

LEFT ORDERED GROUPS WITH NO NONABELIAN FREE SUBGROUPS

PETER A. LINNELL

ABSTRACT. There has been interest recently concerning when a left ordered group is locally indicable. Bergman and Tararin have shown that not all left ordered groups are locally indicable, but all known examples contain a nonabelian free subgroup. We shall show for a large class of groups not containing a nonabelian free subgroup, that any left ordered group in this class is locally indicable. We shall also show that certain free products with an amalgamated cyclic subgroup are left orderable.

1. INTRODUCTION

A group G is left ordered if it has a total ordering \leq such that $x \leq y \Rightarrow gx \leq gy$ whenever $g, x, y \in G$. For much information on left ordered groups, see the books [14, 18]. Of course we say that a group G is *right* ordered if it has a total ordering \leq such that $x \leq y \Rightarrow xg \leq yg$ whenever $g, x, y \in G$. However using the involution $g \mapsto g^{-1}$ of G , it is easy to see that a group is right orderable if and only if it is left orderable.

Recall that a group is *locally indicable* if and only if every finitely generated subgroup $\neq 1$ has an infinite cyclic quotient. Every locally indicable group is left orderable [18, theorem 7.3.1], but the converse is not true, as has been shown by Bergman [2] and Tararin [22]. On the other hand Chiswell and Kropholler [4, theorem A] showed that a solvable-by-finite left ordered group is locally indicable; also Tararin [21, theorem 3] has proved that if $A \triangleleft G \neq 1$ are groups with G/A finitely generated and solvable, A abelian and G left orderable, then G has a quotient isomorphic to an infinite subgroup of \mathbb{Q} . Further results in this direction were obtained in [17]. In [16] it was proved that every left ordered elementary amenable group is locally indicable, and the question was raised of whether every left ordered amenable group is locally indicable. Let NF denote the class of groups which contain no nonabelian free subgroup. We shall consider the following stronger statement.

Conjecture 1.1. *A left ordered NF-group is locally indicable.*

We shall now describe the class of groups for which we shall prove Conjecture 1.1. Let \mathcal{P} denote the group of piecewise linear orientation preserving self homeomorphisms of the unit interval $[0, 1]$ with multiplication defined as composition of functions. Thus if $f, g \in \mathcal{P}$, then f is differentiable at all but a finite number of points, and $(fg)(x) = f(g(x))$ for all $x \in [0, 1]$. Also let NS denote the class of

Date: July 12, 2000.

1991 Mathematics Subject Classification. Primary: 20F60; Secondary: 06F15, 57M07.

Key words and phrases. left ordered group, locally indicable, free group, piecewise linear homeomorphisms, Thompson's group.

groups which have no nonabelian free subsemigroup. Now define \mathcal{C} to be the smallest class of groups which contains NS and \mathcal{P} , and is closed under taking subgroups, homomorphic images, group extensions and directed unions. Clearly \mathcal{C} contains all elementary amenable groups and in particular all solvable by finite groups, and it is not difficult to show that $\mathcal{C} \subseteq \text{NF}$ (see Corollary 4.8). Moreover \mathcal{C} contains groups which are not elementary amenable, such as the ubiquitous Thompson's group (see [3] for more information on this topic). Presumably not every NF-group lies in the class \mathcal{C} , though I know of no explicit example in the literature. We can now state

Theorem 1.2. *A \mathcal{C} -group is left orderable if and only if it is locally indicable.*

Of course the result that G is locally indicable (whether or not $G \in \mathcal{C}$) implies that G is left orderable has already been noted above. For the reverse implication, we prove a stronger result Theorem 4.12 which states that if $F \neq 1$ is a finitely generated left orderable group and $F \supseteq G \in \mathcal{C}$, then there exists a left-relatively convex subgroup $H \neq F$ (see Section 2) such that $H \cap G \triangleleft G$, and $G/H \cap G$ has a self centralizing torsion free normal abelian subgroup $A/H \cap G$ such that G/A is torsion free abelian. Thus in the special case $F = G$ (so G is finitely generated and $\neq 1$), we see that G has a quotient isomorphic to \mathbb{Z} .

In Section 5 we shall use Theorem 1.2 to prove the following result about $\text{Homeo}_+(S^1)$, the group of orientation preserving homeomorphisms of the circle.

Corollary 1.3. *Let G be a finitely generated subgroup of $\text{Homeo}_+(S^1)$ such that $G \in \mathcal{C}$. Then*

- (i) *If G is finite, then G is cyclic.*
- (ii) *If G is infinite, then there exists $K \triangleleft H \triangleleft G$ such that G/H is cyclic and $H/K \cong \mathbb{Z}$.*

This should be compared with [8, theorem 1.1], where by considering the smaller group of orientation preserving C^∞ diffeomorphisms of S^1 , similar but stronger results were obtained.

In Section 6 we shall use some of the techniques in this paper to show that certain free products with amalgamation are left orderable. For example, we shall show in Theorem 6.3 that the free product of a left orderable group and a torsion free nilpotent group with an amalgamated cyclic subgroup is left orderable. In the final section we shall briefly consider some examples of left ordered groups which are not locally indicable.

Part of this work was carried out while I was at the Sonderforschungsbereich in Münster. I would like to thank Wolfgang Lück for organizing my visit to Münster, and the Sonderforschungsbereich for financial support.

2. NOTATION, TERMINOLOGY AND ASSUMED RESULTS

As usual \mathbb{Q} , \mathbb{R} , \mathbb{Z} and \mathbb{N} will denote the rational numbers, real numbers, integers and natural numbers $\{1, 2, \dots\}$ respectively. We shall use the notation G' for the commutator subgroup of the group G , and if $g, x \in G$ and $X \subseteq G$, then $x^g = gxg^{-1}$, $X^g = gXg^{-1}$, $\mathbf{C}_G(X) = \{g \in G \mid x^g = x \text{ for all } x \in X\}$, $\mathbf{C}_G(x) = \mathbf{C}_G(\{x\})$, and $\langle X \rangle$ denotes the subgroup generated by X . Also if $H \leq G$, then $\text{core}_G(H) = \bigcap_{g \in G} H^g$, the largest normal subgroup of G contained in H .

All mappings will be written on the left, in particular all group actions will have the group acting on the left of the set. If G is acting on a set Y and $Z \subseteq Y$, then

$\text{Stab}_G(Z)$ will always denote the pointwise stabilizer of Z in G : thus $\text{Stab}_G(Z) = \{g \in G \mid gz = z \text{ for all } z \in Z\}$, and we write $\text{Stab}_G(y)$ for $\text{Stab}_G(\{y\})$ when $y \in Y$. Also if $H \subseteq G$, then $\text{Fix}_Y(H)$ is the fixed points of H , that is $\{y \in Y \mid hy = y \text{ for all } h \in H\}$, and when $g \in G$ we write $\text{Fix}_Y(g)$ for $\text{Fix}_Y(\{g\})$. Then obviously $\text{Fix}_Y(X) = \text{Fix}_Y(\langle X \rangle)$ whenever $X \subseteq G$.

