(s)}{I s + k e^{-\epsilon s} - \alpha(s)}.
\]

By (2.26), $|\exp(-2\beta(s))| < 1$ when $\text{Re } s \geq 0, s \neq 0$. If $k = 0$, then (2.25) implies that the modulus of the right-hand side of (2.17') is greater than or equal to 1, so (2.17') cannot hold when $\text{Re } s \geq 0, s \neq 0$, and part (i) is proved. If $k > 0$ and $|\tau| < \pi/2\epsilon$, then $|\arg k e^{-\epsilon s}| < \pi/2$, so by (2.25) the modulus of the right-hand side of (2.17') is greater than 1, and (2.17') cannot hold. \(\square\)

Proposition 2.1 shows that for a given $\epsilon > 0$, we can always be assured that (2.17) has no eigensolutions below the frequency $\pi/2\epsilon$. On the other hand, for any $I \geq 0$, by rewriting (2.17) as
\[
k e^{-\epsilon s} = -(\alpha(s) \coth \beta(s) + I s),
\]
we see that any $s_0 \in \mathbb{R}$ with $\text{Re } s_0 > 0$ is a solution for some $\epsilon, k$: choose $s_0$, pick $\epsilon > 0$ to make the arguments of (2.17") agree, and then choose $k > 0$ so that the moduli match. Thus, any robustness result must impose restrictions on the sizes of $k$ and $\epsilon$. 

\[
(2.17')
\]

\[
(2.17\prime)
\]

\[
(2.17\prime\prime)
\]
The asymptotic behavior estimates in Lemmas 2.1–2.3, coupled with Proposition 2.1 for low frequencies, enables us to provide necessary restrictions on k and \( \varepsilon \).

For the case where there is no tip mass we obtain the following theorem.

**Theorem 2.1.** Assume that (1.13) holds and that \( I = 0 \). Then

(i) If \( A'(0^+) = -\infty \) and \( 0 < K < A(0^+)^{1/2} \), then there exists \( \varepsilon_0 = \varepsilon_0(K) > 0 \) such that (2.17) has no solution in \( \{ \text{Re} \, s \geq 0 \} \) whenever \( 0 \leq \varepsilon \leq \varepsilon_0 \) and \( 0 \leq k \leq K \).

(ii) If \( A'(0^+) = -\infty \) and \( k > A(0^+) \), then for each \( \varepsilon > 0 \) (2.17) has infinitely many solutions in \( \{ \text{Re} \, s > 0 \} \).

(iii) Let \( A'(0^+) > -\infty \) and define \( C \) by (2.22). If \( K < A(0^+)^{1/2}(1 - C)/(1 + C) \), then there exists \( \varepsilon_0 = \varepsilon_0(K) > 0 \) so that the conclusion of part (i) holds.

(iv) Let \( A'(0^+) > -\infty \) and define \( C \) by (2.22). If

\[
(2.27) \quad k > A(0^+)^{1/2}(1 - C)/(1 + C),
\]

then there exists a dense open set \( \mathcal{D} \subseteq (0, \infty) \) so that for each \( \varepsilon \in \mathcal{D} \), (2.17) has infinitely many solutions in \( \{ \text{Re} \, s > 0 \} \).

We observe that \( A(0^+) \) is the speed of propagation of shear disturbances in the scaled problem (1.9)-(1.12) \cite{21,27}, while \( C \) defined by (2.22) is the height that a unit jump discontinuity in shear velocity has after it has propagated a distance 2 in a semi-infinite medium of the scaled material \( (C = 0 \text{ when } A'(0^+) = -\infty) \) \cite{21, p. 241,27}. Thus, in the original unscaled problem (1.3)-(1.5) and (1.7), the critical feedback gain \( \kappa \) in (1.7) in the case when \( g'(0^+) > -\infty \) is

\[
\frac{\pi}{2} R^4 \rho \left( \frac{G + g(0^+)}{\rho} \right)^{1/2} 1 - C
\]

where \( (\pi/2) R^4 \rho \) is the moment of inertia about its axis of a segment of the rod of length one, \( ((G + g(0^+))/\rho)^{1/2} \) is the speed of propagation of shear disturbances in the material, and \( C = \exp(\rho^{1/2} L g'(0^+)/G + g(0^+))^{3/2} \) is the height of a unit jump discontinuity in shear velocity after it has propagated a distance 2L in a semi-infinite medium of this material.

We also remark that parts (iii) and (iv) of Theorem 2.1 are exactly analogous to the results due to Datko, Lagnese, and Polis \cite{15} for the damped wave equation

\[
\ddot{u} + 2b \dot{u} + \beta u = E u_{xx} \quad (0 \leq x \leq 1, b, E > 0)
\]

with boundary conditions

\[
u(0, t) = 0, \quad u_x(1, t) = -k u_t(1, t - \varepsilon).
\]

**Proof of Theorem 2.1.** (i) Fix \( K < A(0^+)^{1/2} \). By (2.18) and (2.23) there exist \( M = M(K) < 1 \) and \( \tau_0 = \tau_0(K) > 0 \) such that for \( 0 \leq k \leq K, \varepsilon \geq 0, \)

\[
|e^{-2\beta(s)}| < M \quad \text{and} \quad \left| \frac{k e^{-\varepsilon s} + \alpha(s)}{k e^{-\varepsilon s} - \alpha(s)} \right| > M
\]

when \( s = \sigma + i \tau \) with \( \sigma \geq 0 \) and \( |\tau| \geq \tau_0 \), so (2.17') with \( I = 0 \) has no solution when \( \sigma \geq 0, |\tau| \geq \tau_0 \). Now let \( \varepsilon_0 = \varepsilon_0(K) = \pi/2 \tau_0(K) \) and combine the above with Proposition 2.1 to get that (2.17') \( (I = 0) \) has no solution in \( \text{Re} \, s \geq 0 \) when \( 0 \leq k \leq K, 0 \leq \varepsilon \leq \varepsilon_0 \).

(ii) Fix \( k > A(0^+)^{1/2} \) and \( \varepsilon > 0 \), and define \( \sigma_0 > 0 \) by \( k \exp(-\varepsilon \sigma_0) = A(0^+)^{1/2} \). The numbers \( z_n = \sigma_0 + (2n + 1)\pi i/\varepsilon \) \( (n = 0, 1, 2, \cdots) \) are solutions of

\[
h(s) = \frac{k e^{-\varepsilon s} + A(0^+)^{1/2}}{k e^{-\varepsilon s} - A(0^+)^{1/2}} = 0.
\]
Now a Rouché's theorem argument using (2.18) and (2.23), and the fact that $h(s)$ is periodic with period $2\pi i/\epsilon$, show that (2.17') with $I = 0$ has a sequence of solutions $s_n$ with $s_n - z_n \to 0$ $(n \to \infty)$, and the proof of part (ii) is complete.

(iii) For $0 \leq k \leq K$, $\Re s \geq 0$, 
\[
|kA(0^+)^{-1/2} C \exp\left(-2sA(0^+)^{-1/2}\right) - 1| \leq KA(0^+)^{-1/2} + C < 1
\]
by hypothesis. Combining this with Lemma 2.2 and (2.18), we can find $\tau_0 = \tau_0(K) > 0$ so that 
\[
|k\alpha^{-1}(s) \tanh \beta(s)| < 1 \quad (\sigma \geq 0, |\tau| \geq \tau_0, k \leq K),
\]
so (2.17") (with $I = 0$) has no solutions for $\sigma \geq 0, |\tau| \geq \tau_0$. Then, as in the proof of part (i), let $\epsilon_0(K) = \pi/2\tau_0(K).

(iv) In this case the argument in [15] works virtually unchanged. Fix $\epsilon > 0$. Note that $s$ with $\Re s > 0$ is a solution of (2.17') (with $I = 0$) if and only if it is a zero of 
\[
H(s, \epsilon) = \left[\alpha(s) + k \epsilon^{-\epsilon s}\right] + \exp\left(-2\beta(s)\right) [\alpha(s) - k \epsilon^{-\epsilon s}].
\]
By Lemma 2.2 and (2.18), 
\[
H(s, \epsilon) - A^{1/2}(0^+) G(s, \epsilon) \to 0 \quad \text{as} \ s \to \infty
\]
uniformly in $\Re s \geq 0$, where $G(s, \epsilon)$ is defined by 
\[
G(s, \epsilon) = 1 + C \exp\left(-2sA(0^+)^{-1/2}\right) + kA(0^+)^{-1/2} \epsilon^{-\epsilon s}[1 - C \exp\left(-2sA(0^+)^{-1/2}\right)].
\]
Since $G(s, \epsilon)$ is bounded and uniformly almost periodic in vertical strips, the argument used to prove Lemma 2.3 in [13] shows that if $G(s_0, \epsilon) = 0$ for some $s_0 = \sigma_0 + i\tau_0$ with $\sigma_0 > 0$, then $H(s, \epsilon)$ has an infinite number of zeros $s_n$ in any vertical strip $\{|\Re s - \sigma_0| < \delta\}$, $\delta > 0$. Since (2.27) holds, Lemma 2 of [15] implies that there is a dense open set $\mathcal{D} \subseteq (0, \infty)$ so that when $\epsilon \in \mathcal{D}$, $G(s_0(\epsilon), \epsilon) = 0$ for some $s_0 = s_0(\epsilon)$ with $\sigma_0 > 0$, and the proof of part (iv) is complete. \(\square\)

We remark that for some values of $A(0^+)$ and $A'(0^+)$, and some $k$ satisfying (2.27), there exists $\epsilon > 0$ so that $G(s, \epsilon)$ defined above has no zero in $\{|\Re s > 0\}$. An example of this is sketched in [20]. In particular, when the hypotheses of part (iv) hold, we do not know if it is true that for each $\epsilon > 0$, (2.17) has a solution in $\{|\Re s > 0\}$, or if this is only true for $\epsilon$ in a dense open set $\mathcal{D} \subseteq (0, \infty)$. We suspect that the latter alternative holds.

When a tip mass is present ($I > 0$), the $Is$ terms dominate the right-hand side of (2.17') when $|s|$ is large, and in contrast to Theorem 2.1, we obtain the following theorem.

**Theorem 2.2.** Assume that (1.13) holds and that $I > 0$. Then for each $K > 0$, there exists $\epsilon_0 = \epsilon_0(K) > 0$ such that (2.17) has no solution in $\{|\Re s \geq 0\}$ whenever $0 \leq \epsilon \leq \epsilon_0$ and $0 \leq k \leq K$.

