# LEFT ORDERED AMENABLE AND LOCALLY INDICABLE GROUPS

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Abstract. There has been interest recently concerning when a left ordered group is locally indicable. Chiswell and Kropholler proved that every left ordered solvable-by-finite group is locally indicable, while Bergman gave examples of left ordered groups which are not locally indicable. We shall prove that every left ordered elementary amenable group is locally indicable. Every solvable-by-finite group is elementary amenable, and every elementary amenable group is amenable. We leave it as an open problem to as whether every left ordered amenable group is locally indicable.

## 1. Introduction

A group $G$ is left ordered if it has a total ordering $\leq$ such that $x \leq y \Rightarrow gx \leq gy$ whenever $g, x, y \in G$. For much information on left ordered groups, see the book on lattice ordered groups by Darnel [4] and references therein, and also [7, 10]. Of course we say that a group $G$ is *right* ordered if it has a total ordering $\leq$ such that $x \leq y \Rightarrow xg \leq yg$ whenever $g, x, y \in G$. However using the involution $g \mapsto g^{-1}$ of $G$, it is easy to see that a group is right orderable if and only if it is left orderable. We remark that it is well known that a group is left ordered if and only if it is isomorphic to a subgroup of a lattice ordered group [4, definition 3.1, corollary 29.8]. A good source of left ordered groups are the subgroups of $\mathrm{Aut}(\mathbb{Q})$ and $\mathrm{Aut}(\mathbb{R})$, the group of order automorphisms of $\mathbb{Q}$ and $\mathbb{R}$ respectively. The assertion that these groups are left orderable follows from [4, proposition 29.5]. Proposition 2.5 shows that any countable left ordered group can be embedded in both $\mathrm{Aut}(\mathbb{Q})$ and $\mathrm{Aut}(\mathbb{R})$. That countable left ordered groups can be embedded in $\mathrm{Aut}(\mathbb{R})$ has certainly been well known for some time, see e.g. the first paragraph of [6], but the fact that they can also be embedded in $\mathrm{Aut}(\mathbb{Q})$ may not have been recorded before. This fact could be used to give a different proof of part of the theorem of [6]. In that paper an interesting group $\Gamma$ of intermediate growth is constructed, and part of the theorem is that this group is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{Q})$. We can establish this fact as follows. By the proof of [6, lemma 6], we see that $\Gamma$ has an infinite descending chain of normal subgroups

$$\Gamma = H_0 > H_1 > \cdots$$

such that $H_i/H_{i+1}$ is free abelian for all $i$, and $\bigcap_{i \in \mathbb{N}} H_i = 1$. Since $\mathbb{Z}$ is left orderable, we may apply [10, theorem 7.3.2] twice to deduce that $\Gamma$ is also left orderable. The group $\Gamma$ is finitely generated, so certainly countable, and now we may apply Proposition 2.5 to conclude that $\Gamma$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{Q})$.

1991 *Mathematics Subject Classification.* Primary: 20F60; Secondary: 06F15, 43A07.
*Key words and phrases.* amenable, locally indicable, left ordered group.

We need to recollect various definitions related to the concept of a group being amenable. The group $G$ is *amenable* if and only if whenever $G$ acts on the set $X$, then there is a finitely additive $G$-invariant measure $\mu\colon \mathcal{P}(X) \to [0,\infty]$ such that $\mu(A) = 1$, where $\mathcal{P}(X)$ denotes the set of all subsets of $X$ and $A = X$ [14, theorem 10.3]. We say that $G$ is *supramenable* if it satisfies the stronger condition that $A$ in the last sentence can be taken to be an arbitrary nonempty subset of $X$ [14, theorem 12.2]. Also the class of elementary amenable groups is the smallest class of groups which contains all abelian-by-finite groups, is closed under group extension, and is closed under directed unions. Then we have that every solvable-by-finite group is elementary amenable, every elementary amenable group is amenable [14, theorem 10.4], and every locally nilpotent-by-finite group is supramenable [14, p. 198]. For our purposes it will be convenient to define $\mathcal{C}$ to be the smallest class of groups which contains all supramenable groups, is closed under group extension, and is closed under directed unions. Clearly $\mathcal{C}$ contains all elementary amenable and supramenable groups, and every group in $\mathcal{C}$ is amenable by [14, theorem 10.4].

