Lesson 12

Integrals by Substitution
**Review: Antiderivative Rules**

**Def:** We say $F$ is an *antiderivative* for $f$ if $F'(x) = f(x)$.

<table>
<thead>
<tr>
<th>If $f(x)$ is…</th>
<th>…then an antiderivative is…</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{1}{n+1} x^{n+1}$ except if $n = -1$</td>
</tr>
<tr>
<td>1</td>
<td>$x$ (assuming the variable is $x$!)</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$\sin(x)$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$-\cos(x)$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
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<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
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Antiderivatives are linear. So if $F$ is an antiderivative for $f$ and $G$ is an antiderivative for $g$, then an antiderivative for $b f(x) + c g(x)$ is $b F(x) + c G(x)$.
Review: The Indefinite Integral

We will use the notation \[ \int f(x) \, dx \]

to represent *all possible* antiderivatives of the function \( f(x) \), with respect to the variable \( x \).

Called the *indefinite integral* of \( f(x) \).
The Chain Rule

Recall: \( \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \) . For example: \( \frac{d}{dx} \sin(x^2) = \cos(x^2)2x \)

So \( \int f'(g(x))g'(x) \, dx = f(g(x)) + C \)

Example: Evaluate: \( \int x \cos(x^2) \, dx \)

We will rearrange the integral to get an exact match:

\[
\int x \cos(x^2) \, dx = \int \cos(x^2) x \, dx
\]

So we must also put in a 1/2 to keep the problem the same.

\[
= \frac{1}{2} \int \cos(x^2) 2x \, dx
= \frac{1}{2} \left[ \sin(x^2) \right] + C
\]

Check: \( \frac{d}{dx} \left( \frac{1}{2} \sin(x^2) \right) = \frac{1}{2} \cos(x^2)2x = x \cos(x^2) \) From the chain rule

Looks almost like \( \cos(x^2)2x \), which is the derivative of \( \sin(x^2) \).
Integrals by Substitution

Start with \[ \int f(g(x)) g'(x) \, dx \]

Let \( u = g(x) \). So we get:

\[ \int f(g(x)) g'(x) \, dx = \int f(u) g'(x) \, dx = \int f(u) \, du \]

Now need antiderivative of \( f \), with \( u \) plugged in.

Rewrite the integral using the fact that

\[ \frac{du}{dx} = g'(x) \quad \text{so} \quad du = g'(x) \, dx \]
Suppose we are trying to find \[ \int (3x^2 + 1)^3 x \, dx \]

Here, the inside function is \( u = 3x^2 + 1 \).

So \( du/dx = 6x \), or \( du = 6x \, dx \).

Substitute:

\[
\int (3x^2 + 1)^3 x \, dx = \int u^3 x \, dx
\]

Have to cancel the 6 we put in

(Use the fact that \( u = 3x^2 + 1 \).)

Put in the 6 we need to get \( du \)

Our question wasn’t about \( u \!\)
Integrals by Substitution

1) Choose $u$.
2) Calculate $du$.  $du = \frac{du}{dx} \, dx$
3) Substitute $u$.
   Arrange to have $du$ in your integral also.
   (All $x$s and $dx$s must be replaced!)
4) Solve the new integral.
5) Substitute back in to get $x$ again.

Example: A linear substitution: $\int e^{3x+2} \, dx$

Let $u = 3x + 2$. Then $du = 3 \, dx$.

\[
\int e^{3x+2} \, dx = \int e^u \, dx = \frac{1}{3} \int e^u \, 3 \, dx = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^{3x+2} + C
\]
A Second Approach

Instead of rearranging your integral to produce a \( du \), you can solve for the differential \( dx \) (or \( dt \), or whatever) and substitute for the differential.

You may need to simplify, and you **still must end up with only \( u \) or \( du \) and no \( x \) or \( dx \).**

**Example:** \( \int \cos(5x) \, dx \)

Let \( u = 5x \) so \( du = 5dx \), or \( dx = du/5 \).

\[
\int \cos(5x) \, dx = \int \cos(u) \frac{du}{5} = \frac{1}{5} \int \cos(u) \, du = \frac{1}{5} \sin(u) + C = \frac{1}{5} \sin(5x) + C
\]

Replace \( dx \) with what it’s equal to

Note: This is the same as the book’s rule for \( \cos(kx) \).
Choosing $u$

- Try to choose $u$ to be an inside function. (Think chain rule.)
- Try to choose $u$ so that $du$ is in the problem, except for a constant multiple.

**Example 1:** For $\int (3x^2 + 1)^3 x \, dx$

$u = 3x^2 + 1$ was a good choice because

1. $3x^2 + 1$ is inside the cube.
2. The derivative is $6x$, and we have an $x$.

**Example 2:** For $\int e^{3x+2} \, dx$

$u = 3x + 2$ was a good choice because

1. $3x + 2$ is inside the exponential.
2. The derivative is $3$, which is *only* a constant.
Practice

\[ \int \frac{x}{x^2 + 1} \, dx \]

Let \( u = x^2 + 1 \) \( du = 2x \, dx \)

\[ \int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int \frac{1}{x^2 + 1} \, 2x \, dx = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^2 + 1| + C \]

\[ \int \sin(7x + 2) \, dx \]

Let \( u =? \)

Let \( u = 7x + 2 \) \( \text{So } du = 7 \, dx, \text{ and} \)

\[ \int \sin(7x + 2) \, dx = \frac{1}{7} \int \sin(7x + 2) \, 7 \, dx = \frac{1}{7} \int \sin u \, du = -\frac{1}{7} \cos(7x + 2) + C \]
Practice

\[ \int \sin(x) \cos(x) \, dx. \] (Hint: There are several ways to do this.)