**Proof.** Using (2.19), and Lemma 2.2 when $A'(0^+) > -\infty$ or Lemma 2.3 when $A'(0^+) = -\infty$, we can find $M = M(K) < 1$ and $\tau_0 = \tau_0(K) > 0$ such that the left-hand side of (2.17') has modulus $< M$ and the right-hand side of (2.17') has modulus $> M$ whenever $\sigma \geq 0$ and $|\tau| \geq \tau_0$, $0 \leq k \leq K$, and $0 \leq \epsilon$. Now let $\epsilon_0 = \epsilon_0(K) = \pi/2\tau_0(K)$ and use Proposition 2.1 as in the proof of Theorem 2.1(i) to complete the proof of Theorem 2.2. \(\square\)

In particular, we emphasize that even though the presence of a tip mass makes the feedback mechanism (2.4) with $\epsilon = 0$ ineffective for exponential stabilization of high-frequency vibrations (possibly desirable when $A'(0^+) > -\infty$; see §§ 3 and 4), the presence of a tip mass precludes the extreme sensitivity to time delays exhibited in parts (ii) and (iv) of Theorem 2.1.
3. Effectiveness of boundary feedback: high frequencies. We now examine the influence of the boundary feedback (2.4) with \( \varepsilon = 0 \) on the eigensolutions of (2.17). We assume throughout that assumptions (1.13) and (1.14) hold. Three model examples satisfying these hypotheses are the following:

(I) \( A(t) = E + \gamma e^{-\delta t} (E, \gamma, \delta > 0) \); standard linear solid model.

(II) \( A(t) = E + (\gamma / \Gamma(1 - \alpha)) t^{-\alpha} e^{-\delta t} (0 < \alpha < 1, E, \gamma, \delta > 0) \); \( \Gamma \) = gamma function; a modified “fractional derivative model” with exponential decay as \( t \to \infty \).

(III) \( A(t) = E + (\gamma / \Gamma(1 - \alpha)) \int_{0}^{\infty} \tau^{-\alpha} e^{-\delta \tau} d\tau (0 < \alpha < 1, E, \gamma, \delta > 0) \); an intermediate model with \( A(0^+) < \infty \) and \( A'(0^+) = -\infty \).

(For all three examples, (1.14) holds with any \( \gamma < \delta \)).

Throughout §§ 3 and 4 we assume that \( \varepsilon = 0 \). Recall (Proposition 2.1) that in this case (2.17) has no solution in \( \{ \text{Re} \ s \geq 0 \} \). Since \( \Delta(s) \) is analytic in \( \mathbb{C} \setminus (-\infty, s^*) \), where \( s^* \) is defined by (2.16), \( \Delta(s) \) cannot have a sequence of zeros with a finite limit point in \( \mathbb{C} \setminus (-\infty, s^*) \). It follows from the asymptotic estimates in Theorems 3.1 and 3.2 that for each \( k \geq 0 \), (2.17) has no solution in \( \{ \text{Re} \ s \geq -d \} \) for some \( d = d(k) > 0 \).

In this section we examine the influence of the feedback (2.4) (\( \varepsilon = 0 \)) on the high-frequency eigensolutions of (2.17). Detailed numerical calculations for specific stress relaxation moduli of the forms (I)-(III) that show how the low- to moderate-frequency solutions of (2.17) depend on the feedback gain \( k \) are presented in § 4.

**Theorem 3.1.** Assume that (1.13) holds and that \( A'(0^+) = -\infty \). Let \( k \geq 0 \) and \( \varepsilon = 0 \). Then given \( \sigma_0 > -\infty \), there exists \( \tau_0 = \tau_0(\sigma_0) > 0 \) so that (2.17) has no solutions in \( \sigma \geq \sigma_0, |\tau| \geq \tau_0 \). Moreover, \( \tau_0 \) is independent of \( k \) when \( I = 0 \).

Thus, when \( A'(0^+) = -\infty \), solutions \( s_n \) of (2.17) (\( \varepsilon = 0 \)) always satisfy \( \text{Re} \ s_n \to -\infty \) as \( |\text{Im} \ s_n| \to \infty \), and the corresponding vibrations decay at increasingly high exponential rates. The asymptotic location of these solutions for specific cases of our model examples (II) and (III) is discussed in § 4.

**Proof.** By Lemma 2.3, we can find \( \tau_1 = \tau_1(\sigma_0) > 0 \) so that \( |\exp(-2\beta(s))| \leq \frac{1}{2} \) whenever \( \sigma \geq \sigma_0, |\tau| \geq \tau_1 \). By definition (2.10)

\[
2\Delta(s) = (Is + k + \alpha(s)) e^{\beta(s)} - (Is + k - \alpha(s)) e^{-\beta(s)}.
\]

If \( I = 0 \), we observe that \( \text{Re} \ \alpha(s) > 0 \) for \( s \in \mathbb{C} \setminus (-\infty, 0] \) by (2.8), and obtain

\[
\left| \frac{\exp(\beta(s))}{\Delta(s)} \right| \leq 2 \left( |k + \alpha(s)| - \frac{1}{2} |k - \alpha(s)| \right)\left( |k + \alpha(s)| - \frac{1}{2} |k - \alpha(s)| \right)^{-1} \leq 4|k + \alpha(s)|^{-1} \leq 4|\alpha(s)|^{-1}
\]

when \( \sigma \geq \sigma_0, |\tau| \geq \tau_1 \). In particular, \( \Delta(s) \neq 0 \) in \( \sigma \geq \sigma_0, |\tau| \geq \tau_0 = \tau_1 \).

When \( I > 0 \), \( Is \) is the dominant term of \( Is + k \pm \alpha(s) \) by (2.19). Thus, by possibly increasing \( \tau_0(\sigma_0) \),

\[
\left| \frac{\exp(\beta(s))}{\Delta(s)} \right| \leq 4(\text{I}|s|)^{-1}
\]

when \( \sigma \geq \sigma_0, |\tau| \geq \tau_0 \), and Theorem 3.1 is proved. \( \Box \)

When \( A'(0^+) > -\infty \) we proceed as follows. Define

\[
N = \begin{cases} 
1 & \text{when } I > 0, \\
(k - A(0^+)^{1/2})(k + A(0^+)^{1/2})^{-1} & \text{when } I = 0,
\end{cases}
\]

and define \( G(s) \) by

\[
G(s) = C^{-1} \exp\left(2sA(0^+)^{-1/2}\right) - N,
\]
where $C$ is given by (2.22). Let

$$F(s) = \exp (2\beta(s)) \frac{Is + k - \alpha(s)}{Is + k + \alpha(s)}.$$ 

By (2.18), (2.19), and Lemma 2.2, for each $\sigma_0 > -\infty$

$$F(s) - G(s) \to 0 \quad \text{as } s \to \infty$$

uniformly in $Re \ s \geq \sigma_0$. By considering moduli, we see that as $|Im \ s| \to \infty$, zeros of $F(s)$ must approach the vertical line $Re \ s = \sigma_*$, where

$$\sigma_* = \begin{cases} 
\frac{A'(0^+)}{2A(0^+)} & \text{when } I > 0, \\
\frac{1}{2} \left[ A'(0^+) + A(0^+)^{1/2} \log \left( \frac{k - A(0^+)^{1/2}}{k + A(0^+)^{1/2}} \right) \right] & \text{when } I = 0, \quad k \neq A(0^+)^{1/2}.
\end{cases}$$

In the case where $I = 0$ and $k = A(0^+)^{1/2}$, an argument using Rouché’s theorem shows that $F(s)$ has a sequence of zeros $s_n$ satisfying $s_n - z_n \to 0$ as $|n| \to \infty$, where $z_n$ are the zeros of the limit function $G(s)$ defined in (3.3) and ordered so that $|Im \ z_n|$ increases as $|n|$ increases. Thus we have proved Theorem 3.2.

**Theorem 3.2.** Assume that (1.13) holds and that $A'(0^+) > -\infty$. Let $k \geq 0$ and $\epsilon = 0$.

(i) If $I + |k - A(0^+)^{1/2}| > 0$, define $\sigma_0$ by (3.5). Then given $\sigma_0 > -\infty$ and $d > 0$, there exists $\tau_0 = \tau_0(\sigma_0, d) > 0$ such that all solutions $s = \sigma + \tau r$ of (2.17) in $\sigma \geq \sigma_0, |\tau| \geq \tau_0$ lie in the strip $|\sigma - \sigma_*| \leq d$. Moreover, (2.17) has a sequence of solutions $s_n$ satisfying $s_n - z_n \to 0$ as $|n| \to \infty$, where $z_n$ are the zeros of $G(s)$ defined in (3.3) and ordered so that $|Im \ z_n|$ increases as $|n|$ increases.

(ii) If $I = 0$ and $k = A(0^+)^{1/2}$, then given $\sigma_0 > -\infty$, there exists $\tau_0 = \tau_0(\sigma_0) > 0$ so that (2.17) has no solutions in $\sigma \geq \sigma_0, |\tau| \geq \tau_0$.

In §4 we discuss in more detail the asymptotic location of the high-frequency eigensolutions for the model example (I).

Note that when $I > 0$, the boundary feedback is ineffective at moving the high-frequency eigensolutions of (2.17). From a mechanical point of view, this is to be expected since rapidly oscillating torques have little effect on the concentrated moment of inertia at $x = 1$. Also, as we noted in the Introduction, Theorem 3.2 with $I > 0$ agrees with the recent abstract result of Desch and Miller [16] stating that the essential growth rate of the resolvent operator of certain linear integrodifferential equations in Hilbert space cannot be changed by compact perturbations. The related fact that an infinite-dimensional system of linear ordinary differential equations without damping cannot be exponentially stabilized by compact linear feedback is due to Gibson [17].

4. Numerical investigation of eigenvalues. In §4.1, the numerical methods we use to locate eigenvalues are described. The general picture of the dependence of low modes on the parameters is rather similar for all three models mentioned in §3: the standard linear solid (I), the fractional derivative model (II), and the intermediate model (III). We have chosen the first of these for a detailed discussion (§4.2). Our discussion of the latter two models is less exhaustive (§4.3).

In §4.2.1, the eigenvalues of the standard linear solid model are classified. Their salient features are outlined and illustrated with specific numerical results obtained in a situation where the parameters are $O(1)$. Since we use a spectral method, such a choice of parameters has the best chance of catching all the different types of modes.
If parameters were of very different sizes, then our desired eigenvalues would reflect that, and more spectral modes would be necessary for their resolution.

