ZERO DIVISORS AND $L^p(G)$, II

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Abstract. Let G be a discrete group, let $p \geq 1$, and let $L^p(G)$ denote the Banach space $\{\sum_{g\in G} a_g g \mid \sum_{g\in G} |a_g|^p < \infty\}$. The following problem will be studied: given $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^p(G)$, is $\alpha * \beta \neq 0$? We will concentrate on the case G is a free abelian or free group.

1. Introduction

Let G be a discrete group and let f be a complex-valued function on G. We may represent f as a formal sum $\sum_{g\in G} a_g g$ where $a_g \in \mathbb{C}$ and $f(g) = a_g$. Thus $L^\infty(G)$ will consist of all formal sums $\sum_{g\in G} a_g g$ such that $\sup_{g\in G} |a_g| < \infty$, $C_0(G)$ will consist of those formal sums for which the set $\{g \mid |a_g| > \epsilon\}$ is finite for all $\epsilon > 0$, and for $p \geq 1$, $L^p(G)$ will consist of those formal sums for which $\sum_{g\in G} |a_g|^p < \infty$. Then we have the following inclusions:

$$\mathbb{C}G \subseteq L^p(G) \subseteq C_0(G) \subseteq L^\infty(G).$$

For $\alpha = \sum_{g\in G} a_g g \in L^1(G)$ and $\beta = \sum_{g\in G} b_g g \in L^p(G)$, we define a multiplication $L^1(G) \times L^p(G) \to L^p(G)$ by

$$\alpha * \beta = \sum_{g,h} a_g b_h gh = \sum_{g\in G} \left(\sum_{h\in G} a_{gh^{-1}} b_h \right) g. \tag{1.1}$$

In this paper we consider the following:

Problem 1.1. *Let G be a torsion free group and let $1 \leq p \leq \infty$. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^p(G)$, is $\alpha * \beta \neq 0$?*

Some results on this problem are given in [7, 8]. In this sequel we shall obtain new results for the cases $G = \mathbb{Z}^d$, the free abelian group of rank d, and $G = F_k$, the free group of rank k.

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2. Statement of Main Results

Let $0 \neq \alpha \in L^1(G)$ and let $1 \leq p \in \mathbb{R}$. We shall say that α is a *p-zero divisor* if there exists $\beta \in L^p(G) \setminus 0$ such that $\alpha * \beta = 0$. If $\alpha * \beta \neq 0$ for all $\beta \in C_0(G) \setminus 0$, then we say that α is a *uniform nonzero divisor*.

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Let $2 \leq d \in \mathbb{Z}$. It was shown in [8] that there are p-zero divisors in $\mathbb{C}\mathbb{Z}^d$ for $p > \frac{2d}{d-1}$. In this paper we shall show that this is the best possible by proving

Theorem 2.1. *Let $2 \leq d \in \mathbb{Z}$, $1 \leq p \in \mathbb{R}$, let $0 \neq \alpha \in \mathbb{C}\mathbb{Z}^d$, and let $0 \neq \beta \in L^p(\mathbb{Z}^d)$. If $p \leq \frac{2d}{d-1}$, then $\alpha * \beta \neq 0$.*

Let $\mathbb{T}^d$ denote the d-torus which, except in Section 4, we will view as the cube $[-\pi, \pi]^d$ in $\mathbb{R}^d$ with opposite faces identified, and let $\mathfrak{p}\colon [-\pi,\pi]^d \to \mathbb{T}^d$ denote the natural surjection. For $n \in \mathbb{Z}^d$ and $t \in \mathbb{T}^d$, let $n \cdot t$ indicate the dot product, which is well defined modulo 2π. If $\alpha = \sum_{n \in \mathbb{Z}^d} a_n n \in L^1(\mathbb{Z}^d)$, then for $t \in \mathbb{T}^d$ its Fourier transform $\hat{\alpha}\colon \mathbb{T}^d \to \mathbb{C}$ is defined by

$$\hat{\alpha}(t) = \sum_{n \in \mathbb{Z}^d} a_n e^{-i(n \cdot t)}$$

and we shall let $Z(\alpha) = \{t \in \mathbb{T}^d \mid \hat{\alpha}(t) = 0\}$. We say that M is a hyperplane in $\mathbb{T}^d$ if there exists a hyperplane N in $\mathbb{R}^d$ such that $M = \mathfrak{p}([-\pi,\pi]^d \cap N)$. We will prove the following theorem, which is an improvement over [8, theorem 1].