A totally ordered set X is a set with a binary relation \leq such that for $x, y, z \in X$, either $x \leq y$ or $y \leq x$, $x \leq y$ and $y \leq x$ implies $x = y$, and $x \leq y \leq z$ implies $x \leq z$. Given totally ordered sets X and Y , the map $\theta: X \rightarrow Y$ is said to be order preserving if $x < y$ implies $\theta x < \theta y$ whenever $x, y \in X$. We shall let $\text{Aut}(X)$ denote the group of all order preserving permutations $X \rightarrow X$. Note that if θ is an order preserving bijection $X \rightarrow Y$, then θ^{-1} is also order preserving and thus $\text{Aut}(X)$ is indeed a group. Also if $X \subseteq \mathbb{R}$ and X is given the order induced by the natural order on \mathbb{R} , then the elements of $\text{Aut}(X)$ are homeomorphisms of X .

If (G, \leq) is a left ordered group and $K \leq G$, then we say that K is a convex subgroup of G if $g \in G$, $j, k \in K$ and $j \leq g \leq k$ implies $g \in K$. In this case the left cosets of K in G , which we denote by G/K , is naturally a totally ordered set under the definition $gK < hK$ if and only if $g < h$ and $gK \neq hK$, for $g, h \in G$. Furthermore G then acts as order preserving permutations on G/K according to the rule $g(hK) = ghK$. We say that a subgroup of G is a *left-relatively convex* subgroup [14, p. 127] if it is convex with respect to some left order on G . Conversely suppose G acts faithfully as order preserving permutations on some totally ordered set X . Then, as described in [18, theorem 7.1.2], we can make G into a left ordered group as follows. Well order X , and then for $f, g \in G$ with $f \neq g$, we say that $f < g$ if and only if $f(x) < g(x)$ where x is the least element of X such that $f(x) \neq g(x)$. Note that if $y \in X$ and Y is the set of all elements less than y , then $\text{Stab}_G(Y)$ is a convex subgroup of G under this order and consequently $\text{Stab}_G(Y)$ is a left-relatively convex subgroup. Therefore $\text{Stab}_G(Y_0)$ is a left-relatively convex subgroup of G for any subset Y_0 of X . We need the following basic results about left-relatively convex subgroups.

Lemma 2.1. *Let G be a left ordered group, let H be a normal convex subgroup of G , and let \mathcal{B} be a set of left-relatively convex subgroups of G . Then*

- (i) $\bigcap_{B \in \mathcal{B}} B$ is a left-relatively convex subgroup of G .
- (ii) If \mathcal{B} is totally ordered by inclusion, then $\bigcup_{B \in \mathcal{B}} B$ is a left-relatively convex subgroup of G .
- (iii) If B/H is a left-relatively convex subgroup of G/H , then B is a left-relatively convex subgroup of G .

Proof. For (i) see [14, proposition 5.1.10] or [16, lemma 2.2(i)]. For (ii) see [14, proposition 5.1.7] or [16, lemma 2.2(ii)]. Finally for (iii), see [16, lemma 2.1]. \square

We define PLO to be the class of groups which act faithfully as piecewise linear orientation preserving self homeomorphisms of $[0, 1]$, and PLT the class of groups which act faithfully as piecewise linear orientation preserving self homeomorphisms of $[0, 1]$ which do not have a common fixed point in $(0, 1)$. Thus $\text{PLT} \subset \text{PLO}$, $G \in \text{PLO}$ if and only if G is isomorphic to a subgroup of \mathcal{P} , and $G \in \text{PLT}$ if and only if G acts faithfully as piecewise linear orientation preserving self homeomorphisms of $[0, 1]$, and given $\epsilon > 0$ and $x \in (0, 1)$, there exist $f, g \in G$ such that $f(x) < \epsilon$ and $g(x) > 1 - \epsilon$.

Finally in this section, we need the following refinement of the well known fact that a countable left ordered group can be considered as a subgroup of $\text{Aut}(\mathbb{R})$ (see for example, [23, lemma 2.2]).

Lemma 2.2. *Let G be a countable left ordered group, and let H be a convex subgroup of G such that $H \neq G$. Then there is an order preserving action of G on \mathbb{R} with kernel $\text{core}_G(H)$ such that $\text{Stab}_G(0) = H$ and $\text{Stab}_G(v) \neq G$ for all $v \in \mathbb{R}$.*

Proof. This follows from [16, lemmas 2.4 and 2.3]. \square

3. EXTENSION CLOSED CLASSES OF GROUPS

Very similar results to the next lemma have been proved before, see for example [16, lemma 3.1]. We shall prove a more general result, which will hopefully avoid the need for further similar results. If \mathcal{D} is a class of groups which is closed under taking subgroups, then we shall define $\bar{\mathcal{D}}$ to be the smallest class of groups containing \mathcal{D} which is closed under group extension and is closed under directed unions. For arbitrary classes of groups \mathcal{X} and \mathcal{Y} , we shall let $H \in L\mathcal{X}$ mean that every finite subset of the group H is contained in an \mathcal{X} -subgroup, $H \in Q\mathcal{X}$ to mean that H is isomorphic to a quotient group of an \mathcal{X} -group, and $H \in \mathcal{X}\mathcal{Y}$ mean that H has a normal \mathcal{X} -subgroup X such that $H/X \in \mathcal{Y}$. If \mathcal{X} is subgroup closed, then $H \in L\mathcal{X}$ if and only if every finitely generated subgroup of H is an \mathcal{X} -group. For each ordinal α , the class of groups \mathcal{D}_α is defined inductively by $\mathcal{D}_0 = \{1\}$, $\mathcal{D}_{\alpha+1} = (L\mathcal{D}_\alpha)\mathcal{D}$ and $\mathcal{D}_\beta = \bigcup_{\alpha < \beta} \mathcal{D}_\alpha$ if β is a limit ordinal. Setting $\mathcal{X} = \bigcup_{\alpha \geq 0} \mathcal{D}_\alpha$, we can state

Lemma 3.1. (i) *Each \mathcal{D}_α and \mathcal{X} is subgroup closed.*
(ii) *If \mathcal{D} is quotient group closed, then each \mathcal{D}_α and \mathcal{X} is also quotient group closed.*
(iii) $\mathcal{X} = \bar{\mathcal{D}}$.

Proof. (i) This is easily proved by induction on α , using the fact that \mathcal{D} is subgroup closed.

(ii) This is also easily proved by induction on α , using the fact that \mathcal{D} is quotient group closed.

(iii) Clearly $\mathcal{X} \subseteq \bar{\mathcal{D}}$, $\mathcal{X} \supseteq \mathcal{D}$, and \mathcal{X} is closed under directed unions. Therefore we need to prove that \mathcal{X} is extension closed.

