

A RATIONALITY CRITERION FOR UNBOUNDED OPERATORS

PETER A. LINNELL

ABSTRACT. Let G be a group, let $U(G)$ denote the set of unbounded operators on $L^2(G)$ which are affiliated to the group von Neumann algebra $W(G)$ of G , and let $D(G)$ denote the division closure of $\mathbb{C}G$ in $U(G)$. Thus $D(G)$ is the smallest subring of $U(G)$ containing $\mathbb{C}G$ which is closed under taking inverses. If G is a free group then $D(G)$ is a division ring, and in this case we shall give a criterion for an element of $U(G)$ to be in $D(G)$. This extends a result of Duchamp and Reutenauer, which was concerned with proving a conjecture of Connes.

Un critère de rationalité pour opérateurs non bornés

Résumé – Soient G un groupe, $U(G)$ l'ensemble d'opérateurs non bornés affiliés à l'algèbre de von Neumann de groupe de G , et $D(G)$ la clôture de division de $\mathbb{C}G$ dans $U(G)$. Ainsi $D(G)$ est le plus petit anneau qui est fermé sous l'opération d'inverse. Si G est un groupe libre, nous donnons un critère pour qu'un élément de $U(G)$ soit dans $D(G)$.

Version français abrégée – Soient G un groupe libre, $L^2(G)$ l'espace de Hilbert avec base orthonormée $\{g \mid g \in G\}$, $C_r^*(G)$ l'algèbre réduite de groupe de G , $W(G)$ l'algèbre de von Neumann de groupe de G , $U(G)$ l'ensemble d'opérateurs fermés, affiliés à $W(G)$, $D(G)$ la clôture de division de $\mathbb{C}G$ dans $U(G)$, et $S(G)$ la clôture de division de $\mathbb{C}G$ dans $C_r^*(G)$. Ainsi $D(G)$ est le plus petit anneau qui est fermé sous l'opération d'inverse. J'ai montré dans [9] que $D(G)$ est un anneau de division. Il y a un G -ensemble libre E et un opérateur unitaire $P: L^2(G) \rightarrow L^2(E) \oplus \mathbb{C}$ tel que l'opérateur $P\alpha - \alpha P$ soit de rang fini pour tout $\alpha \in \mathbb{C}G$ [5, p. 341]. Si A est une sous-algèbre de $U(G)$, appelons A_{fin} la plus grande sous-algèbre de A telle que $P\alpha - \alpha P$ soit de rang fini pour tout $\alpha \in A$. Reutenauer et Duchamp ont montré dans [7] que la clôture de division de $\mathbb{C}G$ dans $C_r^*(G)$ est $(W(G))_{\text{fin}}$; ceci a répondu à une question de Connes [5, p. 342]. Nous étendrons ce résultat au $U(G)$.

Définissons les sous-ensembles $R(G)$ et $R'(G)$ de G comme suit. Pour $u \in U(G)$, nous disons que $u \in R(G)$ si et seulement si toutes les fois que $u = s^{-1}a = bt^{-1}$ avec $a, b, s, t \in W(G)$, alors $sPb - aPt$ et $sP^{-1}b - aP^{-1}t$ soit de rang fini, alors que nous disons que $u \in R'(G)$ si et seulement si nous pouvons écrire $u = s^{-1}a = bt^{-1}$ avec $a, b, s, t \in W(G)$ et tels que $sPb - aPt$ et $sP^{-1}b - aP^{-1}t$ ont rang fini. Nous pouvons maintenant énoncer

Théorème 1. $D(G) = R(G) = R'(G)$ et $D(G) \cap W(G) = S(G)$. En outre si $u \in D(G)$, alors nous pouvons écrire $u = s^{-1}a = bt^{-1}$ avec $a, b, s, t \in S(G)$.

Ainsi en particulier, chaque élément de $D(G)$ peut être écrit sous la forme $s^{-1}a$ avec $a, s \in C_r^*(G)$. La démonstration du Théorème 1 dépend cruciallement des résultats de [7]. Une autre description de $D(G)$ est donnée par le résultat suivant.