Theorem 2.2. *Let $\alpha \in \mathbb{C}\mathbb{Z}^d$. Then α is a uniform nonzero divisor if and only if $Z(\alpha)$ is contained in a finite union of hyperplanes in $\mathbb{T}^d$.*

Let $V = \mathfrak{p}\big((-\pi,\pi)^d\big)$, let $\alpha \in L^1(\mathbb{Z}^d)$, let $E = Z(\alpha) \cap V$, and let U be an open subset of $(-\pi,\pi)^{d-1}$. Let $\phi\colon U \to (-\pi,\pi)$ be a smooth map, and suppose $\{\mathfrak{p}(x, \phi(x)) \mid x \in U\} \subseteq E$. If the Hessian matrix

$$\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)$$

of ϕ has constant rank $d - 1 - \nu$ on U where $0 \leq \nu \leq d-1$, then we say that ϕ has constant relative nullity ν. We shall say that $Z(\alpha)$ has *constant relative nullity* ν if every localization ϕ of E has constant relative nullity ν [6, p. 64]. We shall prove

Theorem 2.3. *Let $\alpha \in \mathbb{C}\mathbb{Z}^d$, let $1 \leq p \in \mathbb{R}$, and let $2 \leq d \in \mathbb{Z}$. Suppose that $Z(\alpha)$ is a smooth $(d-1)$-dimensional submanifold of $\mathbb{T}^d$ with constant relative nullity ν such that $0 \leq \nu \leq d-2$. Then α is a p-zero divisor if and only if $p > \frac{2(d-\nu)}{d-1-\nu}$.*

For $k \in \mathbb{Z}_{\geq 0}$, let F_k denote the free group on k generators. It was proven in [7] that if $\alpha \in \mathbb{C}F_k \setminus 0$ and $\beta \in L^2(F_k) \setminus 0$, then $\alpha * \beta \neq 0$. We will give an explicit example to show that if $k \geq 2$, then this result cannot be extended to $L^p(F_k)$ for any $p > 2$. This is a bit surprising in view of Theorem 2.1. We will conclude this paper with some results about p-zero divisors for the free group case.

3. A CHARACTERIZATION OF p-ZERO DIVISORS

Let G be a group, not necessarily discrete, and let $L^p(G)$ be the space of p-integrable functions on G with respect to Haar measure. Let $y \in G$ and let $f \in L^p(G)$. The right translate of f by y will be denoted by f_y, where $f_y(x) = f(xy^{-1})$. Define $T^p[f]$ to be the closure in $L^p(G)$ of all linear combinations of right translates of f. A common problem is to determine when $T^p[f] = L^p(G)$; see [3, 4, 11] for background.

Now suppose that G is also discrete. Given $1 \leq p \in \mathbb{R}$, we shall always let q denote the conjugate index of p. Thus if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$, and if $p = 1$ then

$q = \infty$. Sometimes we shall require $p = \infty$, and then $q = 1$. Let $\alpha = \sum_{g\in G} a_g g \in L^p(G)$, $\beta = \sum_{g\in G} b_g g \in L^q(G)$, and define a map $\langle \cdot,\cdot\rangle \colon L^p(G) \times L^q(G) \to \mathbb{C}$ by

$$\langle \alpha, \beta\rangle = \sum_{g\in G} a_g \overline{b_g}.$$

Fix $h \in L^q(G)$. Then $\langle \cdot, h\rangle$ is a continuous linear functional on $L^p(G)$ and if $p \neq \infty$, then every continuous linear functional on $L^p(G)$ is of this form. We shall use the notation $\tilde{\beta}$ for $\sum_{g\in G} b_g g^{-1}$, $\bar{\beta}$ for $\sum_{g\in G} \overline{b_g} g$, and β^* for $\sum_{g\in G} \overline{b_g} g^{-1}$. Also the same formula in equation (1.1) gives a multiplication $L^p(G) \times L^q(G) \to L^\infty(G)$. Then we have the following elementary lemma, which roughly says that $\alpha * \beta = 0$ if and only if all the translates of α are perpendicular to β.

Lemma 3.1. *Let $1 \leq p \in \mathbb{R}$ or $p = \infty$, let $\alpha \in L^p(G)$, and let $\beta \in L^q(G)$. Then $\alpha * \beta = 0$ if and only if $\langle (\tilde{\alpha})_y, \bar{\beta}\rangle = 0$ for all $y \in G$.*

Proof. Write $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{g\in G} b_g g$. Then

$$\alpha * \beta = \sum_{y\in G} (\sum_{g\in G} a_{yg^{-1}} b_g) y$$

and $\langle (\tilde{\alpha})_y, \bar{\beta}\rangle = \sum_{g\in G} a_{yg^{-1}} b_g$. The result follows. □

The following proposition, which is a generalization of [8, lemma 1], characterizes p-zero divisors in terms of their right translates (the statement of [8, lemma 1] should have the additional condition that $p \neq 1$).

