# TRIVIAL UNITS IN GROUP RINGS

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Abstract. Let $G$ be an arbitrary group and let $U$ be a subgroup of the normalized units in $\mathbb{Z}G$. We show that if $U$ contains $G$ as a subgroup of finite index, then $U = G$. This result can be used to give an alternative proof of a recent result of Marciniak and Sehgal on units in the integral group ring of a crystallographic group.

In the last section of their paper [1], Marciniak and Sehgal discuss units in the integral group ring of a crystallographic group. Their new insight is that the given action of such a group $\Gamma$ as isometries of $\mathbb{R}^n$ extends to an action of the group of normalized units $\mathcal{U}_1(\mathbb{Z}\Gamma)$ as affine transformations of $\mathbb{R}^n$. (Normalized units are units whose value under the augmentation map is 1.) They prove that if $\Gamma$ is torsion free and the group of normalized units acts freely on the ambient affine space, then all such units are trivial, i.e., $\mathcal{U}_1(\mathbb{Z}\Gamma) = \Gamma$. The first step is to observe that if $P$ is an integral point in a compact fundamental domain for $\Gamma$ and $u \in \mathcal{U}_1(\mathbb{Z}\Gamma)$, then there is some $g \in \Gamma$ such that $gu(P)$ is one of the finitely many integral points in the domain. Since the action is fixed point free, the index $[\mathcal{U}_1(\mathbb{Z}\Gamma) : \Gamma]$ is finite. Then the authors use a nontrivial geometric argument to deduce that $\mathcal{U}_1(\mathbb{Z}\Gamma)$ coincides with $\Gamma$.

In this short note, we give an algebraic proof of a stronger general result about unit groups in integral group rings which have finite index over the base group.

**Theorem.** *Let $G$ be an arbitrary group and let $U$ be a subgroup of $\mathcal{U}_1(\mathbb{Z}G)$. If $U$ contains $G$ and $[U : G]$ is finite, then $U = G$.*

Let $\Delta(G)$ denote the finite conjugate subgroup of the group $G$. We first reduce to the case that $G = \Delta(G)$.

**Lemma.** *Let $G$ be a group and let $H$ be a subgroup of $G$ with $[G : H]$ finite. If $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ has 1 in its support and normalizes $H$, then $\alpha$ lies in $\mathbb{Z}\Delta(G)$.*

*Proof.* Let $h \in H$. Then $\alpha h \alpha^{-1} = k$ for some $k \in H$. Since 1 is in the support of $h^{-1}\alpha h$ and $h^{-1}\alpha h = h^{-1}k\alpha$ it is also in the support of $h^{-1}k\alpha$. Therefore there are only finitely many possibilities for $h^{-1}k$ and, hence, only finitely many possibilities for $h^{-1}\alpha h$. It follows that $\alpha$ is centralized by a subgroup of finite index in $G$. We conclude that each element of the support of $\alpha$ lies in $\Delta(G)$. □

Now suppose $U$ and $G$ are groups as described in the theorem. Certainly $G$ contains a subgroup $H$ which is normal and of finite index in $U$. According to the lemma,

$$U = G \cdot (U \cap \mathbb{Z}\Delta(G)).$$

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Notice that $U \cap \mathbb{Z}\Delta(G) = U \cap \mathcal{U}_1(\mathbb{Z}\Delta(G))$ and

$$[U \cap \mathcal{U}_1(\mathbb{Z}\Delta(G)) : G \cap \mathcal{U}_1(\mathbb{Z}\Delta(G))] \leq [U : G].$$

But $G \cap \mathcal{U}_1(\mathbb{Z}\Delta(G)) = \Delta(G)$. Hence

$$[U \cap \mathcal{U}_1(\mathbb{Z}\Delta(G)) : \Delta(G)] < \infty.$$

In this manner, we see that it suffices to prove the theorem with $\Delta(G)$ replacing $G$.

Our main argument will require a basic result about units, due to Higman and Berman, and a basic tool, the trace associated with Hattori, Stallings, and Bass.

**Theorem.** [2, corollary II.1.2], [3, proposition 1.4] *If $\alpha \in \mathbb{Z}G$ has 1 in its support and $\alpha^n = 1$ for some positive integer $n$, then $\alpha = \pm 1$.*

We review some facts about the trace in a form suitable for our purposes. Let $G$ be a group and let $\mathbb{F}$ be a field of characteristic $p > 0$. If $\mathcal{C}$ denotes a conjugacy class of $G$ we define the trace $\operatorname{tr}_{\mathcal{C}} \colon \mathbb{F}G \to \mathbb{F}$ by

$$\operatorname{tr}_{\mathcal{C}}(\sum_{g \in G} \alpha_g g) = \sum_{g \in \mathcal{C}} \alpha_g.$$

If $\mathcal{D}$ is another conjugacy class and it contains an element whose $p$th power is in $\mathcal{C}$, we will write $\mathcal{D}^p = \mathcal{C}$. The simple but powerful equation which leads to so many group ring consequences is

$$\operatorname{tr}_{\mathcal{C}}(\alpha^p) = \sum_{\mathcal{D}^p = \mathcal{C}} (\operatorname{tr}_{\mathcal{D}}(\alpha))^p$$

for all $\alpha \in \mathbb{F}G$. (A proof and applications can be found in [3, section 1.7].)