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Proposition 2. *Soient $u \in U(G)$. Alors $u \in D(G)$ si et seulement s'il y a un sous-espace M de codimension finie dans $L^2(G)$ tels que les restrictions de Pu et uP à M sont égales.*

1. INTRODUCTION

Let H be a Hilbert space and for $u, v \in H$, let $\langle u, v \rangle$ indicate the inner product of u and v . The set of all closed densely defined linear operators acting on the left of H will be denoted by $\mathcal{U}(H)$, and the subset consisting of bounded operators will be denoted by $\mathcal{B}(H)$. The adjoint θ^* of $\theta \in \mathcal{U}(H)$ satisfies $\langle \theta u, v \rangle = \langle u, \theta^* v \rangle$ whenever θu and $\theta^* v$ are defined. Now let G be a group and let $L^2(G)$ denote the Hilbert space with Hilbert basis $\{g \mid g \in G\}$. Thus $L^2(G)$ consists of all formal sums $\sum_{g \in G} a_g g$ where $a_g \in \mathbb{C}$ and $\sum_{g \in G} |a_g|^2 < \infty$, and has inner product defined by

$$\left\langle \sum_{g \in G} a_g g, \sum_{h \in G} b_h h \right\rangle = \sum_{g \in G} a_g \bar{b}_g$$

where $\bar{}$ denotes complex conjugation. If $\alpha = \sum_{g \in G} a_g g \in \mathbb{C}G$ (so $a_g \in \mathbb{C}$ and $a_g = 0$ for all but finitely many g) and $\beta = \sum_{g \in G} b_g g \in L^2(G)$, then

$$\alpha\beta = \sum_{g, h \in G} a_g b_h gh = \sum_{g \in G} \left(\sum_{h \in G} a_{gh^{-1}} b_h \right) g \in L^2(G)$$

and the map $\beta \mapsto \alpha\beta$ (left multiplication by α) is in $\mathcal{B}(L^2(G))$. It follows that we may identify $\mathbb{C}G$ with a subring of $\mathcal{B}(L^2(G))$. By definition the reduced group C^* -algebra $C_r^*(G)$ of G is the norm closure of $\mathbb{C}G$ in $\mathcal{B}(L^2(G))$, and the group von Neumann algebra $W(G)$ of G is the weak closure of $\mathbb{C}G$ in $\mathcal{B}(L^2(G))$; thus $\mathbb{C}G \subseteq C_r^*(G) \subseteq W(G)$ and $W(G)$ is a finite von Neumann algebra. Let $U(G)$ denote the operators in $\mathcal{U}(G)$ which are affiliated to $W(G)$ [1, p. 150]. Then $U(G) = U(G)^*$, $U(G)$ is a $*$ -regular ring [2, definition 1 of §51] containing $W(G)$, and every element of $U(G)$ can be written in the form $s^{-1}a$ and also as^{-1} , where $a \in W(G)$ and s is a nonzero divisor in $W(G)$ [1, theorems 1 and 10]. Using the fact that $U(G)$ is a $*$ -regular ring, we see that s is a nonzero divisor in $W(G)$ if and only if $sw \neq 0$ (or $ws \neq 0$) whenever $0 \neq w \in W(G)$. Furthermore any finite set of elements in $U(G)$ have a common denominator, so for example if $u, v \in U(G)$, then there exist $a, b, s \in W(G)$ with s a nonzero divisor such that $u = s^{-1}a$ and $v = s^{-1}b$.

Suppose H is the Hilbert space sum of an arbitrary number (finite or infinite) of copies of $L^2(G)$. For $h \in H$, we shall write $h = (h_1, h_2, \dots)$ where h_i denotes the i th component of h . If $\theta \in \mathcal{U}(L^2(G))$, then θ defines an element of $\mathcal{U}(H \oplus \mathbb{C})$ according to the rule $\theta(h_1, h_2, \dots, c) = (\theta h_1, \theta h_2, \dots, c)$ where $c \in \mathbb{C}$. The domain of this operator is $\text{dom}(\theta) \times \text{dom}(\theta) \times \dots \times \mathbb{C}$. If $\theta \in U(G)$ and $\theta = s^{-1}a = bt^{-1}$ where $a, b, s, t \in W(G)$ and s, t are nonzero divisors, then $tL^2(G) \subseteq \text{dom}(\theta) = \{h \in L^2(G) \mid ah \in sL^2(G)\}$.