Proposition 3.2. *Let $\alpha \in L^1(G)$ and let $1 < p \in \mathbb{R}$ or $p = \infty$. Then α is a p-zero divisor if and only if $T^q[\tilde{\alpha}] \neq L^q(G)$.*

Proof. The Hahn-Banach theorem tells us that $T^q[\tilde{\alpha}] \neq L^q(G)$ if and only if there exists a nonzero continuous linear functional on $L^q(G)$ which vanishes on $T^q[\tilde{\alpha}]$. The result now follows from Lemma 3.1. □

Remark 3.3. If $p = 1$ in the above Proposition 3.2, we would need to replace $L^q(G)$ with $C_0(G)$, and $T^q[\tilde{\alpha}]$ with the closure in $C_0(G)$ of all linear combinations of right translates of $\tilde{\alpha}$.

4. A KEY PROPOSITION

In this section we prove a proposition that will enable us to prove Theorems 2.1, 2.2 and 2.3.

Let $1 \leq p \in \mathbb{R}$, let $y \in \mathbb{R}^d$ and let $f \in L^p(\mathbb{R}^d)$. We shall use additive notation for the group operation in $\mathbb{R}^d$; thus the right translate of f by y is now given by $f_y = f(x - y)$. We say that f has linearly independent translates if and only if for all $a_1, \dots, a_m \in \mathbb{C}$, not all zero, and for all distinct $y_1, \dots, y_m \in \mathbb{R}^d$,

$$\sum_{i=1}^m a_i f_{y_i} \neq 0.$$

For the rest of this section we shall view $\mathbb{T}^d$ as the unit cube $[0,1]^d$ with opposite faces identified. Let $L^p(\mathbb{T}^d \times \mathbb{Z}^d)$ denote the space of functions on $\mathbb{T}^d \times \mathbb{Z}^d$ which satisfy

$$\int_{t\in\mathbb{T}^d} \sum_{m\in\mathbb{Z}^d} |f(t,m)|^p \, dt < \infty.$$

Then for $\alpha = \sum_{n \in \mathbb{Z}^d} a_n n \in \mathbb{C}\mathbb{Z}^d$ and $f \in L^p(\mathbb{T}^d \times \mathbb{Z}^d)$, we define $\alpha f \in L^p(\mathbb{T}^d \times \mathbb{Z}^d)$ by

$$(\alpha f)(t, m) = \sum_{n \in \mathbb{Z}^d} a_n f(t, m - n),$$

and this yields an action of $\mathbb{C}\mathbb{Z}^d$ on $L^p(\mathbb{T}^d \times \mathbb{Z}^d)$.

Lemma 4.1. *Let* $\alpha \in \mathbb{C}\mathbb{Z}^d$*. Then there exists* $\beta \in L^p(\mathbb{Z}^d) \setminus 0$ *such that* $\alpha * \beta = 0$ *if and only if there exists* $f \in L^p(\mathbb{T}^d \times \mathbb{Z}^d) \setminus 0$ *such that* $\alpha f = 0$*.*

Proof. Let $\beta \in L^p(\mathbb{Z}^d) \setminus 0$ such that $\alpha * \beta = 0$ and define a nonzero function $f \in L^p(\mathbb{T}^d \times \mathbb{Z}^d)$ by $f(t, m) = \beta(m)$. For $n \in \mathbb{Z}^d$, set $b_n = \beta(n)$. Then

$$\begin{aligned}(\alpha f)(t, m) &= \sum_{n \in \mathbb{Z}^d} a_n f(t, m - n) = \sum_{n \in \mathbb{Z}^d} a_n \beta(m - n) \\ &= \sum_{n \in \mathbb{Z}^d} a_n b_{m-n} = (\alpha * \beta)(m) = 0.\end{aligned} \tag{4.1}$$

Conversely suppose there exists $f \in L^p(\mathbb{T}^d \times \mathbb{Z}^d) \setminus 0$ such that $\alpha f = 0$. This means that $(\alpha f)(t, n) = 0$ for all n, for all t except on a set $T_1 \subset \mathbb{T}^d$ of measure zero. Also $\sum_{n \in \mathbb{Z}^d} |f(t, n)|^p < \infty$ for all t except on a set $T_2 \subset \mathbb{T}^d$ of measure zero. Since $f \neq 0$, we may choose $s \in \mathbb{T}^d \setminus (T_1 \cup T_2)$ such that $f(s, n) \neq 0$ for some n. Now define $\beta(n) = f(s, n)$. Then $\beta \in L^p(\mathbb{Z}^d) \setminus 0$ and the calculation in equation (4.1) shows that $\alpha * \beta = 0$. □

For $\alpha = \sum_{n \in \mathbb{Z}^d} a_n n \in \mathbb{C}\mathbb{Z}^d$ and $f \in L^p(\mathbb{R}^d)$, we define $\alpha f \in L^p(\mathbb{R}^d)$ by

$$(\alpha f)(x) = \sum_{n \in \mathbb{Z}^d} a_n f(x - n).$$

If $\alpha \neq 0$ and $\alpha f = 0$, then there is a dependency among the right translates of f, i.e. f does not have linearly independent translates. We are now ready to prove