We show by induction on β that $\mathcal{D}_\alpha \mathcal{D}_\beta \subseteq \mathcal{D}_{\alpha+\beta}$; the case $\beta = 0$ being obvious. If $\beta = \gamma + 1$ for some ordinal γ , then

$$\begin{aligned} \mathcal{D}_\alpha \mathcal{D}_\beta &= \mathcal{D}_\alpha ((L\mathcal{D}_\gamma)\mathcal{D}) \subseteq (\mathcal{D}_\alpha (L\mathcal{D}_\gamma))\mathcal{D} \subseteq (L(\mathcal{D}_\alpha \mathcal{D}_\gamma))\mathcal{D} \\ &\subseteq (L\mathcal{D}_{\alpha+\gamma})\mathcal{D} \quad (\text{by induction}) \\ &= \mathcal{D}_{\alpha+\beta}. \end{aligned}$$

On the other hand if β is a limit ordinal, then $\mathcal{D}_\beta = \bigcup_{\gamma < \beta} \mathcal{D}_\gamma$ and

$$\begin{aligned} \mathcal{D}_\alpha \mathcal{D}_\beta &= \mathcal{D}_\alpha \left(\bigcup_{\gamma < \beta} \mathcal{D}_\gamma \right) = \bigcup_{\gamma < \beta} \mathcal{D}_\alpha \mathcal{D}_\gamma \\ &\subseteq \bigcup_{\gamma < \beta} \mathcal{D}_{\alpha+\gamma} \quad (\text{by induction}) \\ &\subseteq \mathcal{D}_{\alpha+\beta} \end{aligned}$$

as required. \square

For the rest of this paper, we let $\mathcal{D} = \text{NS} \cup \text{Q}(\text{PLO})$. Then clearly $\bar{\mathcal{D}} \subseteq \mathcal{C}$ and \mathcal{D} is closed under taking subgroups, quotient groups, consequently $\mathcal{C} = \bar{\mathcal{D}} = \mathcal{X}$.

4. PROOF OF THE MAIN THEOREM

The statement and proof of the next lemma is just a reformulation of [20, assertion 2.1].

Lemma 4.1. *Let W be a nonempty closed subset of \mathbb{R} and let $\alpha, \beta, z \in \text{Aut}(W)$. Suppose $\text{Fix}_W(z) = \emptyset$ and z commutes with α and β . If $\text{Fix}_W(\alpha) \neq \emptyset \neq \text{Fix}_W(\beta)$ and $\text{Fix}_W(\alpha) \cap \text{Fix}_W(\beta) = \emptyset$, then $\langle \alpha, \beta \rangle$ contains a nonabelian free subgroup.*

Proof. Without loss of generality we may assume that $0, 1 \in W$, $\beta(0) = 0$, and $z(0) = 1$. Then $\beta(1) = 1$. Suppose that $\text{Fix}_W(\alpha) \neq \emptyset$ yet $\text{Fix}_W(\alpha) \cap \text{Fix}_W(\beta) = \emptyset$. We may write $\mathbb{R} \setminus \text{Fix}_W(\alpha)$ and $\mathbb{R} \setminus \text{Fix}_W(\beta)$ as a disjoint union of open intervals, which we shall call \mathcal{A} and \mathcal{B} respectively. Then a finite number n of these intervals will cover $[0, 1]$; let these intervals be (a_0, a_1) , (b_1, b_2) , (a_2, a_3) , (b_3, b_4) , \dots , (b_{n-2}, b_{n-1}) , (a_{n-1}, a_n) (so n is an odd integer) where the (a_{2i}, a_{2i+1}) are intervals in \mathcal{A} , the (b_{2i+1}, b_{2i+2}) are intervals in \mathcal{B} , and $a_0 < 0 < a_1 \leq a_2 < a_3 \leq \dots \leq a_{n-1} < 1 < a_n$, $0 < b_1 < b_2 \leq b_3 < b_4 < \dots < b_{n-1} < 1$. Note that $za_0 = a_{n-1}$ and $za_1 = a_n$, and $a_i \in \text{Fix}_W(\alpha)$ and $b_i \in \text{Fix}_W(\beta)$ for all i . Also $(b_i, a_i) \cap W \neq \emptyset$ if i is odd, and $(a_i, b_i) \cap W \neq \emptyset$ if i is even. To see this let us consider the former case. We have $b_i \in \text{Fix}_W(\beta)$, so certainly $b_i \in W$. Also $b_i \notin \text{Fix}_W(\alpha)$, so by replacing α with α^{-1} if necessary, we may assume that $\alpha(b_i) > b_i$. Then $\alpha(b_i) \in W$ and since $\alpha(a_i) = a_i$, it follows that $\alpha(b_i) \in (b_i, a_i)$. Similarly if i is even, we can show that $(a_i, b_i) \cap W \neq \emptyset$. Now choose $x_i \in (b_i, a_i) \cap W$ if i is odd ($1 \leq i \leq n-2$), and $x_i \in (a_i, b_i) \cap W$ ($2 \leq i \leq n-1$) if i is even. Finally set $x_0 = z^{-1}x_{n-1}$.

Set $P_1 = (x_0, x_1) \cup (x_2, x_3) \cup \dots \cup (x_{n-3}, x_{n-2})$ and $Q_1 = (x_1, x_2) \cup (x_3, x_4) \cup \dots \cup (x_{n-2}, x_{n-1})$, and for $r \in \mathbb{Z}$ define

$$\begin{aligned} z^r P_1 &= (z^r x_0, z^r x_1) \cup (z^r x_2, z^r x_3) \cup \dots \cup (z^r x_{n-3}, z^r x_{n-2}) \\ z^r Q_1 &= (z^r x_1, z^r x_2) \cup (z^r x_3, z^r x_4) \cup \dots \cup (z^r x_{n-2}, z^r x_{n-1}). \end{aligned}$$

Now set $P = \bigcup_{r \in \mathbb{Z}} z^r P_1$ and $Q = \bigcup_{r \in \mathbb{Z}} z^r Q_1$. Observe that $P \cap Q = \emptyset$. Indeed if $y \in P \cap Q$, then by translating by z^r for suitable r , we may assume that $y \in (0, 1)$, and then the result is clear.

If i is even, then $(x_i, x_{i+1}) \subset (a_i, a_{i+1})$. Since $\text{Fix}_W(\alpha) \cap (a_i, a_{i+1}) = \emptyset$, we see that either $\alpha(x) > x$ for all $x \in (a_i, a_{i+1}) \cap W$, or $\alpha(x) < x$ for all $x \in (a_i, a_{i+1}) \cap W$; without loss of generality we may assume that $\alpha(x) > x$. Now $a_i, a_{i+1} \in \text{Fix}_W(\alpha)$, $a_i < x_i < b_i$ and $b_{i+1} < x_{i+1} < a_{i+1}$, hence there exists a positive integer p_i such that $\alpha^r(x_i, x_{i+1}) \subset (x_{i+1}, a_{i+1})$ and $\alpha^{-r}(x_i, x_{i+1}) \subset (a_i, x_i)$ for all $r > p_i$. Let p be the maximum of the of the p_i ($0 \leq i \leq n-3$). Since (x_{i+1}, a_{i+1}) , $(a_i, x_i) \subset Q$, we see that $\alpha^{pr}(x_i, x_{i+1}) \subset Q$ for all i and for all $r \neq 0$, and it follows that $\alpha^{pr}P \subset Q$ for all $r \neq 0$. Similarly there exists a positive integer q such that $\beta^{qr}Q \subset P$ for all $r \neq 0$. It now follows from Klein's Table Tennis lemma [5, p. 130] that $\langle \alpha^p, \beta^q \rangle$ is a free group. \square

Lemma 4.2. *Let W be a nonempty closed subset of \mathbb{R} and let $G, Z \leq \text{Aut}(W)$. Suppose $G \in \text{NF}$, Z centralizes G and $\text{Fix}_W(Z) = \emptyset$. Let $H = \{g \in G \mid \text{Fix}_W(g) \neq \emptyset\}$. Then $G' \subseteq H \triangleleft G$ and $\text{Fix}_W(F) \neq \emptyset$ for every finitely generated subgroup F of H .*

Proof. Let $\{f_1, \dots, f_n\}$ be a finite subset of H , and set $F = \langle f_1, \dots, f_n \rangle$. The result will follow if we can prove that $\text{Fix}_W(F) \neq \emptyset$, because then clearly $H \triangleleft G$, and G/H is abelian by [1, Theorem 2.1]. Choose $a_i \in \text{Fix}_W(f_i)$, and then select $a \in W$ such that $a < a_i$ for all i . Now choose $z \in Z$ such that $za > a_i$ for all i .