Suppose R is a subring of the ring S . Then the division closure of R in S is the smallest subring D of S containing R which is closed under taking inverses in S (i.e. $d \in D$ and d invertible in S implies $d^{-1} \in D$). We shall let $D(G)$ denote the division closure of $\mathbb{C}G$ in $U(G)$. Clearly if there is a division ring E such that $R \subseteq E \subseteq S$, then the division closure of R in S is a division subring of E . Also the rational closure of R in S is the subset T of S defined by the property $t \in T$ if and only if there exists an integer n and $M \in M_n(R)$ such that M is invertible in $M_n(S)$ and t is one of the entries of M^{-1} [3, p. 382]. The rational closure of R in S is

always a subring of S [3, theorem 7.1.2] which contains the division closure of R in S [3, exercise 7.1.4]. Often the division closure is equal to the rational closure, but there are examples when the division closure is strictly contained in the rational closure. However in the case that the division closure is a division ring, then it is clear that the division closure is equal to the rational closure.

For the rest of this paper G will be a free group. By [9, theorem 1.3], $D(G)$ is a division ring and so $D(G)$ is equal to the rational closure of $\mathbb{C}G$ in $U(G)$. If G is free on the set X , then it is shown on page 573 of [9] that $D(G)$ is isomorphic to the free field on X over \mathbb{C} [4, p. 224].

Let G act on the one element set $\{*\}$ according to the rule $g* = 0$ for all $g \in G$. Then there is a free left G -set E and a bijection $\pi: G \rightarrow E \cup \{*\}$ such that $\{b \in G \mid \pi gb \neq g\pi b\}$ is finite for all $g \in G$ [5, p. 341] (E here is T^1 there). Then π extends to a unitary operator $P: L^2(G) \rightarrow L^2(E) \oplus \mathbb{C}$ with the property that $Pa - aP$ has finite rank (i.e. $\text{im}(Pa - aP)$ has finite dimension over \mathbb{C}) for all $a \in \mathbb{C}G$; this can be seen from the proof of [5, lemma IV.5.1(a) on p. 342], where P there is the same as P here. For any subalgebra A of $\mathcal{B}(L^2(G))$, let $A_{\text{fin}} = \{a \in A \mid Pa - aP \text{ has finite rank}\}$ so if $A \supseteq \mathbb{C}G$, then $A_{\text{fin}} \supseteq \mathbb{C}G$. Let $S(G)$ denote the rational closure of $\mathbb{C}G$ in $C_r^*(G)$. Then [5, remark 3 on p. 342] shows that $S(G) \subseteq (C_r^*(G))_{\text{fin}}$, and the question of whether $S(G) = (C_r^*(G))_{\text{fin}}$ is posed there. This question was answered in the affirmative by [7, théorème 7], where the stronger result, that the division closure of $\mathbb{C}G$ in $C_r^*(G)$ is equal to $(W(G))_{\text{fin}}$, was proved so in particular $S(G)$ is also the division closure of $\mathbb{C}G$ in $C_r^*(G)$. Christophe Reutenauer has told me that when he and Duchamp proved this, Connes posed the problem of extending their result to $U(G)$. The purpose of this paper is to give an answer to this problem, and then to give a few simple applications of the result.