Proposition 4.2. *Let* $\alpha \in \mathbb{C}\mathbb{Z}^d$*. Then* α *is a* p*-zero divisor if and only if there exists* $f \in L^p(\mathbb{R}^d) \setminus 0$ *such that* $\alpha f = 0$*.*

Proof. Define a Banach space isomorphism $\zeta \colon L^p(\mathbb{R}^d) \to L^p(\mathbb{T}^d \times \mathbb{Z}^d)$ by $(\zeta f)(t, n) = f(t + n)$ for $f \in L^p(\mathbb{R}^d)$. We want to show that this isomorphism commutes with the action of $\mathbb{C}\mathbb{Z}^d$. Clearly it will be sufficient to show that ζ commutes with the action of $\mathbb{Z}^d$. If $m \in \mathbb{Z}^d$, then

$$\begin{aligned}\big(m(\zeta f)\big)(t, n) &= (\zeta f)(t, n - m) = f(t + n - m) \\ &= (mf)(t + n) \quad = (\zeta(mf))(t, n).\end{aligned}$$

Thus the action of $\mathbb{C}\mathbb{Z}^d$ commutes with ζ. We deduce that for $\alpha \in \mathbb{C}\mathbb{Z}^d$, there exists $f \in L^p(\mathbb{R}^d) \setminus 0$ such that $\alpha f = 0$ if and only if there exists $f' \in L^p(\mathbb{T}^d \times \mathbb{Z}^d) \setminus 0$ such that $\alpha f' = 0$. The proposition now follows from Lemma 4.1. □

Remark 4.3. Replacing $L^p(\mathbb{R}^d)$ by $C_0(\mathbb{R}^d)$ in the above arguments, we can also show that α is a uniform nonzero divisor if and only if $\alpha f \neq 0$ for all $f \in C_0(\mathbb{R}^d) \setminus 0$.

5. Proofs of theorems 2.1, 2.2, and 2.3

The proof of Theorem 2.1 is obtained by combining [11, theorem 3] with Proposition 4.2. The proof of Theorem 2.2 is obtained by combining [3, theorem 2.12] with Remark 4.3.

Before we prove Theorem 2.3, we will need to define the notion of a q-thin set. See [4] for more information on this and other concepts used in this paragraph. Let G be a locally compact abelian group and let X be its character group. Let $\beta \in L^\infty(G)$ and let $\hat{\beta}$ indicate the generalized Fourier transform of β. The key reason for using the generalized Fourier transform is that for $\alpha \in L^1(G)$, we have $\widehat{\alpha * \beta} = \hat{\alpha}\hat{\beta}$ which tells us that $\alpha * \beta = 0$ if and only if $\operatorname{supp} \hat{\beta} \subseteq Z(\alpha)$. Let $E \subseteq X$. We shall say that E is q-thin if $\beta \in C_0(G) \cap L^p(G)$ and $\operatorname{supp} \hat{\beta} \subseteq E$ implies $\beta = 0$. Recall that p is the conjugate index of q. The result of Edwards [4, theorem 2.2] says that if $\alpha \in L^1(\mathbb{Z}^d)$ and $Z(\alpha)$ is q-thin, then $T^q[\alpha] = L^q(G)$. Here our q is used in place of Edwards's p, and our p is used in place of his p'.

We are now ready to prove Theorem 2.3. Suppose $Z(\alpha)$ satisfies the hypothesis of the theorem. Let $\beta \in L^p(\mathbb{Z}^d) \setminus 0$ such that $\alpha * \beta = 0$ and $p \leq \frac{2(d-\nu)}{d-1-\nu}$. Since $\frac{2(d-\nu)}{d-1-\nu} > 1$ and increasing p retains the property $\beta \in L^p(\mathbb{Z}^d)$, we may assume that $p > 1$. Then $\tilde{\alpha} * \tilde{\beta} = 0$ and using Proposition 3.2, we see that $T^q[\alpha] \neq L^q(\mathbb{Z}^d)$. But [4, theorem 2.2] tells us that $Z(\alpha)$ is not q-thin, and this contradicts [6, theorem 1].

Conversely, let T be a smooth, nonzero mass density on $Z(\alpha)$ vanishing near the boundary of $Z(\alpha)$. Using [6, theorem 3], we can construct $\beta \in L^p(\mathbb{R}^d) \setminus 0$ for $p > \frac{2(d-\nu)}{d-1-\nu}$ such that $\hat{\beta} = T$. Then $\operatorname{supp} \hat{\beta} \subseteq Z(\alpha)$, that is $\alpha\beta = 0$. An application of Proposition 4.2 completes the proof of Theorem 2.3.