Suppose the sequence $\{z^r a \mid r > 0\}$ is bounded above, and let L be the least upper bound of the sequence. Then $z^r a_i \in \text{Fix}_W(f_i)$ for all i and L is the least upper bound of the sequence $\{z^r a_i \mid r > 0\}$. We conclude that $L \in \text{Fix}_W(f_i)$ for all i and hence $L \in \text{Fix}_W(F)$. Therefore we may assume that the sequence $\{z^r a \mid r > 0\}$ is not bounded above, and similarly we may assume that the sequence $\{z^r a \mid r < 0\}$ is not bounded below.

Therefore we may assume that $\text{Fix}_W(z) = \emptyset$. It now follows from Lemma 4.1 that $\text{Fix}_W(f) \neq \emptyset$ for all $f \in F$. Since z commutes with all elements of F , we see that every element of F fixes a point in $[a, za)$ and we deduce that the set $\{fa \mid f \in F\}$ is bounded above by za . If M is the supremum of the set $\{fa \mid f \in F\}$, then $M \in \text{Fix}_W(F)$ and the result is proven. \square

The statement and proof of the next lemma is just a reformulation of [19, lemma 3.1]

Lemma 4.3. *Let W be a nonempty closed subset of \mathbb{R} , let $\alpha, \beta \in \text{Aut}(W)$, let $a \in \text{Fix}_W(\alpha)$, and let $b \in \text{Fix}_W(\beta)$. Suppose that $a < b$ and $\text{Fix}_W(\alpha) \cap (a, b] = \emptyset = \text{Fix}_W(\beta) \cap [a, b)$. Then $\langle \alpha, \beta \rangle$ contains a nonabelian free subsemigroup.*

Proof. By replacing α and or β with their inverses if necessary, we may assume that $\alpha(b) < b$ and $\beta(a) > a$. Set $x = \beta(a)$ and note that $a < x < b$. Since $\beta(b) = b$, β has no fixed points on $[a, b)$, and $\beta(a) > a$, we see that there exists a positive integer n such that $\beta^n[a, b) \subseteq (x, b)$. Similarly there exists a positive integer m such that $\alpha^m(a, b] \subseteq (a, x)$.

We now show that the subsemigroup generated by α^m and β^n is free on those generators. Set $\gamma = \alpha^m$ and $\delta = \beta^n$. Suppose to the contrary that two nontrivial distinct finite products π, ρ of the form $\dots \gamma^{n_1} \delta^{n_2} \gamma^{n_3} \delta^{n_4} \dots$, where the n_i are positive integers, yield the same element of $\text{Aut}(W)$. By cancelling on the left, we may assume without loss of generality that $\pi = \gamma \pi_1$ and $\rho = \delta \rho_1$ or 1, where π_1, ρ_1 are also products of the form $\dots \gamma^{n_1} \delta^{n_2} \gamma^{n_3} \delta^{n_4} \dots$. Since $\pi_1 x, \rho_1 x \in (a, b)$, we see that $\gamma \pi_1 x \in (a, x)$ and $\delta \rho_1 x$ or $1x \in [x, b)$. Thus $\pi x \neq \rho x$ and we have a contradiction. We deduce that the subsemigroup generated by α^m and β^n is free on those generators and the result follows. \square

The statement and proof of the next lemma is just a reformulation of [19, lemma 3.6]

Lemma 4.4. *Let W be a nonempty closed subset of \mathbb{R} , let n be a positive integer, let $\alpha_1, \dots, \alpha_n \in \text{Aut}(W)$, and let $G = \langle \alpha_1, \dots, \alpha_n \rangle$. Suppose $G \in \text{NS}$. If $\text{Fix}_W(\alpha_i) \neq \emptyset$ for all i , then $\text{Fix}_W(G) \neq \emptyset$.*

Proof. For each i , we may write $\mathbb{R} \setminus \text{Fix}_W(\alpha_i)$ as a disjoint union of open intervals, say $\bigcup_j I_{ij}$, where each I_{ij} is an open interval. Then $I_{ij} \neq \mathbb{R}$ for all i, j , because $\text{Fix}_W(\alpha_i) \neq \emptyset$. Suppose $\text{Fix}_W(G) = \emptyset$. If $I_{ij} \subseteq I_{kl}$ and $(i, j) \neq (k, l)$ then $i \neq k$, so we may choose i, j such that I_{ij} is not contained in any other open interval. Using the fact that \mathbb{R} is connected, we may now choose k, l so that I_{ij} has nonempty intersection with I_{kl} , and also does not contain I_{kl} . Clearly $i \neq k$. Write $I_{ij} \cap I_{kl} =$

(a, b) , and assume without loss of generality that $a \in I_{kl}$ and $b \in I_{ij}$. The result now follows by applying Lemma 4.3 with $\alpha = \alpha_i$ and $\beta = \alpha_k$. \square

Lemma 4.5. *Let $G \in \text{PLT}$, and let A and B be finitely generated subgroups of G' . Then there exists $g \in G$ such that A^g and B centralize each other.*

Proof. We may view G as a subgroup of the piecewise linear orientation preserving homeomorphisms of $[0, 1]$ such that G fixes no point in $(0, 1)$. Define $H = \{g \in G \mid \text{there exists } \epsilon > 0 \text{ such that } g(t) = t \text{ for all } t \in [0, \epsilon] \cup [1 - \epsilon, 1]\}$. Then obviously $H \triangleleft G$ and we see that $H \supseteq G'$. Therefore if A and B are finitely generated subgroups of G' , there exists $0 < r < s < 1$ such that c is the identity map outside (r, s) for all $c \in A \cup B$. Since G does not fix any point in $(0, 1)$, there exists $g \in G$ such that $gr > s$. Then gAg^{-1} fixes all points outside (gr, gs) and the result follows. \square

Lemma 4.6. *Let $G \in \text{PLO}$ be a finitely generated group. Then there exists a series $G' = G_0 \supseteq G_1 \supseteq \dots \supseteq G_n = 1$ with the property that $G_i \triangleleft G$ and if A, B are finitely generated subgroups of G_{i-1}/G_i , then there exists $g \in G/G_i$ such that A^g and B centralize each other, for $i = 1, \dots, n$.*

Proof. Write $G = \langle g_1, \dots, g_m \rangle$ and consider G as a group of orientation preserving homeomorphisms of $[0, 1]$. Let W' denote the complement $[0, 1] \setminus W$ of a subset W of $[0, 1]$. Since $\text{Fix}_{[0,1]}(g_i)'$ is a finite union of open intervals and $\text{Fix}_{[0,1]}(G)' = \bigcup_i \text{Fix}_{[0,1]}(g_i)'$, we see that $\text{Fix}_{[0,1]}(G)'$ is a finite union of open intervals. Therefore $\text{Fix}_{[0,1]}(G)'$ is a finite union of disjoint open intervals, say

$$(a_1, b_1) \cup (a_2, b_2) \cup \dots \cup (a_n, b_n)$$

where $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq a_3 < \dots \leq a_n < b_n \leq 1$. Set $H_i = \text{Stab}_G([a_i, b_i])$, $G_0 = G'$, and $G_i = G_0 \cap H_1 \cap \dots \cap H_i$ for $i = 1, \dots, n$. Then $G' = G_0 \supseteq G_1 \supseteq \dots \supseteq G_n = 1$ and $G_i \triangleleft G$ for all i . Since $g[a_i, b_i] = [a_i, b_i]$ for all $g \in G$, we see that $G/H_i \leq \text{Aut}([a_i, b_i])$.