Define subsets $R(G), R'(G)$ of $U(G)$ as follows. For $u \in U(G)$, we say that $u \in R(G)$ if and only if whenever $u = s^{-1}a = bt^{-1}$ with $a, b, s, t \in W(G)$, then $sPb - aPt$ and $sP^{-1}b - aP^{-1}t$ have finite rank (the former is a bounded linear operator $L^2(G) \rightarrow L^2(E) \oplus \mathbb{C}$, and the latter is a bounded linear operator $L^2(E) \oplus \mathbb{C} \rightarrow L^2(G)$), while we say that $u \in R'(G)$ if and only if we may write $u = s^{-1}a = bt^{-1}$ with $a, b, s, t \in W(G)$ and such that $sPb - aPt$ and $sP^{-1}b - aP^{-1}t$ have finite rank. As remarked above, there is always at least one way to write $u = s^{-1}a = bt^{-1}$ with $a, b, s, t \in W(G)$, consequently $R(G) \subseteq R'(G)$. We can now state

Theorem 1.1. *$D(G) = R(G) = R'(G)$ and $D(G) \cap W(G) = S(G)$. Furthermore if $u \in D(G)$, then we may write $u = s^{-1}a = bt^{-1}$ with $a, b, s, t \in S(G)$.*

It is easy to read off a number of consequences of this result, for example we can now state that every element of $D(G)$ can be written in the form $s^{-1}a$ with $a, s \in (C_r^*(G))_{\text{fin}}$. It seems plausible that the definition of $R(G)$ could be weakened to requiring only that $sPb - aPt$ has finite rank, but I have been unable prove this. Theorem 1.1 generalizes [7, théorème 7], and the proof depends crucially on the results of [7].

Finally we give two other ways of defining $R(G)$. For the first let

$$F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix} \in \mathcal{B}(L^2(G) \oplus L^2(E) \oplus \mathbb{C}),$$

so F yields a Fredholm module as described on page 341 of [5]. Then for $u \in U(G)$, we can say that $u \in R(G)$ if and only if whenever $u = s^{-1}a = bt^{-1}$ with $a, b, s, t \in$

$W(G)$, then $sFb - aFt$ has finite rank; this is obvious. The second way is described by the following result.

Proposition 1.2. *Let $u \in U(G)$. Then $u \in D(G)$ if and only if there exists a subspace M of finite codimension in $L^2(G)$ such that the restrictions of Pu and uP to M are equal.*

Of course Pu and uP will *not* in general be bounded operators; when we say that two unbounded operators are equal, then we implicitly assume that their domains of definition are equal.

I am very grateful to Christophe Reutenauer for introducing me to the problem studied in this paper, and for some useful discussions.

2. NOTATION, TERMINOLOGY AND ASSUMED RESULTS

Most of the notation and terminology used in this paper has already been defined above. Mappings will be written on the left and \mathbb{C} will denote the complex numbers. All rings will have a 1, and subrings will have the same 1. If E is a set, then $L^2(E)$ will denote the Hilbert space with Hilbert basis $\{e \mid e \in E\}$. We shall let $\text{im } \theta$ and $\ker \theta$ denote the image and kernel of the map θ respectively. If n is a positive integer, then $M_n(R)$ will indicate the n by n matrices over a ring R . A projection in $\mathcal{B}(H)$ is an element e such that $e = e^* = e^2$. If u is an unbounded operator, then $\text{dom}(u)$ will indicate the domain of u , in other words the subspace on which it is defined. We need the following three elementary lemmas.

Lemma 2.1. *Let $\theta: H \rightarrow K$ and $\phi: K \rightarrow L$ be bounded linear maps between Hilbert spaces.*

- (i) *If $\ker \phi = 0$ and $\phi\theta$ has finite rank, then θ also has finite rank.*
- (ii) *If $\text{im } \theta$ is dense in K and $\phi\theta$ has finite rank, then ϕ also has finite rank.*

Proof. (i) is obvious. For (ii), since θH is dense in K , we see that $\phi\theta H$ is dense in ϕK . But $\phi\theta H$ is finite dimensional and therefore closed, hence ϕK is finite dimensional and the result is proven. \square

Lemma 2.2. *Let $\theta \in W(G)$. If θ is a nonzero divisor, then $\ker \theta = 0$ and $\text{im } \theta$ is dense in $L^2(G)$.*

Proof. Since θ is a nonzero divisor in $W(G)$, it is invertible in $U(G)$ and it follows that $\ker \theta = 0$. Also θ^* is a nonzero divisor in $W(G)$, and we deduce from this that $\text{im } \theta$ is dense in $L^2(G)$. \square