6. Free groups and p-zero divisors

Throughout this section, $2 \leq k \in \mathbb{Z}$. In [7] it was proven that if $0 \neq \alpha \in \mathbb{C}F_k$, then α is not a 2-zero divisor. In this section we will give explicit examples to show that this result cannot be extended to $L^p(F_k)$ for any $p > 2$. We will conclude this section by giving sufficient conditions for elements of $L^1_r(F_k)$, the radial functions of $L^1(F_k)$ as defined below, to be p-zero divisors.

Any element x of F_k has a unique expression as a finite product of generators and their inverses, which does not contain any two adjacent factors ww^{-1} or $w^{-1}w$. The number of factors in x is called the *length* of x and is denoted by $|x|$.

A function in $L^\infty(F_k)$ will be called radial if its value depends only on $|x|$. Let $E_n = \{x \in F_k \mid |x| = n\}$, and let e_n indicate the cardinality of E_n. Then $e_n = 2k(2k-1)^{n-1}$ for $n \geq 1$, and $e_0 = 1$. Let χ_n denote the characteristic function of E_n, so as an element of $\mathbb{C}F_k$ we have $\chi_n = \sum_{|x|=n} x$. Then every radial function has the form $\sum_{n=0}^\infty a_n \chi_n$ where $a_n \in \mathbb{C}$. Let $L^p_r(F_k)$ denote the radial functions contained in $L^p(F_k)$ and let $(\mathbb{C}F_k)_r$ denote the radial functions contained in $\mathbb{C}F_k$. Then $L^p_r(F_k)$ is the closure of $(\mathbb{C}F_k)_r$ in $L^p(F_k)$. Let $\omega = \sqrt{2k-1}$. It was shown in [5, chapter 3] that

$$\begin{aligned} \chi_1 * \chi_1 &= \chi_2 + 2k * \chi_0 \\ \chi_1 * \chi_n &= \chi_{n+1} + \omega^2 \chi_{n-1}, \quad n \geq 2, \end{aligned}$$

hence $L^1_r(F_k)$ is a commutative algebra which is generated by χ_0 and χ_1.

Later we will need the following elementary result.

Lemma 6.1. *Let* $x, y \in F_k$ *with* $|x| = |y|$, *and let* $0 \leq m, n \in \mathbb{Z}$. *Then* $\langle \chi_m * x, \chi_n \rangle = \langle \chi_m * y, \chi_n \rangle$.

Proof. We have $\langle \chi_m * x, \chi_n \rangle = \langle x, \chi_m^* * \chi_n \rangle = \langle x, \chi_m * \chi_n \rangle$. By the above remarks, $\chi_m * \chi_n$ is a sum of elements of the form χ_r. Therefore we need only prove that $\langle x, \chi_r \rangle = \langle y, \chi_r \rangle$. But $\langle x, \chi_r \rangle = 1$ if $|x| = r$ and 0 if $|x| \neq r$, and the result follows. □

Let α be a complex-valued function on F_k. Set

$$a_n(\alpha) = \frac{1}{e_n} \sum_{x \in E_n} \alpha(x)$$

and denote by $P(\alpha)$ the radial function $\sum_{n=0}^{\infty} a_n(\alpha)\chi_n$.

Lemma 6.2. *Let* $1 \leq p \in \mathbb{R}$ *or* $p = \infty$, *let* $\alpha \in L_r^1(F_k)$, *and let* $\beta \in L^p(F_k)$. *If* $\alpha * \beta = 0$, *then* $\alpha * P(\beta) = 0$.

Proof. Let $f, h \in \mathbb{C}F_k$. It was shown in [9, lemma 6.1] that $P(f) * P(h) = P(P(f) * h)$. Write $\beta = \sum_{g \in F_k} b_g g$. If $p \neq \infty$ and $0 \leq a_1, \dots, a_n \in \mathbb{R}$, then by Jensen's inequality [10, p. 189] applied to the function x^p for $x > 0$,

$$\left(\frac{a_1 + \cdots + a_n}{n} \right)^p \leq \frac{a_1^p + \cdots + a_n^p}{n},$$

consequently

$$\|P(\beta)\|_p^p = \sum_{n=0}^{\infty} e_n \left| \frac{1}{e_n} \sum_{|g|=n} b_g \right|^p \leq \sum_{g \in F_k} |b_g|^p = \|\beta\|_p^p.$$

Therefore P is a continuous map from $L^p(F_k)$ into $L_r^p(F_k)$ for $p \neq \infty$. It is also continuous for $p = \infty$. The lemma follows because the map $L^1(G) \times L^p(G) \to L^p(G)$ is continuous; specifically $\|\alpha * \beta\|_p \leq \|\alpha\|_1 \|\beta\|_p$. □

For $n \in \mathbb{Z}_{\geq 0}$, define polynomials P_n by

$$P_0(z) = 1, \quad P_1(z) = z, \quad P_2(z) = z^2 - 2k$$
$$\text{and } P_{n+1}(z) = zP_n(z) - \omega^2 P_{n-1}(z) \text{ for } n \geq 2.$$

Let $\alpha = \sum_{n=0}^{\infty} a_n \chi_n \in L_r^1(F_k)$. In [9], Pytlik shows the following.