Recall that a group is *locally indicable* if and only if every finitely generated subgroup $\neq 1$ has an infinite cyclic quotient. Every locally indicable group is left orderable [10, theorem 7.3.1], but the converse is not true, as has been shown by Bergman [1] and Tararin [13]. On the other hand Chiswell and Kropholler [3, theorem A] showed that a solvable-by-finite left ordered group is locally indicable; also Tararin [12, theorem 3] has proved that if $A \lhd G \neq 1$ are groups with $G/A$ finitely generated and solvable, $A$ abelian and $G$ left orderable, then $G$ has a quotient isomorphic to an infinite subgroup of $\mathbb{Q}$. The main result of this paper is

**Theorem 1.1.** *Let $G \in \mathcal{C}$. Then $G$ is left orderable if and only if $G$ is locally indicable.*

Of course the result that $G$ is locally indicable (whether or not $G \in \mathcal{C}$) implies that $G$ is left orderable, has already been noted above. For the reverse implication, we prove a stronger result Theorem 4.6 which states that if $F \neq 1$ is a finitely generated left orderable group and $F \supseteq G \in \mathcal{C}$, then $F$ can be made into a left ordered group in such a way that there exists a convex subgroup $H \neq F$ (see Section 2) such that $H \cap G \lhd G$, and $G/H \cap G$ has a self centralizing torsion free normal abelian subgroup $A/H \cap G$ such that $G/A$ is torsion free abelian. Thus in the special case $F = G$ (so $G$ is finitely generated and $\neq 1$), we see that $G$ has a quotient isomorphic to $\mathbb{Z}$.

We leave it as an open question to as whether Theorem 1.1 remains true in the case $G$ is amenable. Presumably not every amenable group lies in the class $\mathcal{C}$, though I know of no explicit example in the literature. The ubiquitous Thompson group of piecewise linear homeomorphisms of the real line is not in $\mathcal{C}$; this can be established in the same way as [2, theorem 4.10]. This paper depends heavily on viewing left ordered groups as orientation preserving homeomorphisms of $\mathbb{R}$, and it is interesting to relate this paper to the theory which has been developed on Thompson's group: for much information on this group, see [2] and references therein. Finally in Section 5 we give an example (Example 5.1) of a nontrivial left ordered solvable group whose abelianization is a torsion group.

I am very grateful to Pierre de la Harpe for suggesting that I view left ordered groups as orientation preserving homeomorphisms of $\mathbb{R}$, and for his helpful comments in the early stages of this project.

## 2. Notation, Terminology and Assumed Results

As usual $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ will denote the rational numbers, real numbers, integers and natural numbers $\{1, 2, \dots\}$ respectively. We shall use the notation $G'$ for the commutator subgroup of the group $G$, and if $g, x \in G$ and $X \subseteq G$, then $x^g = gxg^{-1}$, $X^g = gXg^{-1}$, and $\mathbf{C}_G(X) = \{g \in G \mid x^g = x \text{ for all } x \in X\}$. Also if $H, K \leqslant G$, then $\operatorname{core}_G(H) = \bigcap_{g\in G} H^g$, the largest normal subgroup of $G$ contained in $H$, and $\langle H, K\rangle$ indicates the smallest subgroup containing $H$ and $K$. If $A$ is a group of automorphisms acting on $G$, then $G \rtimes A$ denotes the split extension of $G$ by $A$; this has multiplication defined by the rule $ga\, hb = g(aha^{-1})\, ab$ for $a, b \in A$ and $g, h \in G$, where $h \mapsto aha^{-1}$ is the automorphism of $G$ determined by $a$.