In §§ 4.2.2 and 4.2.3, we highlight the effectiveness of the feedback parameter \( k \) in damping the system. This is best illustrated by numerical results that concern a nearly elastic situation rather than a situation with \( O(1) \) parameters. In addition, the parameters are chosen so that the asymptotic formula for highly oscillatory modes places these well away from the imaginary axis. We note that in the elastic case, there are modes on the imaginary axis at \( k = 0 \). In our nearly elastic situation, several of the least stable modes at \( k = 0 \) are close to the imaginary axis. A small addition of \( k \) is shown to have a relatively large stabilizing effect on these modes. The effectiveness of small \( k \) decreases as the moment of inertia increases. This is consistent with the expectation that it takes more friction to stop a heavier object.

In § 4.2.3, the optimal value of \( k \) is discussed. This topic is complicated by the fact that, although the least stable mode at \( k = 0 \) is the lowest complex conjugate pair, a higher mode may become the least stable one as \( k \) is increased. For large \( k \), a mode on the negative real axis becomes the least stable one. In order to find the optimal value of \( k \), we compute all relevant eigenvalues for the nearly elastic situation and note the least stable one for each \( k \). This yields that for zero moment of inertia the optimal \( k \) is slightly less than the value at which low modes "loop up" (see, e.g., Fig. 1 in § 4.2.1). For nonzero moment of inertia, the optimal \( k \) is larger due to the fact that high modes lag behind low modes in their response to \( k \). It is interesting that a large portion of the attainable stabilization is achieved while \( k \) is relatively small. The stabilization settles down somewhat for moderate \( k \) and then worsens at large \( k \). Thus, the order of magnitude of stabilization achieved by the optimal \( k \) is also achieved in a wide interval of \( k \).

4.1. Numerical schemes.

4.1.1. Chebyshev-\( \tau \) method. We let \( u(x, t) = e^{\varepsilon t}U(x) \) in the history value version of (1.9)-(1.12) (see (1.1)) with no external torques and \( \varepsilon = 0 \), and obtain

\[
(4.1) \quad s^2 U(x) - \left[ E + s \int_0^\infty a(\tau) e^{-s\tau} d\tau \right] U_{xx}(x) = 0, \quad 0 < x < 1,
\]

\[
(4.2) \quad s^2 I U(1) + sk U(1) + \left[ E + s \int_0^\infty a(\tau) e^{-s\tau} d\tau \right] U_x(1) = 0,
\]

\[
(4.3) \quad U(0) = 0,
\]

\[
(4.4) \quad a(t) = \gamma e^{-\delta t} \quad \text{for the standard linear solid},
\]

\[
(4.5) \quad a(t) = \left( \gamma / \Gamma(1-\alpha) \right) t^{-\alpha} e^{-\delta t}
\]

for the fractional derivative model modified by an exponential factor,

\[
(4.6) \quad a(t) = \frac{\gamma}{\Gamma(1-\alpha)} \int_t^\infty \tau^{-\alpha} e^{-\delta \tau} d\tau \quad \text{for the intermediate model}.
\]

The Chebyshev-\( \tau \) method [18] is useful if (4.1)-(4.3) can be transformed into a matrix eigenvalue problem: determinant \( (A - \lambda(s)B) = 0 \), where \( s \) can be retrieved from \( \lambda(s) \). This applies to the standard linear solid (4.4) and the two models (4.5) and (4.6), where the exponent \( \alpha \) is a rational number. There is a limit to the size of an eigenvalue that can be approximated well with this method. This limit is set by the
storage capacity of the computer, since more Chebyshev modes are required to approximate an eigenfunction belonging to an eigenvalue of a larger magnitude, and by the accuracy available, since the matrix eigenvalue problem tends to become rather ill conditioned when a large number of modes is used. We use Newton’s scheme for the highly oscillatory modes, as well as other modes for which we already have an idea of where to look, e.g., in the various limits of the parameters. On the other hand, it is virtually impossible to obtain a general picture of where all eigenvalues are with Newton’s scheme, whereas the Chebyshev-τ method yields a complete picture of the location of low to moderate modes. The convergence rate of the latter for isolated eigenvalues of $C^\infty$-eigenfunctions is of infinite order [18]. Our computations were done on a VAX 11/785 computer.

In (4.1)-(4.3), we let $U(x) = \sum_{j=0}^{N} u_j T_j(z)$, where $z = 2x-1$ and $T_j(z)$ is the Chebyshev polynomial of degree $j$. For the standard linear solid, the equations transform to $A_0^s + A_1^s s + A_2^s s^2 + A_3^s s^3 = 0$, where

$$A_0^s = \begin{bmatrix} -E\delta U_{xx}(x) \\ E\delta U_x(1) \\ U(0) \end{bmatrix}, \quad A_1^s = \begin{bmatrix} -(\gamma + E) U_{xx}(x) \\ (\gamma + E) U_x(1) + k\delta U(1) \\ 0 \end{bmatrix},$$

$$A_2^s = \begin{bmatrix} \delta U(x) \\ (I\delta + k) U(1) \\ 0 \end{bmatrix}, \quad A_3^s = \begin{bmatrix} U(x) \\ IU(1) \end{bmatrix}.$$ 

When we use the orthogonality of $T_j(z)$, this equation is transformed into $(A_0 + A_1 s + A_2 s^2 + A_3 s^3) y = 0$, where $y$ is the vector of coefficients $u_j$ and $A_i$ is an $N+1$ square matrix with the first $N-1$ rows devoted to the differential equation. Finally, the eigenvalue problem reduces to determinant $(A - sB) = 0$, where

$$A = \begin{bmatrix} 0 & 0 & A_0 \\ 1 & 0 & -A_1 \\ 0 & 1 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -A_3 \end{bmatrix},$$

where $A$ and $B$ are $3(N+1)$ square matrices.

For the fractional derivative model modified by an exponential factor (4.5), we investigate in detail the case $\alpha = \frac{1}{2}$, for which (4.1)-(4.3) transform to $A_0^\alpha + A_1^\alpha \lambda + A_2^\alpha \lambda^2 + A_3^\alpha \lambda^3 + A_4^\alpha \lambda^4 = 0$, where

$$\lambda = \sqrt{s + \delta}, \quad A_0^\alpha = \begin{bmatrix} \gamma\delta U_{xx}(x) \\ -\gamma\delta U_x(1) \end{bmatrix},$$

$$A_1^\alpha = \begin{bmatrix} \delta^2 U(x) - EU_{xx}(x) \\ (I\delta - k)\delta U(1) + EU_x(1) \end{bmatrix}, \quad A_2^\alpha = \begin{bmatrix} -\gamma U_{xx}(x) \\ \gamma U_x(1) \end{bmatrix},$$

$$A_3^\alpha = \begin{bmatrix} -2\delta U(x) \\ (k - 2I\delta) U(1) \end{bmatrix}, \quad A_4^\alpha = \begin{bmatrix} U(x) \\ IU(1) \end{bmatrix}.$$
In a manner analogous to the case of the standard linear solid, we obtain \( \det (A - \lambda(s)B) = 0 \), where

\[
A = \begin{bmatrix}
0 & 0 & 0 & A_0 \\
1 & 0 & 0 & -A_1 \\
0 & 1 & 0 & A_2 \\
0 & 0 & 1 & -A_3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -A_5
\end{bmatrix},
\]

where \( A \) and \( B \) are \( 5(N+1) \) square matrices. After calculating \( \lambda(s) \), we discard those whose real parts are negative since they are not on the branch of interest. We then retrieve the eigenvalues \( s = \lambda^2 - \delta \). This type of procedure can in principle be used for both models (4.5) and (4.6), provided the exponent \( \alpha \) is a rational number.

4.1.2. Newton scheme. A Newton scheme, based on the characteristic equation

\[
e^{2\beta(s)}(Is + \alpha(s) + k) - Is + \alpha(s) - k = 0, \quad \beta(s) = s/\alpha(s),
\]

\[
\alpha(s) = \begin{cases}
\left( E + \frac{\gamma s}{s + \delta} \right)^{1/2} & \text{for the standard linear solid,} \\
\left( E + \frac{\gamma s}{(s + \delta)^{1-\alpha}} \right)^{1/2} & \text{for the fractional derivative model,} \\
\left( E - \frac{\gamma}{(s + \delta)^{1-\alpha}} + \frac{\gamma}{\delta^{1-\alpha}} \right)^{1/2} & \text{for the intermediate model}
\end{cases}
\]

is a cost-effective means of tracking specific eigenvalues when an initial guess is available. Initial guesses are obtained by computing eigenvalues with the Chebyshev-\( \tau \) method for a particular parameter set, or with various limiting cases such as the asymptotics for large imaginary part, the case \( I = k = 0 \), or the case \( |Is + k| \) large compared with \( |\alpha(s)| \). The latter two cases are governed by

\[
s^2 + c_n\alpha^2(s) = 0, \quad n = 1, 2, \ldots,
\]

where \( c_n = (2n-1)^2\pi^2/4 \) and \( n^2\pi^2 \), respectively. These are cubic equations for the standard linear solid:

\[
s^3 + s^2\delta + c_n(E + \gamma)s + c_nE\delta = 0
\]

and quintic equations in \( \lambda = \sqrt{s + \delta} \) for the other two models with exponent \( \alpha = \frac{1}{2} \).

4.2. Standard linear solid model.

4.2.1. Location of eigenvalues. The general picture of the location of eigenvalues must first be obtained in order to keep a tab on the candidates for the least stable mode. At first sight, the picture for the case of zero moment of inertia appears to be quite different from the case of nonzero moment of inertia. However, the picture changes continuously from one case to the other, and it is of interest to present that transformation. We illustrate the main features with numerical results for the situation \( E = \gamma = \delta = 1 \).

For zero moment of inertia, we separate the eigenvalues into four classes below. Candidates for the least stable mode belong to Class 1 for low to moderate \( k \) and Class 4 for moderate to large \( k \). Classes 2 and 3 are subject to bounds that prevent them from approaching the origin.

Class 1. These are the complex conjugate modes. Those with positive imaginary parts are included in Fig. 1 \((I = 0, E = \gamma = \delta = 1)\). At \( k = 0 \) and at \( k = \infty \), these satisfy (4.9). The third root of this equation is discussed under Class 2. The index \( n \) in (4.9) is used to index these modes. The modes at \( k = 0 \) and \( \infty \) are interlaced in the complex
plane as shown in Fig. 1: the circles denote $k=0$ and the squares $k=\infty$. We do not use this notation for modes on the real axis because some lie close together and therefore the figure would lose clarity. The arrows show the trends as $k$ increases. Modes travel away from the imaginary axis as $k$ increases, loop up (around $k=\sqrt{\gamma}+\delta$ in the nearly elastic case), and then travel the other way as $k\to\infty$. Highly oscillatory modes have the asymptotic behavior

\begin{equation}
 s \sim -\frac{\gamma\delta}{2(\gamma+\delta)} + \frac{\sqrt{\gamma+\delta}}{2} \log \left| \frac{k-\sqrt{\gamma+\delta}}{k+\sqrt{\gamma+\delta}} \right| \pm n\pi i \sqrt{\gamma+\delta}, \quad n \to \infty
\end{equation}

for $k \neq \sqrt{\gamma+\delta}$, and if $k = \sqrt{\gamma+\delta}$

\begin{equation}
 s \sim -\frac{\gamma\delta}{2(\gamma+\delta)} + \frac{\sqrt{\gamma+\delta}}{2} \log \left| \frac{\gamma\delta}{4\pi n(\gamma+\delta)^{3/2}} \right| \pm n\pi i \sqrt{\gamma+\delta}.
\end{equation}

Here, $n$ is the index used in (4.9).