1. $X = \{x + iy \in \mathbb{C} \mid (\frac{x}{2k})^2 + (\frac{y}{2k-2})^2 \leq 1\}$ is the spectrum of $L_r^1(F_k)$.
2. The Gelfand transform of α is given by $\hat{\alpha}(z) = \sum_{n=0}^{\infty} a_n P_n(z)$ for $z \in X$.

Let $Z(\alpha) = \{z \in X \mid \hat{\alpha}(z) = 0\}$. For $z \in X$ we define $\phi_z \in L_r^\infty(F_k)$, the space of continuous linear functionals on $L_r^1(F_k)$ [1, p. 34], by

$$\phi_z = \sum_{n=0}^{\infty} \frac{P_n(z)}{e_n} \chi_n.$$

We can now state

Lemma 6.3. *Let* $\alpha \in L_r^1(F_k)$ *and let* $z \in X$. *Then* $\alpha * \overline{\phi_z} = 0$ *if and only if* $z \in Z(\alpha)$.

Proof. Let $\beta \in L^1_r(F_k)$ and write $\beta = \sum_{m=0}^{\infty} b_m \chi_m$. Then

$$\begin{aligned}\langle \beta, \overline{\phi_z} \rangle &= \sum_{m,n} \frac{b_m P_n(z)}{e_n} \langle \chi_m, \chi_n \rangle \\ &= \sum_n b_n P_n(z) = \hat{\beta}(z).\end{aligned}$$

Applying this in the case $\beta = \alpha * \chi_n$, we obtain $\langle \alpha * \chi_n, \overline{\phi_z} \rangle = \hat{\alpha}(z) P_n(z)$. Using Lemma 6.1, we deduce that if $y \in F_k$ and $|y| = n$, then $\langle \alpha * y, \phi_z \rangle = \hat{\alpha}(z) P_n(z)/e_n$. Since $\alpha = \tilde{\alpha}$, the result now follows from Lemma 3.1. $\square$

If $\alpha \in L^1_r(F_k)$, we shall say that $\alpha * \chi_n$ is a radial translate of α. We then set $TR^1[\alpha]$ equal to the closure in $L^1_r(F_k)$ of the set of linear combinations of radial translates of α.

Proposition 6.4. *Let $\alpha \in L^1_r(F_k)$. Then $\alpha * \beta \neq 0$ for all $\beta \in L^\infty(F_k) \setminus 0$ if and only if $Z(\alpha) = \emptyset$.*

Proof. If $z \in Z(\alpha)$, then $\phi_z \in L^\infty(F_k) \setminus 0$ and $\alpha * \overline{\phi_z} = 0$ by Lemma 6.3.

Conversely suppose there exists $\beta \in L^\infty(F_k) \setminus 0$ such that $\alpha * \beta = 0$. Then $\beta(y) \neq 0$ for some $y \in F_k$, so replacing β with $\beta * y^{-1}$, we may assume that $P(\beta) \neq 0$. If $\gamma = \bar{\beta}$, then $\alpha * \bar{\gamma} = 0$ and $P(\gamma) \neq 0$. Using Lemma 6.2 we see that $\alpha * \overline{P(\gamma)} = 0$, and we deduce from Lemma 3.1 that $\langle \alpha_y, P(\gamma) \rangle = 0$ for all $y \in F_k$. It follows that $\langle \alpha * \chi_n, P(\gamma) \rangle = 0$ for all $n \in \mathbb{Z}_{\geq 0}$, consequently $TR^1[\alpha] \neq L^1_r(F_k)$. Let J be a maximal ideal in $L^1_r(F_k)$ which contains $TR^1[\alpha]$. By Gelfand theory there exists $z \in X$ such that $J = \{\delta \in L^1_r(F_k) \mid \hat{\delta}(z) = 0\}$, so $z \in Z(\gamma)$. $\square$

We can now state

Example 6.5. *Let $k \geq 2$. Then χ_1 is a p-zero divisor for all $p > 2$.*

Proof. Since $0 \in Z(\chi_1)$, we see from Lemma 6.3 that $\chi_1 * \phi_0 = 0$. Of course $\phi_0 \neq 0$. We now prove the stronger statement that $\phi_0 \in L^p(F_k)$ for all $p > 2$. We have

$$\phi_0 = \sum_{n=0}^{\infty} \frac{P_n(0)}{e_n} \chi_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2k-1)^n} \chi_{2n}.$$