Now suppose A/G_i and B/G_i are finitely generated subgroups of G_{i-1}/G_i . Then AH_i/H_i and BH_i/H_i are finitely generated subgroups of $G'H_i/H_i$, so by Lemma 4.5 there exists $g \in G$ such that $[A^g, B]$ (the commutator of A^g and B) is contained in H_i . Therefore $[A^g, B] \subseteq H_i \cap G_{i-1} = G_i$ and the result follows. \square

Lemma 4.7. *Let $1 \neq G \in Q(\text{PLO})$ and suppose G is finitely generated. Then there exists $1 < H \triangleleft G$ such that if A, B are finitely generated subgroups of H , then there exists $g \in G$ such that A^g and B centralize each other.*

Proof. Write $G = P/K$ where $P \in \text{PLO}$ and $K \triangleleft P$. If $G = \langle Kp_1, \dots, Kp_d \rangle$, then replacing P with $\langle p_1, \dots, p_d \rangle$ and K with $K \cap \langle p_1, \dots, p_d \rangle$, we may assume that P is finitely generated. By Lemma 4.6, there exists a series $P' = P_0 \supseteq P_1 \supseteq \dots \supseteq P_n = 1$ such that $P_i \triangleleft P$ for all i with the property that if C/P_i and D/P_i are finitely generated subgroups of P_{i-1}/P_i , then there exists $p \in P$ such that C^p/P_i and D/P_i centralize each other. If $P' \subseteq K$, then we may take $H = G$ and we are finished. Otherwise we may let m be the smallest integer such that $P_m \subseteq K$ and set $H = P_{m-1}K/K$, a nontrivial normal subgroup of G . If A and B are finitely generated subgroups of H , then there exist finitely generated subgroups C and D of P_{m-1} such that $A = CK/K$ and $B = DK/K$. Then we can find $p \in P$ such that C^pP_m/P_m and DP_m/P_m centralize each other. If $g = pK$, then A^g and B centralize each other as required. \square

Corollary 4.8. $\mathcal{C} \subseteq \text{NF}$.

Proof. It is easy to see that NF is closed under taking subgroups, quotient groups, group extensions and directed unions. Since $\text{NS} \subseteq \text{NF}$, it will now be sufficient to show that $\mathcal{P} \in \text{NF}$. However if G is the free group on two generators, $1 \neq H \triangleleft G$, $x \in H \setminus H'$ and $y \in H' \setminus 1$, then there is no $g \in G$ such that x^g and y centralize each other. We deduce from Lemma 4.7 that $G \notin \text{PLO}$ and the proof is complete. \square

Lemma 4.9. *Let W be a nonempty closed subset of \mathbb{R} and let $G \leq \text{Aut}(W)$. Suppose $1 \neq G \in \text{Q(PLO)}$ and that G is finitely generated. Then there exists $H \triangleleft G$ such that $H \neq 1$ and $\text{Fix}_W(F) \neq \emptyset$ whenever F is a finitely generated subgroup of H' .*

Proof. Using Lemma 4.7, we can find $H \triangleleft G$ such that $H \neq 1$ with the property that if A, B are finitely generated subgroups of H , then there exists $g \in G$ such that A^g and B centralize each other. The result is obvious if $\text{Fix}_W(A) \neq \emptyset$ for all finitely generated subgroups A of H , so we may assume that there is a finitely generated subgroup A of H such that $\text{Fix}_W(A) = \emptyset$. Let B be any finitely generated subgroup of H . Then there exists $g \in G$ such that A^g and B centralize each other. Since $B \in \text{NF}$ by Corollary 4.8, we deduce from Lemma 4.2 that $\text{Fix}_W(E) \neq \emptyset$ for every finitely generated subgroup E of B' . We conclude that $\text{Fix}_W(F) \neq \emptyset$ for every finitely generated subgroup F of H' as required. \square

Lemma 4.10. *Let W be a nonempty closed subset of \mathbb{R} and let $H \triangleleft G \leq \text{Aut}(W)$ with H solvable. Suppose either $G/H \in \text{NS}$, or $G/H \in \text{Q(PLO)}$ and G/H is finitely generated. Then either $G'' = 1$, or there exists $F \triangleleft G$ such that $F \neq 1$ and $\text{Fix}_W(E) \neq \emptyset$ whenever E is a finitely generated subgroup of F . Furthermore in the former case there exists $A \triangleleft G$ such that A and G/A are torsion free abelian, and $\mathbf{C}_G(A) = A$.*

Proof. First suppose G has a nontrivial normal abelian subgroup A . Here we set $F = \{a \in A \mid \text{Fix}_W(a) \neq \emptyset\}$. Since $A \in \text{NS}$, we see from Lemma 4.4 that $F \leq A$ and $\text{Fix}_W(E) \neq \emptyset$ for all finitely generated subgroups E of F , so we may assume that $F = 1$. Let $C = \mathbf{C}_G(A)$ and let $B = \{b \in C \mid \text{Fix}_W(b) \neq \emptyset\}$. Since $G \in \text{NF}$ by Corollary 4.8, Lemma 4.2 shows that B is a normal subgroup of G and that $\text{Fix}_W(E) \neq \emptyset$ for all finitely generated subgroups E of B . Therefore we may assume that $B = 1$. Using [16, lemma 4.1], we see that C and G/C are abelian and it follows that $G'' = 1$. Thus we may assume that G has no nontrivial normal abelian subgroup, so in particular $H = 1$.