The low eigenvalues at $k=0$ and $\infty$ in Fig. 1 are lined up almost vertically: this need not be the case if the parameters are chosen differently. In fact, when parameters are close to the elastic case, the lowest mode is very close to the imaginary axis and higher modes swing gradually toward the asymptote. The asymptotic formulas apply when the imaginary part of $s$ is much larger than $\delta$. In Fig. 1, $\delta$ is not large. Thus, the approach to the asymptote is fast. Even the low modes behave according to (4.10). On the other hand, a nearly elastic situation such as the one in § 4.2.2 has a large $\delta$, so the asymptote is approached extremely slowly with $n$.

At $k=0$, the first mode in Fig. 1 is the least stable by a small margin. In the nearly elastic case, this margin is large. When $k$ is increased, a higher mode can become the least stable one. That is, a picture of these modes at a fixed nonzero $k$ can show a gradual swing of real parts toward zero as the index $n$ increases, followed by a gradual swing of higher modes in the opposite direction toward the asymptote. We did not
encounter any situation numerically where there was more than one such swing as \( n \) increased. Reasons for having one swing are discussed in §4.2.2.

Class 2. The characteristic equation (4.7) has an essential singularity at the value \( s^* \) defined in (2.16), where \( a(s^*) = 0 \). In this example, \( s^* = -E\delta/(E + \gamma) \). We comment that only those modes with \( \text{Re } s > s^* \) affect the exponential decay rate of the solution \( u(x, t) \) of (2.1)-(2.4) as described in §5 below, since \( u \) is recovered from its Laplace transform by an inversion integral along a vertical line \( \text{Re } s = \mu \), where \( \mu > s^* \). The location of roots to the characteristic equation in the half-plane \( \{ \text{Re } s \leq s^* \} \) is discussed here only for the sake of completeness.

In the interval \((-\delta, s^*)\), there are eigenvalues clustering toward \( s^* \). At \( k = 0 \) and \( \infty \), these are the countable number of real roots of the cubic equation (4.9). These modes move to the right as \( k \) increases. Newton’s scheme does not track these modes well because (obviously) it can converge to any member of the cluster. The Chebyshev-\( \tau \) method resolves the modes furthest away from \( s^* \) well, and it takes more and more Chebyshev modes to converge to members closer to \( s^* \). In Fig. 1, \( s^* = -0.5 \) and the cluster stays very close to this.

Class 3. On the negative real axis, an eigenvalue travels in from \(-\infty \) for \( k > \sqrt{E + \gamma} \) and approaches \(-\delta \) as \( k \to \infty \). This mode is absent when \( k \leq \sqrt{E + \gamma} \). This mode enters the picture in Fig. 1 when \( k \) is approximately 1.6 and is already close to \(-\delta \) when \( k = 100 \).

Class 4. An eigenvalue pops out on the right of the cluster point \( s^* \) when \( k > 0 \) and travels toward the origin as \( k \to \infty \). This mode is absent at \( k = 0 \). This mode is inversely proportional to \( k \) as \( k \to \infty \). Thus, there exists a value of \( k \) beyond which the use of \( k \) does more harm than good. For example, when \( k \) is larger than about 3 in Fig. 1, the Class 4 mode becomes less stable than the system is at \( k = 0 \).

As noted earlier, only those modes belonging to Classes 1 or 4 affect the exponential decay rate of the solution to (2.1)-(2.4) as described in §5.

For nonzero moment of inertia, the only modes present at \( k = 0 \) are those analogous to Classes 1 (complex conjugates) and 2 (cluster). A mode corresponding to Class 4 pops out of the essential singularity \( s^* \) for \( k > -Is^* \) and travels toward the origin. The dependence of these modes on \( k \) is quite different from the case of zero moment of inertia. As \( k \to \infty \), there are modes corresponding to all four classes. Candidates for the least stable mode are again the complex conjugates for low \( k \) and the mode that adopts the Class 4 behavior for large \( k \). The moment of inertia destabilizes the complex conjugates and their asymptote loses its dependence on \( k \) at the leading order:

\[
s \sim \frac{-\gamma\delta}{2(E + \gamma)} + \pi \quad \text{in } \sqrt{E + \gamma} \quad \text{as } n \to \infty,
\]

where the moment of inertia appears at \( O(n^{-1}) \) and the feedback parameter \( k \) at \( O(n^{-2}) \).

We present the overall picture of eigenvalues at large moment of inertia \( I \) and show how this changes as \( I \to 0 \). We again illustrate the main features with numerical results for the case \( E = \gamma = \delta = 1 \). For sufficiently large \( I \), the complex conjugates loop down rather than up as \( k \) increases from zero. The first mode loops down to the negative real axis as shown in Fig. 2 (\( I = 10, E = \gamma = \delta = 1 \)). This pair (the “dropper”) then becomes two real eigenvalues. One travels toward the origin as \( k \to \infty \), just like a Class 4 eigenvalue. The other travels toward the essential singularity \( s^* \). Meanwhile, when \( k > -Is^* \), one eigenvalue (the “popper”) corresponding to Class 4 pops out of \( s^* \) and travels right. The upper diagram in Fig. 2 displays the details around \( s^* \). The popper emerges for \( k > 5 \) and meets one of the droppers at \( k = 5.11 \). The two modes then become complex conjugates to hop over \( s^* \) and all members of the cluster (Class 2) except the one closest to \(-\delta \). Note that there is a relatively large gap between
this member and the rest of the cluster at $k=0$. They land on the axis at $k=5.27$ and become two reals. One travels right as the last member of the cluster. As $k \to \infty$, this cluster behaves just like Class 2. The other eigenvalue heads in the direction of $-\delta$ and at $k=5.5$ and meets the remaining member of the original cluster, and both become complex conjugates to jump over $-\delta$. It takes a considerable amount of $k$ to complete this jump. They land on the axis at $k$ between 17 and 18, and then become two reals. One travels toward $-\delta$ as $k \to \infty$ as a Class 3 mode. The other goes out to $-\infty$ as $k \to \infty$: this mode goes out of the picture altogether as $I \to 0$, and the manner in which it does so will be explained.

The picture of the higher modes at large moment of inertia is exemplified in Fig. 3 ($I=0.4, E=\gamma=\delta=1$). As $k \to \infty$, the $n$th mode, $n=2, 3, 4, \cdots$, approaches the limit of the $(n-1)$th mode of the case $I=0$. As is evident from the characteristic equation (4.7), higher modes vary less with $k$ and they lag behind low modes in their response to $k$. This is reflected in Fig. 3. At any one mode, the variation with $k$ is less the larger the moment of inertia.

When $I$ is decreased from the value in Fig. 2, the trajectory of the lowest complex conjugate pair over $s^*$ gets closer to the two adjacent trajectories over the axis. Eventually, these merge. For example, at $I=8, E=\gamma=\delta=1$, the dropper lands on the axis to the left of $-\delta$ without landing on the right. There are two valleys on its trajectory before it lands: one to the right of $s^*$ and a shallower valley to the left. As $I$ decreases further, the valleys become less pronounced. Thus, in Fig. 3 ($I=0.4, E=\gamma=\delta=1$) no valleys are visible on the trajectory of the dropper. The dropper lands on the axis to the left of $-\delta$ at about $k=2.8$, and becomes two real eigenvalues. One behaves like Class 3 as $k \to \infty$. The other travels to $-\infty$. The popper emerges from $s^*$ for $k>-Is^* = 0.2$. As $k \to \infty$, this eigenvalue behaves like the Class 4 mode. The looping down of the higher modes remains qualitatively the same as at larger $I$. 

**FIG. 2.** $I=10$, $E=\gamma=\delta=1$, $k=0$ to 20, $s^*=-0.5$. The journey of the lowest complex conjugate mode and modes on the real axis.
When the moment of inertia is decreased further, the dropper’s trajectory rises slightly and kisses the trajectory of the mode above it. At that point, the multiplicity of the eigenvalue is two. This is shown in the schematic drawings of Fig. 4, where $I$ decreases from $I^*+$ to $I^*−$. A specific example is the transition from $I = I^*+ = 0.4$ in Fig. 3 to $I = I^*− = 0.01$ in Fig. 5. With successive exchanges as in Fig. 4, the dropping branch propagates upward and the sense of the loops changes from downward to upward. Figure 6 shows the fifth mode dropping at $I = 0.001$. The value of $k$ at which the dropping branch reaches the real axis appears to decrease to $\sqrt{\frac{E}{\gamma}} + 3\delta$ and the junction on the real axis moves out to $-\infty$ as $I \to 0$.

4.2.2. Effectiveness of small $k$. In this section, we focus on a situation where, at $k = 0$, the least stable mode is extremely lightly damped compared with the asymptotic...
behavior of high modes. This occurs, e.g., when the parameters are close to the elastic case. We present numerical evidence to show that a small $k$ can be very effective in damping the system when the moment of inertia is zero, and this effectiveness decreases as the moment of inertia increases. Moreover, since the least stable mode at $k=0$ is less stable the larger the $I$, the end product after the addition of a small $k$ is that the situation at a higher $I$ is still less stable than at a lower $I$. A better decay rate for the larger $I$ may be achieved by using a larger $k$, as we show in §4.2.3. This is in fact consistent with the expectation that it takes more frictional force to stop a heavier object.

As an example of a nearly elastic situation, we take $E=1$, $\gamma=0.01$, $\delta=1000$. Several of the Class 1 modes for $I=k=0$ are listed in Table 1. These are the complex conjugate roots of (4.9). Here, many low modes have real parts that are much smaller than the asymptotic limit $-4.95$ of highly oscillatory modes. Due to the size of $\delta$, the asymptote is achieved extremely slowly: the thousandth mode is still ten percent away. The approach to the asymptote at $I=k=0$ is monotonic with $n$. For our parameters, the low complex conjugates are the least stable modes for small $k$ and no other mode enters into our discussion.