All mappings will be written on the left, in particular all group actions will have the group acting on the left of the set. If $G$ is acting on a set $Y$ and $Z \subseteq Y$, then $\operatorname{Stab}_G(Z)$ will always denote the pointwise stabilizer of $Z$ in $G$: thus $\operatorname{Stab}_G(Z) = \{g \in G \mid gz = z \text{ for all } z \in Z\}$, and we write $\operatorname{Stab}_G(y)$ for $\operatorname{Stab}_G(\{y\})$ when $y \in Y$. Also if $H \subseteq G$, then $\operatorname{Fix}_Y(H)$ or $\operatorname{Fix}(H)$ are the fixed points of $H$, that is $\{y \in Y \mid hy = y \text{ for all } h \in H\}$, and when $g \in G$, we write $\operatorname{Fix}_Y(g)$ for $\operatorname{Fix}_Y(\{g\})$ and $\operatorname{Fix}(g)$ for $\operatorname{Fix}(\{g\})$. Then obviously $\operatorname{Fix}(g) = \operatorname{Fix}(\langle g\rangle)$ for all $g \in G$, where $\langle g\rangle$ denotes the subgroup generated by $g$.

A totally ordered set $X$ is a set with a binary relation $\leq$ on $X$ such that for all $x, y, z \in X$ either $x \leq y$ or $y \leq x$, $x \leq y$ and $y \leq x$ implies $x = y$, and $x \leq y \leq z$ implies $x \leq z$. Given totally ordered sets $X$ and $Y$, then the map $\theta\colon X \to Y$ is said to be order preserving if $x < y$ implies $\theta x < \theta y$ whenever $x, y \in X$. We shall let $\operatorname{Aut}(X)$ denote the group of all order preserving permutations $X \to X$. Note that if $\theta$ is an order preserving bijection $X \to Y$, then $\theta^{-1}$ is also order preserving and thus $\operatorname{Aut}(X)$ is indeed a group. Also if $X \subset \mathbb{R}$ and $X$ is given the order induced by the natural order on $\mathbb{R}$, then the elements of $\operatorname{Aut}(X)$ are homeomorphisms of $X$.

If $G$ is a left ordered group, then an order automorphism of $G$ is a group automorphism of $G$ which is also order preserving, and similarly an order isomorphism is a group isomorphism which preserves order. The positive cone $P$ of $G$ is the set $\{g \in G \mid g \geq 1\}$. Note that if $\theta$ is a group automorphism of $G$ such that $\theta P \subseteq P$, then $\theta$ is an order automorphism of $G$.

If $(G, \leq)$ is a left ordered group and $S \subseteq G$, then the convex hull of $S$ in $G$, $\operatorname{hull}_G(S)$, is the subset $\{g \in G \mid \text{there exist } s, t \in S \text{ such that } s \leq g \leq t\}$. If $K \leqslant G$, then we say that $K$ is a *convex* subgroup of $G$ if $\operatorname{hull}_G(K) = K$. For an arbitrary subgroup $H$ of $G$, it is *not* true in general that $\operatorname{hull}_G(H)$ is a subgroup of $G$. If $K$ is a convex subgroup of $G$ then the left cosets of $K$ in $G$, which we denote by $G/K$, is naturally a totally ordered set under the definition $gK < hK$ if and only if $gK \neq hK$ and $g < h$ for $g, h \in G$. Furthermore $G$ then acts as order preserving permutations on $G/K$ according to the rule $g(hK) = ghK$. Conversely suppose $G$ acts faithfully as order preserving permutations on some totally ordered set $X$. Then, as described in [10, theorem 7.1.2], we can make $G$ into a left ordered group as follows. Well order $X$, and then for $f, g \in G$ with $f \neq g$, we say that $f < g$ if and only if $f(x) < g(x)$ where $x$ is the least element of $X$ such that $f(x) \neq g(x)$. We shall call a left order constructed in this way a *standard* order. Note that if $y \in X$ and $Y$ is the set of all elements less than $y$, then $\operatorname{Stab}_G(Y)$ is a convex subgroup of $G$ under this order.

Following [12, p. 1036], we call the subgroup $B$ of a left orderable group $G$ a *b-subgroup* if it is convex with respect to some left order on $G$ (maybe "left-relatively convex", like the definition on [7, p. 127], is better terminology). Clearly if $B$ is a $b$-subgroup and $g \in G$, then $B^g$ is also a $b$-subgroup of $G$. We record without proof the following elementary result.