Lemma 2.3. *Let $u \in U(G)$. Then there exists $\lambda \in \mathbb{C}$ such that $u - \lambda$ is invertible in $U(G)$.*

Proof. For each $\lambda \in \mathbb{C}$, let $K_\lambda = \{x \in W(G) \mid (u - \lambda)x = 0\}$, a right ideal of $W(G)$. We first show that $K_\lambda = 0$ for some $\lambda \in \mathbb{C}$. Since $W(G)$ is a von Neumann algebra, there is a unique projection $e_\lambda \in W(G)$ such that $e_\lambda W(G) = K_\lambda$. Let $\text{tr}: W(G) \rightarrow \mathbb{C}$ denote the trace map, as described for example in [8, p. 352]. If we write an element α of $W(G)$ in the form $\sum_{g \in G} a_g g$ where $a_g \in \mathbb{C}$, then $\text{tr } \alpha = a_1$. Also if e is a nonzero projection in $W(G)$, then $0 < \text{tr } e \leq 1$. Note that the sum $\sum_\lambda K_\lambda$ is direct, so by using [8, lemma 12], we see that $\sum_{\lambda \in S} \text{tr}(e_\lambda) \leq 1$ for any finite subset S of \mathbb{C} . It follows that the number of λ for which $e_\lambda \neq 0$ is countable, and we deduce that there exists $\lambda \in \mathbb{C}$ (in fact uncountably many such λ) such

that $e_\lambda = 0$. For this λ , we have $K_\lambda = 0$ and $u - \lambda$ is a nonzero divisor in $W(G)$. Since every element of $U(G)$ can be written in the form $s^{-1}a$ and also as^{-1} with $a, s \in W(G)$, we conclude that $u - \lambda$ is a nonzero divisor in $U(G)$. But every element of $U(G)$ is either a zero divisor or invertible, and the result follows. \square

3. PROOFS

It will be clear from the next lemma that $\mathbb{C}G \subseteq R(G) = R'(G)$. We are going to show that $R(G)$ is a subring which is closed under taking inverses and adjoints. This will mean in particular that $R(G)$ is division closed and so will contain the division closure of $\mathbb{C}G$ in $U(G)$. First we show that we need only check the condition $sPb - aPt$, $sP^{-1}b - aP^{-1}t$ have finite rank for one choice of a, b, s, t satisfying $u = s^{-1}a = bt^{-1}$.

Lemma 3.1. *Let $u \in U(G)$ and suppose $u = s^{-1}a = bt^{-1}$, where $a, b, s, t \in W(G)$ and s, t are nonzero divisors. If $sPb - aPt$ and $sP^{-1}b - aP^{-1}t$ have finite rank, then $u \in R(G)$.*

Proof. Suppose $u = s_1^{-1}a_1$. Then we need to show that $s_1Pb - a_1Pt$ and $s_1P^{-1}b - a_1P^{-1}t$ have finite rank. There are nonzero divisors $x, x_1 \in W(G)$ such that $ss_1^{-1} = x^{-1}x_1$. Then $xs = x_1s_1$ and $xa = x_1a_1$, hence

$$x_1(s_1Pb - a_1Pt) = x(sPb - aPt)$$

and we see that $x_1(s_1Pb - a_1Pt)$ has finite rank. Using Lemmas 2.1 and 2.2, we deduce that $s_1Pb - a_1Pt$ has finite rank. Similarly $s_1P^{-1}b - a_1P^{-1}t$ has finite rank. If $u = b_1t_1^{-1}$, then in a similar fashion we can show that $s_1Pb_1 - a_1Pt_1$ and $s_1P^{-1}b_1 - a_1P^{-1}t_1$ have finite rank. This establishes the result. \square

Next we show that $R(G)$ is closed under addition.