We now have two cases to consider, namely $G \in \text{NS}$ and $G \in \text{Q(PLO)}$ and is finitely generated. In the former case the result follows from Lemma 4.4 and [16, lemma 4.1], while in the latter case the result follows from Lemma 4.9. \square

Lemma 4.11. *Let $H \triangleleft G \leq F$ be groups such that F is finitely generated. Assume that G/H has a solvable normal subgroup K/H such that $G/K \in \mathcal{D}$ and is finitely generated. Suppose F is left orderable and there exists a left-relatively convex subgroup B of F such that $H \subseteq B \neq F$. Then there exists a left-relatively convex subgroup B_1 of F such that $B_1 \neq F$, $B_1 \cap G \triangleleft G$, and $G/B_1 \cap G$ has a self centralizing torsion free abelian normal subgroup $B_2/B_1 \cap G$ such that G/B_2 is torsion free abelian.*

Proof. For each $X \subseteq F$, let $\mathbf{c}X$ denote the smallest left-relatively convex subgroup of F containing X , and let $\mathcal{S} = \{I \triangleleft G \mid H \subseteq I \text{ and } \mathbf{c}I \neq F\}$. Then \mathcal{S} is partially

ordered by inclusion. Suppose \mathcal{T} is a nonempty chain in \mathcal{S} . Then $\bigcup_{I \in \mathcal{T}} \mathfrak{c}I$ is a left-relatively convex subgroup of F by Lemma 2.1, which is not the whole of F because F is finitely generated and $\mathfrak{c}I \neq F$ for all $I \in \mathcal{T}$, consequently \mathcal{T} is bounded above by $\bigcup_{I \in \mathcal{T}} I$. But $H \in \mathcal{S}$ because $\mathfrak{c}H \subseteq B \neq F$, hence $\mathcal{S} \neq \emptyset$ and we may apply Zorn's lemma to deduce that \mathcal{S} has a maximal element E say. Set $B_1 = \bigcap_{g \in G} (\mathfrak{c}E)^g$, which by Lemma 2.1 is a left-relatively convex subgroup of F , so using the maximality of E we see that $B_1 \cap G = E$ (thus $B_1 = \mathfrak{c}E$). If G/E has a self centralizing torsion free abelian normal subgroup B_2/E such that G/B_2 is torsion free abelian, then we are finished so we assume that this is not the case.

By Lemma 2.2, there is an order preserving action of F on \mathbb{R} with kernel $\text{core}_F(\mathfrak{c}E)$ such that $\text{Stab}_F(0) = \mathfrak{c}E$, and $\text{Stab}_F(v) \neq F$ for all $v \in \mathbb{R}$. Replacing F with $F/\text{core}_F(\mathfrak{c}E)$ and using Lemma 2.1, we may assume that $\text{core}_F(\mathfrak{c}E) = 1$.

Let $W = \text{Fix}_{\mathbb{R}}(E)$. Then W is a nonempty closed subset of \mathbb{R} , and G/E is naturally a subgroup of $\text{Aut}(W)$. Using the hypotheses of the Lemma, there is a normal solvable subgroup K_1/E of G/E such that $G/K_1 \in \mathcal{D}$ and is finitely generated. By Lemma 4.10, there is a nontrivial normal subgroup A/E of G/E such that $\text{Fix}_W(C) \neq \emptyset$ whenever C is a finitely generated subgroup of A/E . Write $A = \bigcup_{i \in \mathbb{N}} A_i$ where $E \leq A_1 \leq A_2 \leq \dots$ and A_i/E is finitely generated for all i (if A/E is finitely generated, we may choose $A_i = A$ for all i), and $X_i = \text{Fix}_{\mathbb{R}}(A_i)$. Then $X_i \neq \emptyset$ for all i , and $\text{Stab}_F(X_1) \leq \text{Stab}_F(X_2) \leq \dots$ is an ascending chain of left-relatively convex subgroups of F with the property that $A_i \subseteq \text{Stab}_F(X_i) \neq F$ for all i . Furthermore $\bigcup_{i \in \mathbb{N}} \text{Stab}_F(X_i)$ is a left-relatively convex subgroup by Lemma 2.1, which cannot be F itself because F is finitely generated. We deduce that $\mathfrak{c}A \neq F$ which contradicts the maximality of E and finishes the proof. \square

Theorem 4.12. *Let $G \leq F \neq 1$ be groups such that $G \in \mathcal{C}$ and F is finitely generated and left orderable. Then there exists a left-relatively convex subgroup B of F such that $B \neq F$, $B \cap G \triangleleft G$, and $G/B \cap G$ has a self centralizing torsion free abelian normal subgroup $A/B \cap G$ such that G/A is torsion free abelian.*

Proof. We shall prove the result by transfinite induction on G , so by Lemma 3.1 choose the least ordinal α such that $G \in \mathcal{D}_\alpha$ and assume that the result is true whenever $H \in \mathcal{D}_\beta$ and $\beta < \alpha$. Now α cannot be a limit ordinal, and the result is clearly true if $\alpha = 0$. Therefore we may assume that $\alpha = \gamma + 1$ for some ordinal γ , and then there exists $H \triangleleft G$ such that $G/H \in \mathcal{D}$ and $H \in \text{L}\mathcal{D}_\gamma$. Using Lemma 3.1, we may write $H = \bigcup_{i \in \mathbb{N}} H_i$ where $H_1 \leq H_2 \leq \dots \leq H$ and every subgroup of H_i is in \mathcal{D}_γ for all i . For each $X \subseteq F$, let $\mathfrak{c}X$ denote the smallest left-relatively convex subgroup of F containing X .

First consider the case G/H is finitely generated. We have an ascending chain of left-relatively convex subgroups $\mathfrak{c}(H_1'') \leq \mathfrak{c}(H_2'') \leq \dots$, so their union is also a left-relatively convex subgroup by Lemma 2.1 which contains H'' . The inductive hypothesis shows that $\mathfrak{c}(H_i'') \neq F$ for all i and since F is finitely generated, we deduce that $\mathfrak{c}(H'') \neq F$. But G/H'' has the solvable normal subgroup H/H'' such that $G/H \in \mathcal{D}$, so the result follows from an application of Lemma 4.11.

Finally we need to consider the case G/H is not finitely generated. Here we write $G = \bigcup_{i \in \mathbb{N}} G_i$, where $H \leq G_i \leq G$ and G_i/H is finitely generated for all i . We now have an ascending chain of left relatively convex subgroups $\mathfrak{c}(G_1'') \leq \mathfrak{c}(G_2'') \leq \dots$, so their union is also a left-relatively convex subgroup by Lemma 2.1 which contains G'' . By the case G/H is finitely generated considered in the previous paragraph,

we know that $\mathfrak{c}(G''_i) \neq F$ for all i and since F is finitely generated, we deduce that $\mathfrak{c}(G'') \neq F$. Another application of Lemma 4.11 completes the proof. \square

5. GROUPS OF HOMEOMORPHISMS OF THE CIRCLE

Proof of Corollary 1.3.

Proof. By [23, lemma 2.3], we may lift the action of G on S^1 to an action of a group H on \mathbb{R} ; specifically H is a left orderable group with a central subgroup Z such that $Z \cong \mathbb{Z}$ and $H/Z \cong G$. Note that H is finitely generated because G is finitely generated. If G is finite, then H is a torsion free group with an infinite central cyclic subgroup of finite index and it follows that $H \cong \mathbb{Z}$. We deduce that G is cyclic. Therefore we may assume that G is infinite.

By Theorem 1.2 H has a normal subgroup K such that $H/K \cong \mathbb{Z}$. If $K \cap Z \neq 1$, then $Z/K \cap Z$ is finite, consequently KZ/K is a finite subgroup of H/K and we deduce that $H/KZ \cong \mathbb{Z}$. It follows that G has an infinite cyclic quotient, so we may assume that $K \cap Z = 1$.