Therefore

$$\begin{aligned}\sum_{g \in F_k} |\phi_0(g)|^p &= 1 + \sum_{n=1}^{\infty} \frac{e_{2n}}{(2k-1)^{pn}} = 1 + \sum_{n=1}^{\infty} \frac{2k(2k-1)^{2n-1}}{(2k-1)^{pn}} \\ &= 1 + \frac{2k}{2k-1} \sum_{n=1}^{\infty} \frac{1}{(2k-1)^{n(p-2)}}\end{aligned}$$

and the result follows. $\square$

We can use the above result to prove that the nonsymmetric sum of generators in F_k is a p-zero divisor for all $p > 2$ in the case k is even and $k > 2$. Specifically we have

Example 6.6. *Let $k > 3$ and let $\{x_1, \dots, x_k\}$ be a set of generators for F_k. If k is even, then $x_1 + \dots + x_k$ is a p-zero divisor for all $p > 2$.*

To establish this, we need some results about free groups.

Lemma 6.7. *Let $0 < n \in \mathbb{Z}$ and let F be the free group on $x_1, \dots, x_n$. Then no nontrivial word in the $2n-1$ elements $x_1^2, \dots, x_n^2, x_1x_2, x_2x_3, \dots, x_{n-1}x_n$ is the identity; in particular these $2n-1$ elements generate a free group of rank $2n-1$.*

Proof. The result is clearly true if $n = 1$, so we may suppose that $n > 1$. We shall use induction on n, so assume that the result is true with $n-1$ in place of n. Let T denote the Cayley graph of F with respect to the generators $x_1, \dots, x_n$. Thus the vertices of T are the elements of F, and $f, g \in F$ are joined by an edge if and only if $f = gx_i^{\pm 1}$ for some i. Suppose a nontrivial word in $x_1^2, \dots, x_n^2, x_1x_2, x_2x_3, \dots, x_{n-1}x_n$ is the identity, and choose such a word w with shortest possible length.

Note that w must involve x_1^2, because F is the free product of the group generated by $x_2, \dots, x_n$ and the group generated by x_1x_2. By conjugating and taking inverses if necessary, we may assume without loss of generality that w begins with x_1^2.

Write $w = w_1 \dots w_m$, where $w_1 = x_1^2$, and each of the w_i are one of the above $2n-1$ elements. Let us consider the path whose $(2i+1)$th vertex is $w_1 \dots w_i$. Note that $w = 1$, but $w_1 \dots w_i \neq 1$ for $0 < i < m$.

Observe that the path of length 2 from x_1^2 to $x_1^2 w_2$ cannot go through x_1 (just go through the $4n-2$ possibilities for w_2, noting that $w_2 \neq x_1^{-2}$). Now remove the edge joining x_1 and x_1^2. Since T is a tree [2, I.8.2 theorem], the resulting graph will become two trees; one component T_1 containing 1 and the other component T_2 containing x_1^2. Since the length 2 path from x_1^2 to $x_1^2 w_2$ did not go through x_1, for $i \geq 1$ the path $w_1 w_2 \dots w_i$ remains in T_2 at least until it passes through x_1^2 again. Also the path must pass through x_1^2 again in order to get back to 1. Since the paths $w_1 \dots w_i$ all have even length (all the w_i are words of length 2), it follows that $w_1 \dots w_l = x_1^2$ for some $l \in \mathbb{Z}$, where $2 \leq l < m$. We deduce that $w_2 \dots w_l = 1$, which contradicts the minimality of the length of w. □

Corollary 6.8. *Let $n \in \mathbb{Z}_{\geq 1}$ and let F be the free group on $x_1, \dots, x_n$. Then no nontrivial word in the $2n-1$ elements $x_1^2, \dots, x_n^2, x_1^{-1}x_2, x_2^{-1}x_3, \dots, x_{n-1}^{-1}x_n$ is the identity; in particular these $2n-1$ elements generate a free group of rank $2n-1$.*

Proof. This follows immediately from Lemma 6.7: replace $x_i x_{i+1}$ with $x_i^{-2} x_i x_{i+1}$ for all $i < n$. □

Corollary 6.9. *Let $n \in \mathbb{Z}_{\geq 1}$ and let F be the free group on $x_1, \dots, x_n, w$. Then the elements $wx_1, wx_1^{-1}, \dots, wx_n, wx_n^{-1}$ generate a free subgroup of rank $2n$.*