For zero moment of inertia, Fig. 7 shows the dramatic shift in the first several modes as a response to a small value of the feedback gain parameter $k$. At $k=0.01$ and 0.1, their real parts become $O(k)$. For example, at $k=0.01$, low modes are lined up at approximately $\text{Re } s = -0.01$ and higher modes curve back toward the asymptote. When $k=0.1$, the low modes have shifted to $\text{Re } s = -0.1$. The modes still approach the asymptote $\text{Re } s = -4.96$ in a monotonic way. When $k=0.1$, there is an $O(10^4)$ stabilization for the worst mode of the system.
$E = \gamma = \delta = 1, \ I = 0.001, \ s^* = -0.5, \ k = 0 \ to \ \infty$. The first four branches loop up, the fifth branch drops down and higher branches loop down with $k$. Complex conjugates at $k = 0$ are denoted by circles and at $k = \infty$ by squares.

**Table 1**

**Standard linear solid model.**

$k = 0, \ E = 1, \ \gamma = .01, \ \delta = 1000, \ I = 0$.

<table>
<thead>
<tr>
<th>Index $n$</th>
<th>Complex conjugates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.125E - 4 \pm 1.57i$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.11E - 3 \pm 4.71i$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.309E - 3 \pm 7.85i$</td>
</tr>
<tr>
<td>10</td>
<td>$-0.445E - 2 \pm 29.9i$</td>
</tr>
<tr>
<td>50</td>
<td>$-0.118 \pm 156i$</td>
</tr>
<tr>
<td>100</td>
<td>$-0.445 \pm 313i$</td>
</tr>
<tr>
<td>200</td>
<td>$-1.41 \pm 628i$</td>
</tr>
<tr>
<td>300</td>
<td>$-2.35 \pm 943i$</td>
</tr>
<tr>
<td>600</td>
<td>$-3.88 \pm 1891i$</td>
</tr>
<tr>
<td>900</td>
<td>$-4.41 \pm 2838i$</td>
</tr>
</tbody>
</table>

The inclusion of moment of inertia is displayed in Figs. 8–10 for $I = 0.1, 1.0,$ and $10.0$, respectively. It is evident from these figures that, for our values of $k$, the addition of the moment of inertia shifts these eigenvalues down and to the right toward the origin, and decreases the effect of $k$ on the modes.

Figure 8 ($I = 0.1$) is drawn to the same scale as Fig. 7 ($I = 0$) and illustrates the addition of a small amount of $I$. In Fig. 8, the situation at $k = 0$ is, of course, similar
to that in Fig. 7. The worst mode at \( k = 0 \) is again the lowest with real part \( O(10^{-5}) \). However, the picture for \( k \neq 0 \) is obviously different from that for \( I = 0 \). Higher modes change little as \( k \) varies from 0 to 0.1. On the other hand, low modes are affected as dramatically as the case \( I = 0 \). We remark that at \( k = 0.1 \), intermediate modes swing back toward the imaginary axis and higher ones swing away toward the asymptote. Thus, the location of the worst mode moves around as \( k \) varies. After an extensive search, we find that the eleventh mode is the least stable at \( k = 0.1 \) with \( s = -0.014 + 34.8i \), and a few neighboring modes have comparable real parts. Therefore, at \( I = 0.1 \), the feedback parameter \( k \) has stabilized the worst decay rate by \( O(10^3) \).

At first glance, we may become concerned that there may be other swing-backs of higher modes toward the imaginary axis. This does not occur for our parameters. This is because \( |\alpha(s)| \) is almost a constant if \( |s| \) is sufficiently large, and for our parameters, this holds even at moderate \( |s| \). In addition, when \( |Is| \) is much larger than the constant \( |\alpha(s)| \), the characteristic equation becomes \( e^{2\beta(s)} = 1 \), regardless of \( k \). The real parts of the roots of this cubic equation ((4.9) with \( c_n = n^2 \pi^2 \)) approach the asymptote as a monotonic function of \( n \). Thus, the swing-back in Fig. 8 occurs at modes where \( |s| \) is only moderately large; then the eigenvalues swing back to approach the solutions of the cubic equation.

In Fig. 9, the low modes are less stable than those of Figs. 7 (\( I = 0 \)) and 8 (\( I = 0.1 \)). At \( I = 1.0 \), the worst mode at \( k = 0 \) is the first at \( s = -0.37E - 5 \pm 0.86i \). At \( k = 0.01 \), it is the third at \( s = -0.44E - 3 \pm 6.4i \). At \( k = 0.1 \), it is the fifth at \( s = -0.14E - 2 \pm 12.6i \). Thus, there is an \( O(10^5) \) improvement. In the scale of Fig. 10, only the first mode appears to vary but there is also much stabilization of the second mode. Here, the worst mode at \( k = 0 \) is the first at \( s = -0.5E - 6 \pm 0.31i \). At \( k = 0.01 \), it is the second at
I = 0.1, E = 1, \gamma = 0.01, \delta = 1000

**Fig. 8.** $I = 0.1, E = 1, \gamma = 0.01, \delta = 1000$. The figure shows the stabilization of the complex conjugate modes as $k$ is increased from 0 to 0.1. Other modes are out of this picture for this interval of $k$.

$s = -.6E - 4 \pm 3.2i$. At $k = 0.1$, it is again the second at $s = -.15E - 3 \pm 3.2i$. The overall stabilization is slightly worse than at $I = 1.0$ but is still $O(10^2)$.

### 4.2.3. The optimal choice for $k$

We search for an optimal choice for $k$ in the nearly elastic situation discussed in the previous section. A number of questions arise, such as: by what order of magnitude does an optimal $k$ stabilize the system? Is this magnitude sensitive to small changes in $k$? Is it possible to choose $k$ so that all modes are stabilized to the same level as the highly oscillatory modes? We investigate these questions with reference to the parameters $E = 1$, $\gamma = .01$, $\delta = 1000$. We show that if the moment of inertia is zero, it is possible to choose $k$ so that the system is almost as stabilized as the high modes are at $k = 0$. For nonzero moment of inertia, the maximum amount of stabilization is not as great and the optimal $k$ may be larger. For both $I = 0$ and $I \neq 0$, the magnitude of improvement achieved with the optimal $k$ is not very sensitive to changes in $k$. The complex conjugate modes of Class 1 and the popper of Class 4 enter into our discussion.

At $I = 0$, $E = 1$, $\gamma = 0.01$, $\delta = 1000$, the dependence of low modes on $k$ is qualitatively similar to Fig. 1, but with the lowest Class 1 modes being markedly closer to the imaginary axis at $k = 0$ and $\infty$. For $k > 0$, the Class 4 mode pops out of the essential singularity $s^* = -990.099 \cdots$, which is quite a distance away from the Class 1 modes. The low Class 1 modes loop up at about $\text{Re } s = -6$. The popper travels toward the origin as $k$ increases, first slowly and then speeding up when $k$ is about 1. When $k$ is slightly larger than 1, the popper is close to the real parts of the low Class 1 modes; then, for large $k$, it is inversely proportional to $k$. 
FIG. 9. \( I = 1.0, E = 1, \gamma = 0.01, \delta = 1000 \). The stabilization of the complex conjugate modes is displayed for \( k \) varying between 0 and 0.1. Other modes are out of this picture for this interval of \( k \).

Since the asymptote for the high modes is maximally stabilized at \( k = \sqrt{E + \gamma} \), we may guess that this value of \( k \) is optimal. However, when \( k \) has reached \( \sqrt{E + \gamma} = 1.005 \), the low modes have rounded their loops and are on their way back to the imaginary axis. (In situations that are not nearly elastic, it is possible that low modes round their loops at larger \( k \) and that \( \sqrt{E + \gamma} \) is not a guideline.) When \( k \) is slightly less than \( \sqrt{E + \gamma} \), the low modes appear to be as much damped as possible, and the high modes even more. There is a slight swing of intermediate modes toward the imaginary axis. The maximum amount of damping possible for each mode is different and accounts for the swing-back. Table 2 lists eigenvalues for \( k = 1 \), which is close to optimal. There is a swing-back around the 50th to 100th modes. The worst mode is around the 70th with real part around -4, which is comparable to the asymptote for high modes at \( k = 0 \). Referring to Table 1 for \( k = 0 \), the optimal \( k \) improves the worst mode by \( O(10^5) \). This order of magnitude in improvement can be achieved as long as the worst modes are pushed back to \( \text{Re} \, s = -1.0 \). In this sense, a wide interval of \( k \), e.g., 0.7 to 1.3, achieves the same magnitude of improvement over the situation with zero \( k \).

As an example of the situation with moment of inertia, we consider \( I = 0.1, E = 1, \gamma = .01, \delta = 1000 \). At \( k = 0 \), the worst mode is the first complex conjugate mode with \( s = -1E^{-4} \pm 1.4i \). The complex conjugates loop down as \( k \to \infty \), as in Figs. 2 and 3. At \( k = 1.23 \), the lowest complex conjugate mode drops to the real axis at \( \text{Re} \, s \approx -2 \), which is quite far from the cluster point \( s^* \). This behavior is reminiscent of Fig. 2. One eigenvalue travels toward \( s^* \) and will meet the popper. The other travels toward the origin and becomes the least stable mode for \( k \approx 13 \). For reasons that have been discussed in § 4.2.2, the lowest modes are affected first by the variation in \( k \) and higher
The stabilization of the complex conjugate modes is displayed for $k$ varying from 0 to 0.1. Other modes are out of this picture for this interval of $k$.

Table 2

<table>
<thead>
<tr>
<th>$n$</th>
<th>Complex conjugates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-5.53 \pm 2.90i$</td>
</tr>
<tr>
<td>2</td>
<td>$-5.41 \pm 5.87i$</td>
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<tr>
<td>3</td>
<td>$-5.28 \pm 8.90i$</td>
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<td>10</td>
<td>$-4.73 \pm 30.69i$</td>
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<td>50</td>
<td>$-4.05 \pm 156i$</td>
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<tr>
<td>100</td>
<td>$-4.05 \pm 313i$</td>
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<tr>
<td>200</td>
<td>$-4.74 \pm 628i$</td>
</tr>
<tr>
<td>300</td>
<td>$-5.55 \pm 944i$</td>
</tr>
<tr>
<td>600</td>
<td>$-6.96 \pm 1891i$</td>
</tr>
<tr>
<td>900</td>
<td>$-7.46 \pm 2839i$</td>
</tr>
</tbody>
</table>

modes require larger $k$ in order to move. Due to this lag, one of the higher modes is the least stable for most of $k \in [0, 13]$.

