Lines and Planes

Lines

Let \( l \) be a line parallel to vector \( \vec{v} \).

\[ \vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \]

Let \( P_0(a, b, c) \) be a point on the line and \( P(x, y, z) \) be any other point on the line.

\[ \overrightarrow{P_0P} \] is a vector parallel to the line.

\[ \overrightarrow{P_0P} \parallel \vec{v} \] so

The parametric equations of a line are:

\[ x = a + v_1 t \]

\[ y = b + v_2 t \]

\[ z = c + v_3 t \]
To find the equation of a line you need a point on the line and a vector parallel to the line.

ex. Find the equation of the line through $P(1, 5, 3)$ parallel to vector $\vec{v} = \langle 2, \ 0, \ 1 \rangle = 2\hat{i} + \hat{k}$.

ex. Find the equation of the line through points $P(1, 2, 3)$ and $Q(3, 6, 9)$. 
ex. Find a vector parallel to the line

\[ x = 9 - t \]
\[ y = 3 \]
\[ z = 6t \]

ex. Determine which of the following line equations represent the same line as \( x = 9 - t \quad y = 3 \quad z = 6t \)

\[ x_1 = 7 + 2t \quad x_2 = 11 - 3t \quad x_3 = 9 + t \]
\[ y_1 = 3 \quad y_2 = 3 \quad y_3 = 3 \]
\[ z_1 = 12 - 12t \quad z_2 = 10 + 18t \quad z_3 = 6t \]
Planes

Let $P_0(a, b, c)$ be a point in the plane. If $P(x, y, z)$ is any other point in the plane, then $\overrightarrow{P_0P}$ is a vector that lies in the plane. Let $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ be a vector perpendicular (normal) to the plane. Then
The equation of a plane is \( n_1 x + n_2 y + n_3 z = n \cdot \vec{p} \) where \( \vec{n} \) is the normal vector and \( \vec{p} \) is the vector formed by the point.

To find the equation of a plane you need a point on the plane and a vector normal (perpendicular) to the plane.

ex. Find the equation of the plane through \( P(2, 6, 3) \) which is normal to \( \vec{n} = \langle 3, -2, 1 \rangle \).

ex. Find the equation of the plane through \( P(2, 6, 3), Q(-1, 0, 1) \) and \( R(4, 4, 4) \).
ex. Find a normal vector to the plane \( z = 2x + y + 1 \).

Do:

i. Find the equation of the line through \( P(1, 1, 1) \) perpendicular to the plane \( 2x - 3y + 5z = 4 \).

ii. Find the equation of the plane through \( P(2, 5, 1) \) parallel to the plane \( z = -3 \).

\[
\begin{align*}
\text{a. i} & : & x &= 3 - 4t \\
& & y &= -2 + 6t \\
& & z &= 6 - 10t \\
\text{ii. z} &= 1 \\
\text{b. i} & : & x &= 3 - 4t \\
& & y &= -2 + 6t \\
& & z &= 6 - 10t \\
& & \text{ii. } -3z &= 1 \\
\text{c. i} & : & x &= 1 + 2t \\
& & y &= 1 + 3t \\
& & z &= 1 + 5t \\
\text{ii. z} &= 1 \\
\text{d. i} & : & x &= 1 + 2t \\
& & y &= 1 + 3t \\
& & z &= 1 + 5t \\
& & \text{ii. } -3z &= 1
\end{align*}
\]
ex. Determine if the following two lines are parallel, perpendicular, intersect at a non right angle, or are skew.

\[ x_1 = 3 + 5t \quad x_2 = 2 + s \]
\[ y_1 = 1 - 5t \quad y_2 = -7 - s \]
\[ z_1 = 10t \quad z_2 = 3 - 2s \]
ex.

\[
\begin{align*}
x_1 &= t \\
y_1 &= 2 + 2t \\
z_1 &= 3 - t
\end{align*}
\]

\[
\begin{align*}
x_2 &= 1 + s \\
y_2 &= 4 + 4s \\
z_2 &= 2 + 2s
\end{align*}
\]
A beam is to be constructed between the point \( P = H \ 3, - 8, - 6L \)
and the surface defined by the equation
\[-2x - 2y - 2z = -16\]
with units in feet, at a cost of $24 per foot. What is the approximate minimum cost of the beam?
Do: i. Find the equation of the line of intersection of planes  
\( x + y + z = 1 \) and \( x - 2z = 0 \).

ii. Is the line \( x = -2 + \frac{1}{2}t, y = -2 + t, z = 0 \) the same line as \( x = 2 + \frac{1}{2}t, y = 2 + t, z = 0 \)?

\[
\begin{align*}
x &= 2t \\
y &= 1 - 3t \\
z &= t
\end{align*}
\]

a. i \( z = t \) \hspace{1cm} ii yes

\[
\begin{align*}
x &= 2 - 2t \\
y &= -1 + 3t \\
z &= 1 - t
\end{align*}
\]

c. i \( z = 1 - t \) \hspace{1cm} ii yes

d. i \( z = 1 - t \) \hspace{1cm} ii no