Note that H/KZ is a finite cyclic group. Since KZ has finite index in the finitely generated group H , we see that K is finitely generated. Moreover K is infinite, so by Theorem 1.2 there exists $L \triangleleft K$ such that $K/L \cong \mathbb{Z}$. But

$$\frac{KZ}{LZ} \cong \frac{K}{(LZ) \cap K} = \frac{K}{L(Z \cap K)} = \frac{K}{L}$$

and the result follows. \square

6. FREE PRODUCTS WITH AMALGAMATION

Lemma 6.1. *Let $G \leq \text{Aut}(\mathbb{R})$. Then there is an action α of G on \mathbb{R} by orientation preserving homeomorphisms with the following properties.*

- (i) *If $c \in G$ and $c(r) > r$ for all $r \in \mathbb{R}$, then $(\alpha c)r < r$ for all $r \in \mathbb{R}$.*
- (ii) *If $c \in G$ and $c(r) < r$ for all $r \in \mathbb{R}$, then $(\alpha c)r > r$ for all $r \in \mathbb{R}$.*

Proof. Define an action α of G on \mathbb{R} by $(\alpha g)r = -g(-r)$ for $g \in G$. This action has the required properties. \square

Lemma 6.2. *Let $G \leq \text{Aut}(\mathbb{R})$, let H be a left ordered group, let $1 \neq c \in G$, let $C = \langle c \rangle$, and let $1 \neq h \in H$. Identify C with $\langle h \rangle$ via the isomorphism $c^n \mapsto h^n : C \rightarrow \langle h \rangle$ for $n \in \mathbb{Z}$. Suppose $\text{Fix}_{\mathbb{R}}(c) = \emptyset$. Then $G *_C H$ is left orderable.*

Proof. Write $H = \bigcup_i H_i$, where the H_i are finitely generated subgroups containing h . Then $G *_C H = \bigcup_i G *_C H_i$, and if each of the $G *_C H_i$ is left orderable, then so is $G *_C H$ by [18, 7.3.2]. Therefore we may assume that H is finitely generated. Using Lemma 2.2, we can view H as a subgroup of $\text{Aut}(\mathbb{R})$.

We may write $\mathbb{R} \setminus \text{Fix}_{\mathbb{R}}(h)$ as a countable disjoint union of nonempty open sets, say $\bigcup_i P_i$. On each P_i , either $h(x) > x$ for all $x \in P_i$, or $h(x) < x$ for all $x \in P_i$. Using Lemma 6.1, for each i there is an action of G on P_i by orientation preserving homeomorphisms with the property that either $h(x) > x$ and $c(x) > x$ for all x , or $h(x) < x$ and $c(x) < x$ for all x . Then by [11, theorem 10] we may assume that $h = c$ on P_i . We have now defined an action of G on $\bigcup_i P_i$, and we extend this to an action α on the whole of \mathbb{R} by defining αg to be the identity on $\text{Fix}_{\mathbb{R}}(h)$ for all $g \in G$. Clearly $\alpha(G) \subseteq \text{Aut}(\mathbb{R})$ and $\alpha(G) \cong G$. Thus we can define a group homomorphism $\theta : G *_C H \rightarrow \text{Aut}(\mathbb{R})$ by $\theta g = \alpha g$ for $g \in G$ and $\theta h = h$ for $h \in H$, because $\alpha(c^n) = h^n$ for $n \in \mathbb{Z}$. The result now follows from [14, theorem 6.2.3]. \square

Theorem 6.3. *Let G be a left ordered group, let H be a torsion free nilpotent group, and let C be a cyclic group. Then $G *_C H$ is left orderable.*

Proof. If $C = 1$ then the result follows from [18, §2.4 on p. 37 and theorem 7.3.2], so we may assume that C is infinite cyclic. We will assume that C is a subgroup of H and write $C = \langle c \rangle$, where $1 \neq c \in H$. Let $1 \neq g \in G$ and identify C with $\langle g \rangle$ via the isomorphism $c^n \mapsto g^n$ for $n \in \mathbb{Z}$. We need to prove that $G *_C H$ is left orderable.

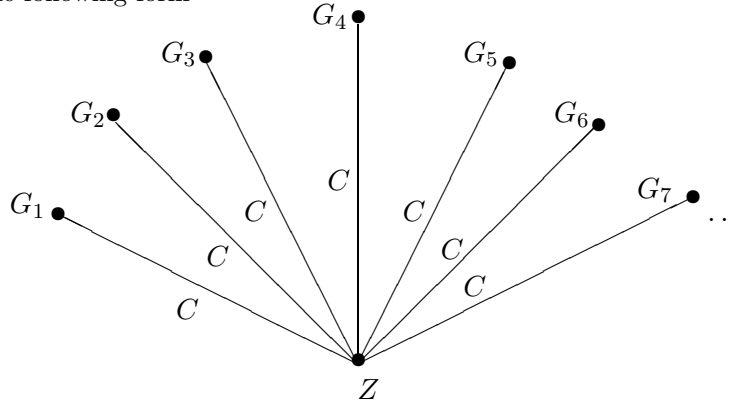
Write $H = \bigcup_i H_i$, where the H_i are finitely generated subgroups containing C . Then $G *_C H = \bigcup_i G *_C H_i$, and if each of the $G *_C H_i$ is left orderable, then so is $G *_C H$ by [18, 7.3.2]. Therefore we may assume that H is finitely generated. We shall use induction on the Hirsch length of H (so if $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = H$ is a normal series for H with H_i/H_{i-1} infinite cyclic for all i , then n is the Hirsch length of H).

First suppose the Hirsch length of H is 1. This means that H is infinite cyclic, say $H = \langle h \rangle$ where h has infinite order. Then we can view H as a subgroup of $\text{Aut}(\mathbb{R})$ by letting H act on \mathbb{R} according to the rule $h(r) = r+1$ for all $r \in \mathbb{R}$. Then $\text{Fix}_{\mathbb{R}}(c) = \emptyset$ and the result follows from Lemma 6.2. Therefore we may assume that the Hirsch length of H is at least 2.

Let Z be a nontrivial cyclic central subgroup of H such that H/Z is torsion free. Then H/Z is left orderable because H/Z is a torsion free nilpotent group [18, §2.4 on p. 37].

Suppose $c \notin Z$. We have an epimorphism $G *_C H \twoheadrightarrow G *_C H/Z$. Let K be the kernel of this map. Then $K \cap G = 1$ and $K \cap H = Z$, so applying [7, I.7.7] we see that K is free and consequently left orderable. By induction $G *_C H/Z$ is left orderable, and so the result follows from [18, 7.3.2].

Finally we need to consider the case $c \in Z$. We have an epimorphism $G *_C H \twoheadrightarrow H/Z$. Let K be the kernel of this map, and let $F = G *_C H$. With $G *_C H$ we have an associated standard tree T [7, I.3.4 definitions], and F acts on this tree. The vertices of T are the left cosets fG and fH , and the edges are the left cosets fC , where $f \in F$. A fundamental F -transversal [7, 2.6 proposition] for T consists of the vertices G, H and the edge C . Let X be a transversal for Z in H . Then a fundamental K -transversal T_0 for T consists of the vertices xG and $xH = H$, and the edges xC , where $x \in X$. The stabilizers of the vertices of T_0 are of the form G^x and $Z^x = Z$, and the stabilizers of the edges are of the form $C^x = C$ for $x \in X$. It follows that K is the fundamental group of a graph of groups [7, I.3.4 definitions] of the following form



where each G_i is of the form G^x for some $x \in X$ (where x depends on i). We can now define an epimorphism $\theta: K \rightarrow G *_C Z$ by $\theta g = x^{-1}gx$ for $g \in G_i$ and $\theta z = z$ for $z \in Z$. The kernel of this map is a free group and hence left orderable. Also $G *_C Z$ is left orderable by induction. We now apply [18, theorem 7.3.2] twice to first deduce that K is left orderable, and then $G *_C H$ is left orderable, as required. \square

Problem 6.4. *Is the free product of two left orderable groups with an amalgamated cyclic subgroup left orderable?*

7. EXAMPLES OF LEFT ORDERED GROUPS WHICH ARE NOT LOCALLY INDICABLE

If G is a group, we shall let $\Delta(G)$ indicate the finite conjugate center of G , that is $\{g \in G \mid \mathbf{C}_G(g) \text{ has finite index in } G\}$. Let n be a positive integer and let B_n denote the Braid group on n strings with standard generators $\sigma_1, \dots, \sigma_{n-1}$.