Lemma 3.2. *Let $u, v \in R(G)$. Then $u + v \in R(G)$.*

Proof. Write $u = s^{-1}a = bt^{-1}$ and $v = s^{-1}c = dt^{-1}$, where $a, b, c, d, s, t \in W(G)$ and s, t are nonzero divisors. Then $sPb - aPt$, $sP^{-1}b - aP^{-1}t$, $sPd - cPt$, $sP^{-1}d - cP^{-1}t$ have finite rank and $u + v = s^{-1}(a + c) = (b + d)t^{-1}$, consequently

$$sP(b + d) - (a + c)Pt = (sPb - aPt) + (sPd - cPt)$$

has finite rank. Similarly

$$sP^{-1}(b + d) - (a + c)P^{-1}t = (sP^{-1}b - aP^{-1}t) + (sP^{-1}d - cP^{-1}t)$$

has finite rank. Using Lemma 3.1, we deduce that $u + v \in R(G)$. \square

Now we show that $R(G)$ is closed under multiplication.

Lemma 3.3. *Let $u, v \in R(G)$. Then $uv \in R(G)$.*

Proof. By Lemma 2.3, there exist $\lambda, \mu \in \mathbb{C}$ such that $u - \lambda$, $v - \mu$ are invertible in $U(G)$, so using Lemma 3.2, we may assume that u, v are invertible in $U(G)$. This means when we write $u = s^{-1}a$ with $a, s \in W(G)$, not only s but also a are nonzero divisors in $W(G)$. Write $v = t^{-1}b$ where b, t are nonzero divisors in $W(G)$, and then write $at^{-1} = w^{-1}c$ where c, w are nonzero divisors in $W(G)$. Then

$$uv = s^{-1}at^{-1}b = (ws)^{-1}(wa)(ct)^{-1}(cb)$$

and $wa = ct$. Thus we may write $u = p^{-1}q$, $v = q^{-1}r$ where p, q, r are nonzero divisors in $W(G)$, and similarly we may write $u = xy^{-1}$, $v = yz^{-1}$ where x, y, z are nonzero divisors in $W(G)$. Then $uv = p^{-1}r = xz^{-1}$. Since $u, v \in R(G)$, we have

$$pPx - qPy, qPy - rPz$$

have finite rank and hence $pPx - rPz = (pPx - qPy) + (qPy - rPz)$ has finite rank. Similarly $pP^{-1}x - rP^{-1}z$ has finite rank and an application of Lemma 3.1 completes the proof. \square

Now we show that $R(G)$ is closed under taking inverses.

Lemma 3.4. *Let $u \in R(G)$. If u is invertible in $U(G)$, then $u^{-1} \in R(G)$.*

Proof. Write $u = s^{-1}a = bt^{-1}$ where $a, b, s, t \in W(G)$, all nonzero divisors because u is invertible in $U(G)$. Then $u^{-1} = a^{-1}s = tb^{-1}$. Since $u \in R(G)$, we know that

$$sPb - aPt, sP^{-1}b - aP^{-1}t$$

have finite rank. Therefore

$$aPt - sPb, aP^{-1}t - sP^{-1}b$$

have finite rank. The result now follows from Lemma 3.1. \square

Finally we show that $R(G)$ is closed under the adjoint operation.

Lemma 3.5. *Let $u \in R(G)$. Then $u^* \in R(G)$.*

Proof. Write $u = s^{-1}a = bt^{-1}$, where $a, b, s, t \in W(G)$ and s, t are nonzero divisors. Then $u^* = (t^*)^{-1}b^* = a^*(s^*)^{-1}$. Since $u \in R(G)$, we know that $sPb - aPt$ and $sP^{-1}b - aP^{-1}t$ have finite rank. If T is a bounded linear map between Hilbert spaces with finite rank, then T^* also has finite rank. Furthermore $P^* = P^{-1}$ because P is a unitary operator. Therefore $t^*P^{-1}a^* - b^*P^{-1}s^*$ and $t^*Pa^* - b^*Ps^*$ have finite rank. The result now follows from Lemma 3.1. \square

It now follows from Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 that $R(G)$ is a subring of $U(G)$ containing $D(G)$ which is closed under the $*$ operation and taking inverses.