Let \( u = ? \)

**Method 1**

\[ u = \sin(x) \implies du = \cos(x) \, dx \]

\[ \int \sin(x) \cos(x) \, dx = \int u \, du \]

\[ = \frac{u^2}{2} + C = \frac{\sin^2(x)}{2} + C \]

**Method 2**

\[ u = \cos(x) \implies du = -\sin(x) \, dx \]

\[ -1 \int \cos(x) (-\sin(x)) \, dx = \int u \, du \]

\[ = -\frac{u^2}{2} + C = \frac{-\cos^2(x)}{2} + C \]

What’s the difference?

\[ \left( \frac{1}{2} \sin^2(x) \right) - \left( -\frac{1}{2} \cos^2(x) \right) = \frac{1}{2} \left( \sin^2(x) + \cos^2(x) \right) = \frac{1}{2} \]

This is 1!

That is, the difference is a constant.
Practice

\[ \int \frac{e^x}{1 + e^x} \, dx \quad \text{Let } u =? \quad u = 1 + e^x \quad \Rightarrow du = e^x \, dx \]

\[ \int \frac{e^x}{1 + e^x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |1 + e^x| + C \]

\[ \int x \sin(x^2) \, dx \quad \text{Let } u =? \quad u = x^2 \quad \Rightarrow du = 2x \]

\[ \int x \sin(x^2) \, dx = \frac{1}{2} \int \sin(x^2) \, 2x \, dx = \frac{1}{2} \int \sin(u) \, du = -\frac{1}{2} \cos(x^2) + C \]
Doesn’t Fit All

1- We can’t use $u$–substitution to solve everything. For example:

Let $u = x^2$

$du = 2x \, dx$

We need $2x$ this time, not just 2.

We CANNOT multiply by a variable to adjust our integral.

We cannot complete this problem.

2- For the same reason, we can’t do the following by $u$–substitution:

But we already knew how to do this!

$\int (x^2 - 3)^3 \, dx = \int x^6 - 9x^4 + 27x^2 - 27 \, dx = \frac{1}{7}x^7 - \frac{9}{5}x^5 + 9x^3 - 27x + C$
There are lots of integrals we will never learn how to solve…

In fact, there are a number of functions without elementary antiderivatives. That’s why numerical integrations are useful!

Morals

• No one technique works for everything.
• Don’t forget things we already know!
problem

Last time, we determined that

\[
\int (3x + 1)^2 \, dx = 3x^3 + 3x^2 + x + C
\]

Now use \( u \)-substitution to compute the same integral.

Do you get the same result? (Don’t just assume or claim you do; multiply out your result to show it!)

If you don’t get exactly the same answer, is it a problem? Why or why not?
Solution

Last time, we determined that

\[ \int (3x + 1)^2 \, dx = 3x^3 + 3x^2 + x + C \]

\[ u = 3x + 1 \quad \int (3x + 1)^2 \, dx = \frac{1}{3} \int (3x + 1)^2 \, 3 \, dx = \frac{1}{3} \int u^2 \, du = \frac{1}{3} \frac{u^3}{3} + C \]

\[ = \frac{1}{9} (3x + 1)^3 + C \]

Are these answers equal?

\[ \frac{1}{9} (3x + 1)^3 + C = \frac{1}{9} (27x^3 + 27x^2 + 9x + 1) + C = 3x^3 + 3x^2 + x + \frac{1}{9} + C \]

\[ = 3x^3 + 3x^2 + x + C_1 \]

Yes. They may differ by a constant.
Summary

• Make a $u$-substitution: find $u$ and $du$, then transform to an integral we can do. Be sure to transform back to the original variable!

• Rearrange to get $du$

• Substitution will still not solve every integral. (Nor is it always needed!)
Antiderivative Practice

Problem 1  \[ \int 2 \sin(3t) - e^{4t} + 4t \, dt \]  Use basic formulas:
\[ \int 2 \sin(3t) - e^{4t} \, dt = -\frac{2}{3} \cos(3t) - \frac{1}{4} e^{4t} + \frac{1}{\ln(4)} 4t + C \]

Problem 2  \[ \int \frac{z^2 + 2z}{z^2} \, dz \]  Simplify algebraically first, then integrate.
\[ \int \frac{z^2 + 2z}{z^2} \, dz = \int \frac{z^2}{z^2} + \frac{2z}{z^2} \, dz = \int 1 + \frac{2}{z} \, dz = z + 2 \ln|z| + C \]

Problem 3  \[ \int \frac{6t}{4 + t^2} \, dt \]  Make a substitution: Let \( u = 4 + t^2 \), so \( du = 2tdt \).
\[ \int \frac{6t}{4 + t^2} \, dt = 3 \int \frac{2t}{4 + t^2} \, dt = 3 \int \frac{1}{u} \, du = 3 \ln|u| + C = 3 \ln|4 + t^2| + C \]
Antiderivative Practice

Problem 4  \[ \int \frac{2}{y \ln(ky)} \, dy \] Make a substitution: \( u = \ln(ky) \), so \( du = \frac{dy}{y} \).

\[
\int \frac{2}{y \ln(ky)} \, dy = \int \frac{2}{\ln(ky)} \frac{1}{y} \, dy = \int \frac{2}{u} \, du = 2 \ln|u| + C = 2 \ln|\ln(ky)| + C
\]

Problem 5  Find the particular function \( F(x) \) such that \( F'(x) = x^2 \) and the graph of \( F(x) \) passes through \((1, 2)\).

The general antiderivative is \( \int x^2 \, dx = \frac{1}{3} x^3 + C \)

Then to find \( C \), we must have \( F(1) = \frac{1}{3} 1^3 + C = 2 \)

Thus, \( C = 5/3 \), and our function is \( F(x) = \frac{1}{3} x^3 + \frac{5}{3} \)