Dehornoy [6] (see also [9]) has proven that the Braid group B_n on n strings is left orderable. Therefore B'_n is also left orderable. It is not difficult to see that B'_n is a finitely generated perfect group with trivial center for $n \geq 5$; we shall give a proof of this (probably known) result in Lemma 7.2 below. Thus for $n \geq 5$, we see that B'_n is a nontrivial finitely generated left orderable perfect group with trivial center; see the last paragraph of [2, p. 248].

Lemma 7.1. *Let n be a positive integer. Then $\Delta(B_n) = \mathbf{Z}(B_n)$.*

Proof. Obviously $\mathbf{Z}(B_n) \subseteq \Delta(B_n)$. Conversely suppose $\beta \in \Delta(B_n)$. Then the centralizer of β in B_n has finite index in B_n , consequently it contains a normal subgroup C of finite index r in B_n . Thus $\sigma_i^r \in C$ for all i , so by [10, 2.2 theorem] we see that $\sigma_i \beta = \beta \sigma_i$ for all i . Therefore $\beta \in \mathbf{Z}(B_n)$ and the result is proven. \square

Lemma 7.2. *Let n be a positive integer. Then B'_n is finitely generated and $\mathbf{Z}(B'_n) = 1$. Furthermore if $n \geq 5$, then $B''_n = B'_n$.*

Proof. The result is trivial if $n \leq 2$, so we may assume that $n \geq 3$. Let $Z = \mathbf{Z}(B_n)$. Then [10, 2.5 corollary] shows that $Z = \langle (\sigma_1 \dots \sigma_{n-1})^n \rangle$, and we now see from [13, exercise 7, p. 47] that $Z \cap B'_n = 1$. Also $B_n/B'_n \cong \mathbb{Z}$ from [15, p. 757] and we deduce that $B'_n Z$ has finite index in B_n . Therefore B'_n is finitely generated and $\mathbf{Z}(B'_n) \subseteq \Delta(B_n)$. But $\Delta(B_n) = Z$ by Lemma 7.1 and the first part is proven. Finally if $n \geq 5$, then $B''_n = B'_n$ from [15, p. 757]. \square

Let \tilde{G} denote the group of piecewise linear homeomorphisms of \mathbb{R} which satisfy $g(x+1) = g(x) + 1$ for all $g \in \tilde{G}$ and $x \in \mathbb{R}$, as described in [12]. Thus \tilde{G} is a finitely generated perfect group with infinite cyclic center Z generated by the map $x \mapsto x + 1$ for $x \in \mathbb{R}$, and \tilde{G}/Z is a simple group, called T in [3]. Then the free product $\tilde{G} * \tilde{G}$ is a finitely generated perfect group with trivial center, and is left orderable by [18, theorem 7.3.2].

REFERENCES

- [1] L. A. Beklaryan. On the classification of the orientation-preserving homeomorphism groups of \mathbb{R} . I. Invariant measures. *Mat. Sb.*, 187(3):23–54, 1996.
- [2] George M. Bergman. Right orderable groups that are not locally indicable. *Pacific J. Math.*, 147(2):243–248, 1991.
- [3] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.
- [4] I. M. Chiswell and P. H. Kropholler. Soluble right orderable groups are locally indicable. *Canad. Math. Bull.*, 36(1):22–29, 1993.

- [5] Pierre de la Harpe. Free groups in linear groups. *Enseign. Math. (2)*, 29(1-2):129–144, 1983.
- [6] Patrick Dehornoy. Braid groups and left distributive operations. *Trans. Amer. Math. Soc.*, 345(1):115–150, 1994.
- [7] Warren Dicks and M. J. Dunwoody. *Groups acting on graphs*. Cambridge University Press, Cambridge, 1989.
- [8] Benson Farb and Peter Shalen. Groups of real-analytic diffeomorphisms of the circle. preprint.
- [9] R. Fenn, M. T. Greene, D. Rolfsen, C. Rourke, and B. Wiest. Ordering the braid groups. *Pacific J. Math.*, 191(1):49–74, 1999.
- [10] Roger Fenn, Dale Rolfsen, and Jun Zhu. Centralisers in the braid group and singular braid monoid. *Enseign. Math. (2)*, 42(1-2):75–96, 1996.
- [11] N. J. Fine and G. E. Schweigert. On the group of homeomorphisms of an arc. *Ann. of Math. (2)*, 62:237–253, 1955.
- [12] Étienne Ghys and Vlad Sergiescu. Sur un groupe remarquable de difféomorphismes du cercle. *Comment. Math. Helv.*, 62(2):185–239, 1987.
- [13] Vagn Lundsgaard Hansen. *Braids and coverings: selected topics*. Cambridge University Press, Cambridge, 1989. With appendices by Lars Gæde and Hugh R. Morton.
- [14] Valeriĭ M. Kopytov and Nikolaĭ Ya. Medvedev. *Right-ordered groups*. Consultants Bureau, New York, 1996.
- [15] V. Y. Lin. Artin braids and the groups and spaces connected with them. *J. Soviet Math.*, 18:736–788, 1982.
- [16] Peter A. Linnell. Left ordered amenable and locally indicable groups. *J. London Math. Soc. (2)*, 60(1):133–142, 1999.
- [17] Patrizia Longobardi, Mercedes Maj, and Akbar Rhemtulla. When is a right orderable group locally indicable? *Proc. Amer. Math. Soc.*, 128(3):637–641, 2000.
- [18] Roberta Botto Mura and Akbar Rhemtulla. *Orderable groups*. Marcel Dekker Inc., New York, 1977. Lecture Notes in Pure and Applied Mathematics, Vol. 27.
- [19] V. V. Solodov. Homeomorphisms of a straight line and foliations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(5):1047–1061, 1136, 1982.
- [20] V. V. Solodov. Homeomorphisms of the circle and foliations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(3):599–613, 1984.
- [21] V. M. Tararin. On convex subgroups of right-ordered groups. *Siberian Math. J.*, 35:1036–1039, 1994.
- [22] V. M. Tararin. On the theory of right orderable groups. *Math. Notes*, 54:833–834, 1994.
- [23] Dave Witte. Arithmetic groups of higher \mathbb{Q} -rank cannot act on 1-manifolds. *Proc. Amer. Math. Soc.*, 122(2):333–340, 1994.

MATH, VPI, BLACKSBURG, VA 24061-0123, USA
 E-mail address: linnell@math.vt.edu
 URL: <http://www.math.vt.edu/people/linnell/>