Proof of Theorem 1.1. We have already shown that $R(G) = R'(G)$, and we see from [7, propositions 5 and 9] and Lemma 3.1 that $R(G) \cap W(G) = S(G)$. Now let $u \in R(G)$ and set $a = (1 + uu^*)^{-1}u$ and $s = (1 + uu^*)^{-1}$. Then a, s are well defined elements of $W(G)$ and s is a nonzero divisor, by [6, 15.12.6]. Since $R(G)$ is a subring of $U(G)$ closed under taking inverses and adjoints, we see that $a, s \in R(G) \cap W(G)$. The result follows. \square

Proof of Proposition 1.2. First suppose $u \in D(G)$. Then by Theorem 1.1, there exist $a, s \in S(G)$ such that $u = s^{-1}a$. Then $\text{dom}(Pu) = \{x \in L^2(G) \mid ax \in sL^2(G)\}$ and $\text{dom}(uP) = \{x \in L^2(G) \mid aPx \in sPL^2(G)\}$. Since $Pa - aP$ and $Ps - sP$ have finite rank, there are subspaces M_1 and M_2 of finite codimension in $L^2(G)$ such that $Pa - aP$ is zero on M_1 and $Ps - sP$ is zero on M_2 . Then $P^{-1}sPM_2 = sM_2$, hence there are subspaces N_1, N_2 of finite codimension in $L^2(G)$ such that $N_1 \cap sL^2(G) \subseteq sM_2$ and $N_2 \cap P^{-1}sPL^2(G) \subseteq sL^2(G)$. Now choose subspaces M_3, M_4 of finite codimension in $L^2(G)$ such that $aM_3 \subseteq N_1$ and $aM_4 \subseteq N_2$, and set $M = M_1 \cap M_3 \cap M_4$.

Suppose $x \in M \cap \text{dom}(Pu)$. Then $ax = sl$ for some $l \in M_2$ and so $Paax = Psl$. Using the property that $Pa - aP$ is zero on M_1 and $Ps - sP$ is zero on M_2 , we see

that $aPx = sPl$ and we deduce that $x \in \text{dom}(uP)$. Conversely if $x \in M \cap \text{dom}(uP)$, then $aPx = sPl$ for some $l \in L^2(G)$. Using the property that $Pa - aP$ is zero on M_1 , we see that $ax = P^{-1}sPl$. But $aM_4 \cap P^{-1}sPL^2(G) \subseteq sL^2(G)$, so $ax \in sL^2(G)$ and we deduce that $x \in \text{dom}(u)$. Therefore $M \cap \text{dom}(Pu) = M \cap \text{dom}(uP)$. Finally for $x \in M \cap \text{dom}(u)$, we have $Pux = y$ where $sP^{-1}y = ax$ and $uPx = z$ where $sz = aPx$. Thus $sP^{-1}y \in sM_2$ because $aM_3 \cap sL^2(G) \subseteq sM_2$. Since $\ker s = 0$ by Lemma 2.2, we see that $P^{-1}y \in M_2$ and hence $sy = Pax$. Also $Pax = aPx$ because $x \in M_1$. Therefore $sy = sz$ and since $\ker s = 0$ by Lemma 2.2, we deduce that $y = z$. We conclude that $Pu = uP$ on M .

Conversely suppose there is a subspace M of finite codimension in $L^2(G)$ such that $Pu = uP$ on M . Write $u = s^{-1}a = bt^{-1}$ where $a, b, s, t \in W(G)$ and let N be a subspace of finite codimension in $L^2(G)$ such that $tN \subseteq M$. Then $tN \subset \text{dom}(u)$ and $sPu = suP$ on $M \cap \text{dom}(u)$, so $sPbn = aPtn$ for all $n \in N$ and we deduce that $sPb - aPt$ has finite rank. Furthermore $uP^{-1} = P^{-1}u$ on PM , a subspace of finite codimension in $L^2(E) \oplus \mathbb{C}$, so by a similar argument we see that $sP^{-1}b - aP^{-1}t$ also has finite rank. Therefore $u \in R'(G)$ and we conclude from Theorem 1.1 that $u \in D(G)$, as required. \square

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MATH, VPI, BLACKSBURG, VA 24061–0123, USA

E-mail address: linnell@math.vt.edu

URL: <http://www.math.vt.edu/people/linnell/>