At $k = 1.2$, low complex conjugates start looping back toward the imaginary axis, but unlike the case in which $I = 0$, this does not indicate the optimal $k$. A search shows that $k$ of about 8 is optimal. At $k = 8$, the worst mode is the 27th complex conjugate pair with $s = -.94E - 1 \pm 78.6i$. The improvement over the case of zero $k$ is then $O(10^6)$. However, an improvement of $O(10^5)$ is gained for any $k$ between 0.1 and 100, i.e., once the worst mode has been pushed to $\text{Re } s = O(0.01)$, the magnitude of improvement
is rather insensitive to $k$. For larger $I$, we find that the modes move around less, so that we expect the worst modes at larger $I$ to be worse than at lower $I$. Thus, we expect the overall improvement achieved by an optimal $k$ to diminish as $I$ increases.

4.3. The fractional derivative model and the intermediate model. We begin with the fractional derivative model modified by an exponential factor (4.5), and end this section with a brief discussion of the intermediate model (4.6). Our results are based on computations for $\alpha = \frac{1}{2}$ in (4.5) and (4.6). For both models, the types of eigenvalues that arise and their journeys through the complex plane as parameters are varied are found to be reminiscent of the standard linear solid model.

For the fractional derivative model, our results concern the case $\alpha = \frac{1}{2}$, $E = 1$, $\gamma = 0.01$, $\delta = 5$. These parameters were chosen for the same reasons as those stated at the beginning of § 4 to justify the choice used in §§ 4.2.2 and 4.2.3: at $k = 0$, the system is close to criticality. For zero moment of inertia, the general picture of the eigenvalues as $k$ varies from 0 to $\infty$ is analogous to Fig. 1, but the Class 3 mode is absent. At $k = 0$, there are modes corresponding to Classes 1, 2, and 4. The difference from the standard linear solid, apart from the asymptotics, is that there is now a branch cut along $(-\infty, -\delta)$ and no eigenvalues stay on that portion of the axis. The Class 2 cluster lies in $(-\delta, s^*)$, where $s^*$ is the essential singularity $(1 - \sqrt{1 + 4\delta^2/E^2})E^2/2\gamma^2$. For $k > -Is^*$, a Class 4 mode pops out of $s^*$.

At small $k$, low complex conjugate modes are the least damped. The first several modes are listed in Table 3. The asymptotic formula for highly oscillatory modes is

$$s_n \sim \frac{\pi^2 (2n - 1)^2 \gamma}{4} \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2} \right)$$

as $n \to \infty$.

Observe that the highly oscillatory modes exhibit frequency proportional or "structural" damping. This behavior is approached slowly if parameters are nearly elastic, and fast if parameters are all moderate. Thus, for our parameters, even $s_{10,000} = -18300.00 + 51500i$ is not yet close to this formula. As $k$ increases, a higher mode may be the least stable mode. For $k > 1$, the Class 4 popper becomes the least stable mode. Since the situation of interest is nearly elastic, and the modes are qualitatively similar to those of the standard linear solid, the search for an optimal $k$ focuses on the low to moderate complex conjugates and the popper. We find that $k$ close to one (i.e., slightly less than $\sqrt{E + \gamma}$) is again optimal and the worst mode recedes to real part $-2.6$, an improvement of $O(10^3)$. The effect of $k$ on the worst modes is shown in Table 3.

For nonzero moment of inertia, the overall journey of the modes is similar to the standard linear solid except that the branch cut $(-\infty, -\delta)$ is free of eigenvalues. For

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k = 0$</th>
<th>$k = 0.01$</th>
<th>$k = 0.1$</th>
<th>$k = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-.53E -2$</td>
<td>$-.15E -1$</td>
<td>$-.11$</td>
<td>$-2.67$</td>
</tr>
<tr>
<td>2</td>
<td>$-.4E -1$</td>
<td>$-.5E -1$</td>
<td>$-.14$</td>
<td>$-2.62$</td>
</tr>
<tr>
<td>3</td>
<td>$-.9E -1$</td>
<td>$-1.0$</td>
<td>$-.19$</td>
<td>$-2.58$</td>
</tr>
<tr>
<td>4</td>
<td>$-.15$</td>
<td>$-.16$</td>
<td>$-.25$</td>
<td>$-2.58$</td>
</tr>
<tr>
<td>5</td>
<td>$-.21$</td>
<td>$-.22$</td>
<td>$-.31$</td>
<td>$-2.59$</td>
</tr>
<tr>
<td>6</td>
<td>$-.28$</td>
<td>$-.39$</td>
<td>$-.38$</td>
<td>$-2.62$</td>
</tr>
<tr>
<td>7</td>
<td>$-.36$</td>
<td>$-.37$</td>
<td>$-.46$</td>
<td>$-2.66$</td>
</tr>
</tbody>
</table>
example, computations at $I = 0.1$ reveal the overall picture to be analogous to Fig. 3 ($I = 0.4, E = \gamma = \delta = 1$ for the standard linear solid) for modes higher than the first. The first branch drops to the negative real axis to the right of $s^*$ as in Fig. 2. The main difference is that after the eigenvalues jump over $s^*$ and $-\delta$, they travel out, moving away from the real axis. For large negative real part, these complex conjugates have $\text{Re } s \sim -k/I$ as $k \to \infty$ with nonzero imaginary parts. Figure 11 displays the effectiveness of $k$, both small and large, on the least stable modes. Here, $I = 0.1$ and $k = 0, 0.1$ and 1.0. This figure is analogous to Fig. 8 ($I = 0.1, E = 1, \gamma = 0.01, \delta = 1000, k = 0, 0.01, 0.1$) for the standard linear solid. The modes on the real axis are out of this picture for our values of $k$. The worst mode at $k = 0$ is $-0.004 \pm 1.4i$ and this is stabilized 20-fold with $k = 0.1$. After the first several modes, however, the effectiveness of small $k$ wears off and higher modes are close to the roots of $e^{2\beta(s)} = 1$, regardless of $k$. Just as in Figs. 8–10, increasing the moment of inertia results in a decrease in the effect of $k$, but the order of magnitude of improvement does not drop drastically. At $I = 10$, there is still a 20-fold improvement in the worst decay rate. In Fig. 11, the first two modes at $k = 1$ are out of the picture while intermediate modes display the same type of swing-back as in Fig. 8 for $k = 0.1$. After the modes have swung back, they approach the roots of $e^{2\beta(s)} = 1$.

We search for an optimal $k$ for the case $I = 0.1$. At $k = 0$, the least stable mode is the lowest, but just as with the standard linear solid, when $k$ is nonzero, the lowest is not always the least stable. Just as in Fig. 2, the lowest conjugate pair drops to the axis to the right of $s^*$. The junction occurs at $k = 1.23$. One mode travels toward $s^*$

\[ \text{Fig. 11. Fractional derivative model with } \alpha = \frac{1}{2}, I = 0.1, E = 1, \gamma = 0.01, \delta = 5. \text{ The stabilization of the complex conjugate modes is displayed for } k \text{ varying from } 0 \text{ to } 1. \text{ Other modes are out of this picture for this interval of } k. \]
and meets the popper. The other travels toward the origin. Meanwhile, one of the higher modes is the worst mode. The optimal $k$ is close to $k = 1.8$, where the worst mode is $-0.55 + 13i$. For $k$ larger than about 2, one of the complex conjugates that dropped to the real axis becomes the least stable mode, approaching the origin as $k \to \infty$. The optimal $k$ yields an $O(10^2)$ improvement in stabilization. This is an order of magnitude less than the improvement at $k = 0$. These magnitudes of improvement are attainable in a wide interval of $k$ around the optimal value.

For the intermediate model, the main features are similar to the fractional derivative model already presented. For example, at zero moment of inertia, there are modes corresponding to Classes 1, 2, and 4 of the standard linear solid, and Class 3 is absent because there is a branch cut at $(-\infty, -\delta)$. The asymptotic behavior of high modes is different from the other models:

$$\text{Im} s_n \sim \pi n \sqrt{E + \gamma/\delta} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \text{Re} s_n \sim -\gamma \pi^2 n^2 \sin \frac{\pi}{4}/(2 \text{Im} s_n \sqrt{\text{Im} s_n}).$$

Thus,

$$\text{Re} \frac{s_n}{\sqrt{\text{Im} s_n}} \to -\gamma \sin \frac{\pi}{4} \left( \frac{E + \gamma/\delta}{\sqrt{\delta}} \right).$$

The Class 4 mode pops out of the essential singularity $s^* = -\delta + \gamma^2 \delta/(E \sqrt{\delta} + \gamma)^2$ for $k > -k^*$.  

5. Existence and exponential decay. In this section we verify that our boundary stabilization problem has a solution (in a weak sense), whose rate of decay as $t \to \infty$ corresponds in the expected way to the location of the solutions of the characteristic equation (2.17) studied in §§ 3 and 4. We consider the problem (2.1)–(2.4), and we assume $k$, $\epsilon$, $I \equiv 0$, and

$$u_0 \in AC[0, 1] \quad \text{with} \quad u_0(0) = 0.$$

We assume that the stress relaxation modulus $A(t)$ satisfies (1.13) and (1.14).

In this section, $H^0$ denotes the Hilbert space $L^2(0, 1)$ with norm $\| \cdot \|$; boldface denotes an element or operator in $H^0$. We assume that $t \to F_0(t) \equiv F_0(\cdot, t) \in L^1_{\text{loc}}(\mathbb{R}^+, H^0)$, $f_0 \in L^1_{\text{loc}}(\mathbb{R}^+)$, and the Laplace transforms

$$\hat{f}_0(s) = \int_0^\infty e^{-st} f_0(t) \, dt \quad \text{and} \quad \hat{F}_0(s) = \int_0^\infty e^{-st} F_0(t) \, dt$$

converge for $\text{Re} s > 0$. Moreover,

$$s\hat{f}_0(s) \text{ and } s\hat{F}_0(s) \text{ have analytic extensions to } \{ \text{Re } s > -\eta \}' \text{ that are bounded in each half-plane } \{ \text{Re } s > -\eta' \}, \quad \eta' < \eta.$$

If, for example, $F_0(x, t) = u_1(x) + \int_0^t F(x, \tau) \, d\tau$, where $u_1$ and $F$ arise from an initial jump in $u$, and a memory term, respectively, as in § 1, then $u_1 \in H^0$ and

$$|u_n(1, t)| + \int_0^1 u_n^2(x, t) \, dx \leq M < \infty \quad (t < 0),$$

together with (1.13), (1.14), will ensure that $F_0$ satisfies our requirements.