**Lemma 2.1.** *Let $H \lhd G$ be groups such that $H$ and $G/H$ are left ordered. Then we can make $G$ into a left ordered group by defining $f < g$ if and only if $f^{-1}gH > H$ in $G/H$, or $f^{-1}gH = H$ and $f^{-1}g > 1$ in $H$. In particular if $B/H$ is a $b$-subgroup of $G/H$, then $B$ is a $b$-subgroup of $G$.*

It now follows from Lemma 2.1 and the remarks at the end of the previous paragraph that if the left orderable group $G$ acts as order preserving permutations on some totally ordered set $X$ and $Y \subseteq X$, then $\operatorname{Stab}_G(Y)$ is a $b$-subgroup of $G$. The following result has already been established by Tararin [12, p. 1036], and is proved in [7, propositions 5.1.10, 5.1.7]. Note that (i) tells us that for each $X \subseteq G$, there exists a least $b$-subgroup of $G$ containing $X$.

**Lemma 2.2.** *Let $G$ be a left orderable group and let $\mathcal{B}$ be a set of $b$-subgroups of $G$. Then*

(i) *$\bigcap_{B\in\mathcal{B}} B$ is a $b$-subgroup of $G$.*
(ii) *If $\mathcal{B}$ is totally ordered by inclusion, then $\bigcup_{B\in\mathcal{B}} B$ is a $b$-subgroup of $G$.*

*Proof.* For (i), we can make $\bigcup_{B\in\mathcal{B}} G/B$ into a totally ordered set by well ordering $\mathcal{B}$ and using the lexicographic ordering, and then $G$ acts as order preserving permutations on this set. The stabilizer of $\bigcup_{B\in\mathcal{B}} B$ is $\bigcap_{B\in\mathcal{B}} B$.

For (ii), we use the subsemigroup criterion [3, lemma 8], and follow the proof of the corresponding statement for orderable groups as in [10, theorem 1.4.1(b), theorem 1.4.5]. □

The next two lemmas are routine; since the first one is well known and easy to establish, we omit its proof.

**Lemma 2.3.** *Let $G \leqslant \operatorname{Aut}(\mathbb{Q})$. Then each $g \in G$ extends by continuity to an element of $\operatorname{Aut}(\mathbb{R})$, consequently $G \leqslant \operatorname{Aut}(\mathbb{R})$. Moreover if $G0$ is neither bounded above nor below as a subset of $\mathbb{Q}$, then $\operatorname{Stab}_G(v) \neq G$ for all $v \in \mathbb{R}$.*

**Lemma 2.4.** *Let $G$ be a countable left ordered group, and let $H$ be a convex subgroup of $G$ such that $H \neq G$. Then there exists a group homomorphism $\theta \colon G \to \operatorname{Aut}(\mathbb{Q})$ such that $\operatorname{Stab}_{\theta G}(0) = \theta H$, $\ker\theta = \operatorname{core}_G(H)$, and $(\theta G)\,0$ is neither bounded above nor below as a subset of $\mathbb{Q}$.*

*Proof.* Set $X = \mathbb{Q} \times G/H$ and give $X$ the lexicographic ordering; thus for $p, q \in \mathbb{Q}$ and $f, g \in G$, we have $(p, fH) < (q, gH)$ if and only if $f^{-1}g \notin H$ and $f < g$, or $f^{-1}g \in H$ and $p < q$. We claim that $X$ is isomorphic to $\mathbb{Q}$ as an ordered set. Since $X$ is countable and has no greatest or least element, it will by [5, problem 13B.1] be sufficient to show that $X$ has no pair of consecutive elements. Suppose then that $(p, fH) < (q, gH)$, and we need to find $x \in X$ such that $(p, fH) < x < (q, gH)$. If $fH < gH$, then we may let $x = (p+1, fH)$ while if $fH = gH$, then $p < q$ so we can choose $t \in \mathbb{Q}$ such that $p < t < q$ and we may let $x = (t, fH)$. This establishes the claim. Therefore there exists an order preserving bijection $\alpha \colon X \to \mathbb{Q}$, and clearly we may choose $\alpha(0, H) = 0$.