Proof. The above elements generate the subgroup generated by

$$x_1^2, \dots, x_n^2, x_1^{-1}x_2, x_2^{-1}x_3, \dots, x_{n-1}^{-1}x_n, wx_1.$$

The result follows from Corollary 6.8. □

Proof of Example 6.6. Let $G = F_k$ and let F be the free group on $y_1, \dots, y_k, w$. By Corollary 6.9 there is a monomorphism $\theta \colon G \to F$ determined by the formula

$$\theta(x_1) = wy_1, \quad \theta(x_2) = wy_1^{-1}, \quad \dots, \quad \theta(x_k) = wy_{k/2}^{-1}.$$

Note that θ induces a Banach space monomorphism $L^p(G) \to L^p(F)$. Set $\alpha = wy_1 + wy_1^{-1} + \dots + wy_{k/2} + wy_{k/2}^{-1}$. Since $y_1 + y_1^{-1} + \dots + y_{k/2} + y_{k/2}^{-1}$ is a p-zero divisor by Example 6.5, we see that α is a p-zero divisor, say $\alpha * \beta = 0$ where $0 \neq \beta \in L^p(F)$. Write $F = \bigcup_{t \in T} \theta(G)t$ where T is a right transversal for $\theta(G)$ in F. Then $\beta = \sum_{t \in T} \beta_t t$ where $\beta_t \in L^p(\theta(G))$ for all t. Also $\alpha * \beta_t = 0$ for all t and

$\beta_s \neq 0$ for some $s \in T$. Define $\gamma \in L^p(G)$ by $\theta(\gamma) = \beta_s$. Then $0 \neq \gamma \in L^p(G)$ and $(x_1 + \cdots + x_k) * \gamma = 0$ as required. ☐

We conclude with some information on the existence of p-zero divisors in $L^1_r(F_k)$. Let $\alpha \in L^1_r(F_k)$ and define $p(\alpha)$ as follows. If $Z(\alpha) \cap (-2k, 2k) = \emptyset$, then set $p(\alpha) = \infty$. If $Z(\alpha) \cap (-2k, 2k) \neq \emptyset$, then set $m(\alpha) = \min\{|t| \mid t \in Z(\alpha) \cap (-2k, 2k)\}$. If $m(\alpha) \in [0, 2\omega]$, then set $p(\alpha) = 2$. Finally if $m(\alpha) \in (2\omega, 2k)$, then let $p(\alpha)$ be the positive root of the following equation in p:

$$m(\alpha) = \sqrt{2k-1}\big((2k-1)^{\frac{1}{2}-\frac{1}{p}} + (2k-1)^{\frac{1}{p}-\frac{1}{2}}\big).$$

We can now state

Proposition 6.10. *Let $\alpha \in L^1_r(F_k)$. Then α is a p-zero divisor for all $p > p(\alpha)$.*

Proof. Let $t \in (-2k, 2k)$ such that $m(\alpha) = |t|$ and suppose $p > p(\alpha)$. Since ϕ_t is a positive definite function by [9, lemma 6.1], we can apply [1, theorem 2(a)] to deduce that $\phi_t \in L^p_r(F_k)$. By Lemma 6.3 $\alpha * \phi_t = 0$ and the result is proven. ☐

References

[1] Joel M. Cohen and Leonede De-Michele. The radial Fourier-Stieltjes algebra of free groups. In *Operator algebras and K-theory (San Francisco, Calif., 1981)*, pages 33–40. Amer. Math. Soc., Providence, R.I., 1982.

[2] Warren Dicks and M. J. Dunwoody. *Groups acting on graphs.* Cambridge University Press, Cambridge, 1989.

[3] G. A. Edgar and J. M. Rosenblatt. Difference equations over locally compact abelian groups. *Trans. Amer. Math. Soc.*, 253:273–289, 1979.

[4] R. E. Edwards. Spans of translates in $L^p(G)$. *J. Austral. Math. Soc.*, 5:216–233, 1965.

[5] Alessandro Figà-Talamanca and Massimo A. Picardello. *Harmonic analysis on free groups.* Marcel Dekker Inc., New York, 1983.

[6] Kang Hui Guo. On the p-thin problem for hypersurfaces of $\mathbf{R}^n$ with zero Gaussian curvature. *Canad. Math. Bull.*, 36(1):64–73, 1993.

[7] Peter A. Linnell. Zero divisors and $L^2(G)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 315(1):49–53, 1992.

[8] Michael J. Puls. Zero divisors and $L^p(G)$. *Proc. Amer. Math. Soc.*, 126(3):721–728, 1998.

[9] T. Pytlik. Radial functions on free groups and a decomposition of the regular representation into irreducible components. *J. Reine Angew. Math.*, 326:124–135, 1981.

[10] A. Wayne Roberts and Dale E. Varberg. *Convex functions.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973. Pure and Applied Mathematics, Vol. 57.

[11] Joseph Rosenblatt. Linear independence of translations. *J. Austral. Math. Soc. Ser. A*, 59(1):131–133, 1995.

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