Let $\mathcal{D} = H^2 \cap H_0^1 \subset H^0$, and let $L : \mathcal{D} \to H^0$ be the operator $d^2/dx^2$. The adjoint of $L$ is $L^*: H^0 \to \mathcal{D}^* = H^{-2}$. We let $x$ denote the identity function $x(x) = x$ in $H^0$.

Define $R(t) : H^0 \to H^0$ by letting $w(t) = R(t)w_0$ be the solution of

$$w'(t) = \int_0^t A(t - \tau)Lw(\tau) \, d\tau, \quad w(0) = w_0.$$
for \( w_0 \in \mathbb{D} \). It is known [6] that, under our hypotheses on \( A(t) \), \( R(t) \) extends to a bounded operator on \( H^0 \), and its operator norm satisfies

\[
\|R(t)\| \leq 1 \quad (0 \leq t < \infty), \quad \int_0^\infty \|R(t)\| \, dt < \infty.
\]

Furthermore,

\[
\hat{R}(s) = \hat{A}(s)^{-1}(\beta^2(s)I - L)^{-1}.
\]

Define \( U(x, s) \) by (2.11), (2.12), for each \( s \) such that \( \Delta(s) \neq 0 \) and \( 0 \leq x \leq 1 \). Let \( \Phi(s) = U(1, s) \). By the elementary theory of boundary value problems, \( U \) is uniquely determined as the solution of

\[
U_{xx}(x, s) - \beta^2 U(x, s) = -[u_0(x) + \hat{F}_0(x, s)]/\hat{A}(s) \quad \text{(a.e.)},
\]

\[
U(\cdot, s) - x\Phi(s) \in \mathbb{D}.
\]

To show that \( U \) is indeed the transform of a weak solution of (2.1)-(2.4), we shall show that \( U(\cdot, s) \) belongs to a Hardy space \( \mathcal{H}^2(\{\Re s > \mu\}, H^0) \) with values in \( H^0 \).

The next result gives the required estimates.

**Lemma 5.1.** (i) Under the general assumptions of this section, suppose \( \Delta(s) \neq 0 \) \( (\Re s \geq \mu, 0 \leq x \leq 1) \). When \( I = 0, \varepsilon > 0, \) and \( A(0^+) < \infty, \) assume in addition that

\[
|u(x, s)| \leq Q(\Re s \geq \mu, 0 \leq x \leq 1),
\]

where \( Q \) is a constant that depends on \( \mu \), \( \mu_i \), and \( \varepsilon \), as well as on \( u_0, f_0, \) and \( F_0 \).

(ii) If \( \varepsilon = 0 \), then (5.7) holds for \( \mu < 0 \), provided that \( \Delta(s) \neq 0 \) in some larger half-plane \( \{\Re s \geq \mu_1\}, \mu_1 < \mu \).

Clearly, (5.7) implies \( U(\cdot, s) \in \mathcal{H}^2(\{\Re s > \mu\}, H^0) \), and \( \Phi \in \mathcal{H}^2(\{\Re s > \mu\}) \).

The proof of Lemma 5.1 involves estimates such as those employed in locating the zeros of \( \Delta \); this proof appears in §6. From Lemma 5.1 we can deduce our result on existence and asymptotic decay.

**Theorem 5.1.** Under the conditions of Lemma 5.1, there exist functions \( u \) and \( \varphi \), with \( t \rightarrow e^{-\mu t}u(t) \in L^2(\mathbb{R}^+, H^0), t \rightarrow e^{-\mu t}\varphi(t) \in L^2(\mathbb{R}^+) \), such that \( \hat{u}(s) = U(\cdot, s) \) and \( \hat{\varphi} = \Phi \). Moreover, \( u \) is the unique solution in \( L_0^1(\mathbb{R}^+, H^0) \) of

\[
u(t) = \int_0^t B(t-\tau)L^*[u(\tau) - x\varphi(\tau)] \, d\tau + \int_0^t F_0(\tau) \, d\tau + u_0
\]

\[
(B(t) = \int_0^t A(\tau) \, d\tau), \text{ and we have the representation}
\]

\[
u(t) = R(t)u_0 + \int_0^t R(t-\tau)F_0(\tau) \, d\tau + x\varphi(t) - z(t),
\]

where \( z \) is defined by its transform

\[
\hat{z}(s) = s\Phi(s)[\hat{R}(\cdot)x]^\wedge(s).
\]

**Remarks.** Equation (5.8) is a weak, integrated form of (2.1)-(2.3), with the right boundary condition \( u(1, t) = \varphi(t) \) in place of (2.4).
Representation (5.9) can be used to define the weak solution $u$ when, instead of (5.7), we have only $\Phi \in \mathcal{H}^2$. This would happen in certain cases where (5.1) is weakened to $u_0 \in L^2(0,1) \cap AC[x_0,1]$, with $0 < x_0 < 1$, so that the integration by parts estimate of (6.15) in §6 works only near $x = 1$.

Stronger hypotheses yield other representations and additional smoothness for $u$ and $\varphi$. If, for example, $A(t) = E + \gamma t^{-1/2} e^{-\delta t}$, then $\beta(s) \sim s^{3/4}(s \to \infty)$, and $R'(t) L^{-\nu} \in L^1(\mathbb{R}^+, H^0)$ ($\nu > 0$) [7], and $u$ can be represented as

$$u(t) = R(t)u_0 + \int_0^t R(t - \tau) F_0(\tau) d\tau - \int_0^t R'(t - \tau) \varphi(\tau) d\tau.$$ 

If, in addition, $F_0 = 0$, $f_0 = 0$, and $u_0$ is in $C^j[0,1]$ with vanishing derivatives of order $0, 1, \cdots, j - 1$ at $x = 0, 1$, then we can improve the estimate of Lemma 5.1 by continuing to integrate by parts as in (6.15) and get

$$|\Phi(s)| \lesssim \frac{Q}{|s|^{3j/4}} \left( |s| + |s|^{1/4} \right)^{-1}(s \to \infty).$$

By $\mathcal{H}^2$ theory, $\varphi$ can then have derivatives in $L^2(\mathbb{R}^+)$. See [21], [27] for a systematic examination of regularity in related problems.

When $\mu \geq 0$, complete monotonicity can be replaced in Theorem 5.1 by the weaker conditions mentioned (for $g$) in §1. When $k = 0$ and in the fixed-end case ($k = 0$), the problem is self-adjoint, and stronger results can be obtained directly through separation of variables, as in [19].

Finally, if $\Delta(s_0) = 0$ with Re $s_0 > 0$, and if we take $u_0(1) = 0$, $F_0 = 0$, $f_0 = 0$,

$$\int_0^1 G(1, y, s_0) u_0(y) dy = -\int_0^1 \sinh \beta(s_0) y u_0(y) dy \neq 0,$$

we see from (2.11) that $\Phi$ is unbounded at $s_0$, and a solution with $u(1, \cdot) \in L^2(\mathbb{R}^+)$ cannot exist.

**Proof of Theorem 5.1.** By elementary $\mathcal{H}^2$ theory, the existence of $u$ and $\varphi$ in the appropriate $L^2$ spaces, with transforms $U$ and $\Phi$, respectively, is an immediate consequence of (5.7). Let $v(t)$ denote the right-hand side of (5.9). By (5.3) and (5.7), relation (5.10) defines $z \in L^2_{loc}(\mathbb{R}^+, H^0)$. Moreover, the second term of $v$ has the transform $\hat{R}(s)$ of $F_0(s)$ in $\mathcal{H}^2(\{ \text{Re } s > 0 \}, H^0)$, so it belongs to $L^2_{loc}(\mathbb{R}^+, H^0)$; the same is clearly true for the other terms of $v$, and, indeed, $t \rightarrow e^{-\mu_0 t} v(t) \in L^2(\mathbb{R}^+, H^0)$, with $\mu_0 = \max \{ \mu, 0 \}$.

By evaluating Fourier coefficients with respect to the basis $\{ \sin n \pi x \}_{n=-1}^{\infty}$ in $H^0$ and taking transforms, we see that $v$ is the unique solution of (5.8). Finally, taking transforms in (5.9) and using (5.4), we see that

$$\hat{\varphi}(s) - s \Phi(s) = (L - \beta^2 I)^{-1} [x \Phi(s) - \hat{F}_0(s) - u_0] / \hat{A}(s).$$

Thus $\hat{\varphi}(s) - s \Phi(s) \in \mathcal{D}$ and

$$\left( \frac{\partial^2}{\partial x^2} - \beta^2 \right) \hat{\varphi}(s)(x) = -(\hat{F}_0(x, s) + u_0(x)) / \hat{A}(s) \quad \text{a.e.,}$$

and we are back to (5.5), which determines $U$ uniquely. This proves that $\hat{\varphi}(s) = U(\cdot, s) = \hat{u}(s)$ (Re $s > \mu_0$), and the proof is complete. $\square$
6. Proofs of Lemmas 2.3 and 5.1.

Proof of Lemma 2.3. Fix $\sigma_0 < 0$. By (2.8), (2.9), and (2.24),

\[
\beta^2(s) = \frac{s}{A(s)} = \frac{\alpha \varphi - \tau \psi + i(\tau \varphi + \sigma \psi)}{\phi^2 + \psi^2} \quad (s = \sigma + i\tau).
\]

Here to simplify notation we have written $\hat{A}(s) = \varphi_\sigma(\tau) - i\psi_\sigma(\tau) = \varphi - i\psi$. We begin by showing that there exists $\tau_0 = \tau_0(\sigma_0) > 0$ such that

\[
\text{Im} \beta^2(s) > 0 \quad \text{for} \quad \tau \geq \tau_0, \quad 0 > \sigma \geq \sigma_0.
\]

By (6.1) and (2.24), (6.2) also holds whenever $\tau > 0$ and $\sigma \geq 0$. Once (6.2) is proved, we note that, since $\beta(\hat{s}) = \hat{\beta}(\hat{s})$, the formula $\beta(s) = (s/\hat{A}(s))^{1/2}$ (principal square root) is valid whenever $\text{Re} \ s \geq \sigma_0$ and $|s| \geq R$ for some sufficiently large $R$. In particular, $\exp(-2\beta(s))$ is bounded and analytic for $\text{Re} \ s \geq \sigma_0, |s| \geq R$, so by Lindelöf’s theorem and $\beta(\hat{s}) = \hat{\beta}(\hat{s})$, it suffices to prove that

\[
\text{Re} \beta(\sigma_0 + i\tau) \to \infty \quad \text{as} \ \tau \to \infty.
\]

Returning to the proof of (6.2), note that by (2.24)

\[
\tau(\tau \varphi + \sigma \psi) = \int_{0}^{\infty} \frac{\tau^2(x + 2\sigma)}{(\sigma + x)^2 + \tau^2} \, d\mu(x)
\]

\[
\geq \int_{0}^{\infty} \frac{\tau^2}{(\sigma + x)^2 + \tau^2} \, d\mu(x) + \int_{-\infty}^{0} \frac{\tau^2}{(\sigma + x)^2 + \tau^2} \, d\mu(x)
\]

when $\sigma_0 \leq \sigma \leq 0$, $\tau \geq -2\sigma_0$. By the dominated convergence theorem the first integral on the right-hand side of this inequality tends to $2\sigma \int_{0}^{2\sigma} d\mu(x)$, as $\tau \to \infty$. The second integral is bounded from below by

\[
2^{-1} \int_{-2\sigma}^{\tau} (x + 2\sigma) \, d\mu(x) \geq 4^{-1} \int_{-4\sigma_0}^{\tau} x \, d\mu(x) \to \infty
\]

as $\tau \to \infty$ by (2.13). Thus,

\[
\tau(\tau \varphi + \sigma \psi) \to \infty \quad \text{as} \ \tau \to \infty \quad \text{uniformly for} \ \sigma_0 \leq \sigma \leq 0.
\]

In particular, (6.2) is proved.