Now $G$ acts on $X$ as order preserving maps in the natural way, namely $g(q, fH) = (q, gfH)$. Clearly $\operatorname{Stab}_G(0, H) = H$, the kernel of the action is $\operatorname{core}_G(H)$, and the set $\{g(0, H) \mid g \in G\}$ is neither bounded above nor below in $X$. Thus we may define an action of $G$ on $\mathbb{Q}$ by $gq := \alpha(g(\alpha^{-1}q))$, and the result follows. □

Finally in this section, we record the following corollary of the previous two results.

**Proposition 2.5.** *Let $G$ be a countable left orderable group. Then $G$ can be embedded as a subgroup in both* $\mathrm{Aut}(\mathbb{Q})$ *and* $\mathrm{Aut}(\mathbb{R})$.

*Proof.* Using Lemma 2.4 in the case $H = 1$, we see that $G$ can be embedded in $\mathrm{Aut}(\mathbb{Q})$, and then an application of Lemma 2.3 shows that $G$ can also be embedded in $\mathrm{Aut}(\mathbb{R})$. □

## 3. Elementary Amenable and Supramenable Groups

To prove Theorem 4.6, we need a description of the class of groups $\mathcal{C}$ similar to the class of elementary amenable groups given in [9, section 3]. If $\mathcal{X}$ and $\mathcal{Y}$ are classes of groups, then $H \in \mathrm{L}\,\mathcal{X}$ means that every finite subset of the group $H$ is contained in an $\mathcal{X}$-subgroup, $H \in \mathcal{X}\mathcal{Y}$ means that $H$ has a normal $\mathcal{X}$-subgroup $X$ such that $H/X \in \mathcal{Y}$, and we shall let $\mathcal{S}$ denote the class of supramenable groups. Also for each ordinal $\alpha$, the class of groups $\mathcal{X}_\alpha$ is defined inductively by $\mathcal{X}_0 = \{1\}$, $\mathcal{X}_{\alpha+1} = (\mathrm{L}\,\mathcal{X}_\alpha)\mathcal{S}$ and $\mathcal{X}_\beta = \bigcup_{\alpha<\beta} \mathcal{X}_\alpha$ if $\beta$ is a limit ordinal. Setting $\mathcal{X} = \bigcup_{\alpha \geq 0} \mathcal{X}_\alpha$, we can state

**Lemma 3.1.** (i) *Each $\mathcal{X}_\alpha$ is subgroup closed.*
(ii) $\mathcal{X} = \mathcal{C}$.

*Proof.* (i) This is easily proved by induction on $\alpha$, using the fact that $\mathcal{S}$ is subgroup closed.

(ii) Clearly $\mathcal{X} \subseteq \mathcal{C}$, $\mathcal{X} \supseteq \mathcal{S}$, and $\mathcal{X}$ is closed under directed unions. Therefore we need to prove that $\mathcal{X}$ is extension closed.

We show by induction on $\beta$ that $\mathcal{X}_\alpha\mathcal{X}_\beta \subseteq \mathcal{X}_{\alpha+\beta}$; the case $\beta = 0$ being obvious. If $\beta = \gamma + 1$ for some ordinal $\gamma$, then

$$\begin{aligned}\mathcal{X}_\alpha\mathcal{X}_\beta &= \mathcal{X}_\alpha((\mathrm{L}\,\mathcal{X}_\gamma)\mathcal{S}) \subseteq (\mathcal{X}_\alpha(\mathrm{L}\,\mathcal{X}_\gamma))\mathcal{S} \subseteq (\mathrm{L}(\mathcal{X}_\alpha\mathcal{X}_\gamma))\mathcal{S} \\ &\subseteq (\mathrm{L}\,\mathcal{X}_{\alpha+\gamma})\mathcal{S} \quad \text{(by induction)} \\ &= \mathcal{X}_{\alpha+\beta}.\end{aligned}$$