Assume that $G$ has a normal subgroup $N$ such that $G/N$ is torsion free. Then any $G$-conjugacy class which meets $N$ is entirely contained in $N$; moreover, $\mathcal{C} \subseteq N$ and $\mathcal{D}^p = \mathcal{C}$ implies $\mathcal{D} \subseteq N$. We conclude that

$$\sum_{\mathcal{C} \subseteq N} \operatorname{tr}_{\mathcal{C}}(\alpha^p) = \sum_{\mathcal{D} \subseteq N} (\operatorname{tr}_{\mathcal{D}}(\alpha))^p.$$

The sum has a simpler interpretation of use to us. Let $\varepsilon_N \colon \mathbb{Z}G \to \mathbb{Z}$ denote the truncated augmentation map,

$$\varepsilon_N(\sum_{g \in G} \alpha_g g) = \sum_{g \in N} \alpha_g.$$

Then, under the hypotheses on $N$, we have the $p$-power formula

$$\varepsilon_N(\alpha^p) \equiv (\varepsilon_N(\alpha))^p \mod p$$

for all $\alpha \in \mathbb{Z}G$.

Now let us prove the main theorem. We are assuming that $G = \Delta(G)$. There is no loss of generality in taking $G$ finitely generated. Then the elements of finite order in $G$ constitute a finite characteristic subgroup $\Delta^+(G)$ and $G/\Delta^+(G)$ is torsion free abelian. Let $T$ be a transversal for $\Delta^+(G)$ in $G$ which contains 1.

Recall that $U \leqslant \mathcal{U}_1(\mathbb{Z}G)$, that $U$ contains $G$, and that $[U : G] < \infty$. Since each member in the support of an element of $\mathbb{Z}G$ is centralized by a subgroup of finite index in $G$, we see that $U$ is a finitely generated finite conjugate group, as well. Certainly $\Delta^+(U) \supseteq \Delta^+(G)$. There are two cases to consider.

First suppose that $\Delta^+(U) = \Delta^+(G)$. Since $\Delta^+(G)$ is normal in $U$ and $U/\Delta^+(G)$ is abelian, $G$ is normal in $U$. If $U = G$ we are done. Suppose, to the contrary, that there is a prime $p$ and $\alpha \in U \setminus G$ such that $\alpha^p \in G$. By assumption, $\varepsilon_G(\alpha) = 1$.

Thus we can find some $t \in T$ so that $\varepsilon_{\Delta^+(G)}(\alpha t^{-1}) \not\equiv 0 \mod p$. Notice that $(\alpha t^{-1})^p \in G$ by the normality of $G$ in $U$. An application of the $p$-power formula yields $\varepsilon_{\Delta^+(G)}((\alpha t^{-1})^p) \not\equiv 0 \mod p$. We conclude that $(\alpha t^{-1})^p \in \Delta^+(G)$. But this means $\alpha t^{-1}$ has finite order, whence we reach the contradiction $\alpha \in \Delta^+(G)T \subseteq G$.

Finally, suppose that $\Delta^+(U) > \Delta^+(G)$. By choosing an element in $\Delta^+(U)$ which has a minimal power larger than 1 in $\Delta^+(G)$ we see that there exists a prime $p$ and $\alpha \in \Delta^+(U) \setminus \Delta^+(G)$ such that $\alpha^p \in \Delta^+(G)$. This time we know that

$$\varepsilon_{\Delta^+(G)}(\alpha^p) = 1 \not\equiv 0 \mod p,$$

so a second application of the $p$-power formula tells us that $\varepsilon_{\Delta^+(G)}(\alpha)$ is nonzero. Thus there is some $h \in \Delta^+(G)$ which lies in the support of $\alpha$; equivalently, 1 is in the support of $h^{-1}\alpha$. But $h^{-1}\alpha$ has finite order as it is a member of $\Delta^+(U)$. By the theorem on units of Higman and Berman, $h^{-1}\alpha = \varepsilon_G(h^{-1}\alpha) \cdot 1$. Since $h^{-1}\alpha$ is a normalized unit, we obtain the contradiction $\alpha = h \in \Delta^+(G)$.

## References


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