We now turn to the proof of (6.3). In order to simplify notation, we drop the subscript and show that (6.3) holds for a fixed $\sigma = \sigma_0 < 0$. We begin by showing that

\[
\sigma \varphi_\sigma(\tau) \to 0 \quad \text{and} \quad \tau \psi_\sigma(\tau) \to A(0^+) \quad \text{as} \ \tau \to \infty.
\]

To verify the first limit in (6.5), write

\[
\sigma \varphi_\sigma(\tau) = \int_{0}^{\infty} \frac{\sigma(\sigma + x)(1 + x)}{(\sigma + x)^2 + \tau^2} \, d\mu(x) / 1 + x.
\]

Since

\[
\hat{A}(0) = \int_{0}^{\infty} \frac{d\mu(x)}{x} < \infty,
\]

the dominated convergence theorem shows that $\sigma \varphi_\sigma(\tau) \to 0$ as $\tau \to \infty$ and the first part of (6.5) is proved.
If $A(0^+)<\infty$, it follows from (2.24), (2.13), and the dominated convergence theorem that
\( \tau \phi_\sigma(\tau) \to A(0^+) \) as $\tau \to \infty$. If $A(0^+) = \infty$, then by (2.24) and (2.13)
\[
\tau \phi_\sigma(\tau) \equiv \frac{1}{2} \int_0^{\tau - \sigma} d\mu(x) \to \infty \quad \text{as } \tau \to \infty,
\]
and the second part of (6.5) is proved.

Writing $\beta^2(s) = \rho_\sigma(\tau) e^{i \theta_\sigma(\tau)} = \rho(\tau) e^{i \theta(\tau)}(s = \sigma + i \tau)$, we have by (6.1), (6.2), and (6.5) that
\[
\rho(\tau) = |s/\hat{A}(s)|, \quad \tan \theta(\tau) = \frac{\tau \phi + \sigma \psi}{\tau \psi - \sigma},
\]
with $\theta(\tau) \in (\pi/2, \pi)$ for all large positive $\tau$.

For $\tau$ such that $\tau \phi + \sigma \psi \geq \tau \psi - \sigma \phi > 0$, $\pi/2 < \theta(\tau) \leq 3\pi/4$, and $\text{Re} \beta(\sigma + i \tau) \geq |s/\hat{A}(s)|^{1/2} \cos 3\pi/8 \to \infty (\tau \to \infty)$.

For the more difficult case where $\tau$ is such that
\[
0 < \tau \phi + \sigma \psi < \tau \psi - \sigma \phi,
\]
we use the identity $2\cos^2 \theta/2 = 1 + \cos \theta$ and the estimate
\[
(1 + x^2)^{-1/2} \leq 1 - (x^2/4)(|x| < 1)
\]
to get
\[
\text{Re} \beta(\sigma + i \tau) \geq 8^{-1/2}|s/\hat{A}(s)|^{1/2} \left[ \frac{\tau \phi + \sigma \psi}{\tau \psi - \sigma} \right].
\]

If $A(0^+) < \infty$, note that (6.8) implies
\[
\text{Re} \beta(\sigma + i \tau) \geq 8^{-1/2}|s/\hat{A}(s)|^{-1/2} \frac{\tau (\tau \phi + \sigma \psi)}{\tau \psi - \sigma \phi};
\]
then use (6.4), (6.5), and $s\hat{A}(s) \to A(0^+) (\tau \to \infty)$ to conclude that (6.3) ($\sigma_0 = \sigma$) holds in this case.

Finally, consider the case where $A(0^+) = \infty$. Since (6.7) is assumed to hold, $\phi_\sigma(\tau) \equiv 2 \psi_\sigma(\tau)$ for all sufficiently large $\tau$. Combining this with (6.5) and (6.8), we can find a constant $M > 0$ so that
\[
\text{Re} \beta(\sigma + i \tau) \geq M \frac{(\tau \psi)^{1/2}}{\tau \psi^2} (\tau \phi + \sigma \psi)
\]
for all large $\tau$. Note that
\[
\int_0^\infty \frac{\tau}{(\sigma + x)^2 + \tau^2} d\mu(x) \equiv (2\pi)^{-1} \int_0^{\tau - \sigma} d\mu(x),
\]
and since $A(0^+) = \infty$, it follows from (2.13) that
\[
\psi_\sigma(\tau) \equiv 2 \int_0^\infty \frac{\tau}{(\sigma + x)^2 + \tau^2} d\mu(x)
\]
for all large $\tau$. Also, by (6.6) and the inequality due to Schwarz,

$$
\left\{ \int_0^\infty \frac{\tau}{(\sigma + x)^2 + \tau^2} d\mu(x) \right\}^2
$$

(6.11)

$$
\leq \left\{ \int_0^\infty \frac{\tau^2}{x[(\sigma + x)^2 + \tau^2]} d\mu(x) \right\} \left\{ \int_0^\infty \frac{x}{(\sigma + x)^2 + \tau^2} d\mu(x) \right\}
$$

$$
\leq \hat{a}(0) \int_0^\infty \frac{x}{(\sigma + x)^2 + \tau^2} d\mu(x).
$$

Finally, the technique used to prove (6.4) can also be used to show that

$$
\tau \phi + \sigma \psi \geq 2^{-1} \int_0^\infty \frac{\tau x}{(\sigma + x)^2 + \tau^2} d\mu(x)
$$

for all large $\tau$. Now, combining this inequality with (6.9)-(6.11) and using (6.5), we obtain

$$
\text{Re } (\sigma + i\tau) \geq \frac{M}{8\hat{a}(0)} (\tau \phi, (\tau))^{1/2} \to \infty \quad \text{as } \tau \to \infty,
$$

and the proof of (6.3) is complete. As we noted earlier, Lemma 2.3 is proved.

Proof of Lemma 5.1. We claim first that

$$
\frac{1}{|\Delta(s)|} \leq \frac{Q|e^{-\beta}|}{I|s| + |\alpha|} \quad (\text{Re } s > \mu).
$$

(Throughout this proof, $Q$ denotes some constant as in the statement of Lemma 5.1.) Since $\Delta(s) \neq 0$ for $\text{Re } s \geq \mu$, we are concerned only with $\text{Re } s \geq \mu$ and $|s|$ large. Write

$$
2\Delta(s) = e^\beta (Is + k e^{-\epsilon s} + \alpha) \left( 1 - e^{-2\beta} \frac{Is + k e^{-\epsilon s} - \alpha}{Is + k e^{-\epsilon s} + \alpha} \right).
$$

(6.13)

First suppose that $\epsilon = 0$, so that $\mu$ can be negative. By (2.15) and (2.16), $\mu > s^*$ and $\text{Re } \mu > 0$ (Re $s \geq \mu$). When $A'(0^+) = -\infty$, (6.12) is immediate from (3.1), (3.2), and (2.19). When $A'(0^+) = -\infty$, an infinite sequence of zeros of $\Delta$ is asymptotic to the line Re $s = \sigma_*$ of (3.5), so $\mu > \sigma_*$ except in the special case $I = 0$, $k = A(0^{1/2})$, where $\sigma_*$ is undefined. Hence, by (3.4) and (2.21), the last factor on the right in (6.13) is bounded away from zero for Re $s \geq \mu$ and $|s|$ large. Using (2.18) and Re $\alpha > 0$, we get (6.12).

Now suppose $\epsilon > 0$, so that $\mu \geq 0$. When $I > 0$ or $A(0^+) = \infty$, (6.12) is evident from (6.13), together with (2.18), (2.19), and Lemma 2.3.

If, on the other hand, $A(0^+) < \infty$ and $I = 0$, we have

$$
2\Delta(s) = e^\beta \alpha (1 + e^{-2\beta}) \left[ 1 + \frac{k e^{-\epsilon s}}{\alpha} \left( 1 - e^{-2\beta} \right) \right].
$$

(6.14)

When $A'(0^+) = -\infty$, we have $e^{-2\beta} \to 0$ and $\alpha \to A(0^{1/2})$ as $|s| \to \infty$ (Lemmas 2.1 and 2.3), and (5.6) yields (6.12). When $A'(0^+) > -\infty$, Lemmas 2.1 and 2.2 and (5.6) give us

$$
\lim \sup_{|s| \to \infty} \left| \frac{k e^{-\epsilon s}}{\alpha} \left( 1 - e^{-2\beta} \right) \right| \leq \frac{k}{A(0^{1/2})} \left( \frac{1 - C}{1 + C} \right) < 1,
$$

and (6.12) follows from (6.14). This establishes (6.12) in all cases.
Now consider $U(x, s)$ in (2.11). Since (5.2) holds and $\beta(s) = E^{-1/2}s + o(s)$ ($s \to 0$), $U$ is analytic and locally bounded in $\{\Re s \geq \mu\}$, uniformly in $0 \leq x \leq 1$. Observe next that

$$
\int_0^1 G(x, y, s)u_0(y) \, dy = G_1(x, 1, s)u_0(1) - \int_0^1 G_1(x, y, s)u(y) \, dy,
$$

where $G_1(x, y, s) = \int_0^\tau G(x, \tau, s) \, d\tau = O[(\epsilon^\beta/\beta)(Is/\alpha + 1)](|s| \to \infty)$, $\Re s \geq \mu$, uniformly in $0 \leq x, y \leq 1$. By (6.12) and (5.2), and since $\alpha\beta = s$, we get (5.7). This proves Lemma 5.1. \(\square\)

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