On the other hand if $\beta$ is a limit ordinal, then $\mathcal{X}_\beta = \bigcup_{\gamma<\beta} \mathcal{X}_\gamma$ and

$$\begin{aligned}\mathcal{X}_\alpha\mathcal{X}_\beta &= \mathcal{X}_\alpha\left(\bigcup_{\gamma<\beta}\mathcal{X}_\gamma\right) = \bigcup_{\gamma<\beta}\mathcal{X}_\alpha\mathcal{X}_\gamma \\ &\subseteq \bigcup_{\gamma<\beta}\mathcal{X}_{\alpha+\gamma} \quad \text{(by induction)} \\ &\subseteq \mathcal{X}_{\alpha+\beta}\end{aligned}$$

as required. □

## 4. Proof of Theorem 4.6

**Lemma 4.1.** *Let $W$ be a closed subset of $\mathbb{R}$ and let $H \lhd G \leqslant \mathrm{Aut}(W)$. Suppose $\mathrm{Fix}(h) = \emptyset$ for all $h \in H\backslash 1$. Then $H$ and $G/\,\boldsymbol{C}_G(H)$ are torsion free abelian groups.*

*Proof.* Select a standard order $\leq$ on $\operatorname{Aut}(W)$ by well ordering $W$ in some way, and let $w$ denote the least element of $W$ under this ordering. Since $\operatorname{Fix}(h) = \emptyset$ for all $h \in H \backslash 1$ and $W$ is closed, we see that $\{h^n w \mid n \in \mathbb{Z}\}$ is bounded neither above nor below for all $h \in H\backslash 1$, consequently $g \in \operatorname{hull}_G(\langle h \rangle)$ for all $h \in H\backslash 1$ and $g \in G$. We can now apply the result of Conrad and Hölder, [10, theorem 7.2.1], to deduce that $H$ is order isomorphic to a subgroup of $\mathbb{R}$ with its natural order, in particular $H$ is abelian. Finally [3, lemma 6] tells us that $h > 1$ implies $h^g > 1$, hence $P^g \subseteq P$ where $P$ denotes the positive cone of $H$. Therefore $G/\mathbf{C}_G(H)$ acts as order automorphisms of $H$ and the result follows from [10, theorem 1.5.1]. □

The proofs of the next two lemmas are very similar to that of [8, proposition 1].

**Lemma 4.2.** *Let $W$ be a nonempty closed subset of $\mathbb{R}$, and let $H, K \leqslant G \leqslant \operatorname{Aut}(W)$. Suppose $G$ is supramenable, $G = \langle H, K \rangle$, and $\operatorname{Fix}_W(H), \operatorname{Fix}_W(K) \neq \emptyset$. Then $\operatorname{Fix}_W(G) \neq \emptyset$.*

*Proof.* If $w \in \operatorname{Fix}_W(H) \cap \operatorname{Fix}_W(K)$, then $w \in \operatorname{Fix}_W(G)$, so without loss of generality we may assume that $0 \in \operatorname{Fix}_W(H)$ and $1 \in \operatorname{Fix}_W(K)$. Also if $G0$ is bounded above, then $\sup_{g \in G} g(0) \in \operatorname{Fix}_W(G)$, so we may further assume that $G0$ is not bounded above. For $a, b \in \mathbb{R}$, we shall use the notation $[a, b)$ for the set $\{w \in W \mid a \leq w < b\}$. Choose $\alpha \in G$ such that $\alpha(0) > 1$. By [14, theorem 12.2], there exists a finitely additive $G$-invariant measure $\mu \colon \mathcal{P}(W) \to [0, \infty]$ such that $\mu[0, \alpha(0)) = 1$, where $\mathcal{P}(W)$ denotes the set of all subsets of $W$. Let $F$ be the set of $g \in G$ such that

$$\begin{aligned} \mu[0, g(0)) &= 0 \quad \text{if } g(0) \geq 0, \\ \mu[g(0), 0) &= 0 \quad \text{if } g(0) \leq 0. \end{aligned}$$

We claim that $H, K \subseteq F \leqslant G$; once this is established, it will follow that $F = G$. Obviously $H \subseteq F$ because $h(0) = 0$ for all $h \in H$. Now we show that $K \subseteq F$, so let $k \in K$. If $k(0) < 0$, then $k^{-1}(0) > 0$ and $\mu[k(0), 0) = \mu[0, k^{-1}(0))$, so replacing $k$ with $k^{-1}$ if necessary we may assume that $k(0) \geq 0$. Since $1 \in \operatorname{Fix}_W(K)$, we see that $k^n(0) < \alpha(0)$ and hence $\mu[0, k^n(0)) \leq 1$ for all $n \in \mathbb{N}$. But $\mu[k^n(0), k^{n+1}(0)) = \mu[0, k(0))$ for all $n \in \mathbb{N}$, consequently $\mu[0, k^n(0)) = n\mu[0, k(0))$ and we conclude that $\mu[0, k(0)) = 0$. Thus $K \subseteq F$.

Finally we must show that $F$ is a subgroup. Clearly $1 \in F$. Next suppose $f, g \in F$. If $f(0) \leq 0$ and $g(0) \geq 0$, then $0 \leq f^{-1}(0) \leq f^{-1}g(0)$ and hence

$$\mu[0, f^{-1}g(0)) = \mu[0, f^{-1}(0)) + \mu[f^{-1}(0), f^{-1}g(0)) = \mu[f(0), 0) + \mu[0, g(0)) = 0,$$

while if $f(0), g(0) \geq 0$, then

$$\begin{aligned} \mu[0, f^{-1}g(0)) &= \mu[f(0), g(0)) \leq \mu[0, g(0)) = 0 \quad \text{if } f(0) \leq g(0), \\ \mu[f^{-1}g(0), 0) &= \mu[g(0), f(0)) \leq \mu[0, f(0)) = 0 \quad \text{if } f(0) \geq g(0) \end{aligned}$$

and in both cases we see that $f^{-1}g \in F$. There are two other cases to consider and by similar arguments, we see that $f^{-1}g \in F$. Therefore $F \leqslant G$ and we conclude that $F = G$. In particular $\alpha \in F$ which is a contradiction because $\mu[0, \alpha(0)) \neq 0$, and the result is proven. □

**Lemma 4.3.** *Let $W$ be a nonempty closed subset of $\mathbb{R}$, and let $z \in G \leqslant \operatorname{Aut}(W)$. Suppose $G$ is amenable, $z$ is in the center of $G$, and $\operatorname{Fix}_W(z) = \emptyset$. Then $\{g \in G \mid \operatorname{Fix}_W(g) \neq \emptyset\} \lhd G$.*

*Proof.* Without loss of generality we may assume that $z(0) = 1$. Note that $\{z^i(0) \mid i \in \mathbb{Z}\}$ is neither bounded above nor below. Set $F = \{g \in G \mid \mathrm{Fix}_W(g) \neq \emptyset\}$. We claim that if $g \in F$, then $\sup_{n\in\mathbb{Z}} g^n(0) < 1$. Indeed if $w \in \mathrm{Fix}_W(g)$ then $z^i(0) \leq w < z^{i+1}(0)$ for some $i \in \mathbb{Z}$, hence $0 \leq z^{-i}(w) < z(0) = 1$. But $z^{-i}(w) \in \mathrm{Fix}_W(z^{-i}gz^i) = \mathrm{Fix}_W(g)$ and $\{g^i(0) \mid i \in \mathbb{Z}\}$ is bounded above by $z^{-i}(w)$, and the claim is established.

According to [14, theorem 10.3], there exists a finitely additive $G$-invariant measure $\mu\colon \mathcal{P}(W) \to [0,1]$ such that $\mu(W) = 1$, where $\mathcal{P}(W)$ denotes the set of all subsets of $W$. For $a, b \in \mathbb{R}$, we shall use the notation $[a, b)$ for the set $\{w \in W \mid a \leq w < b\}$. For $g \in G$, define

$$W_g = \bigcup_{i\in\mathbb{Z}} [z^i(0), gz^i(0)) \quad \text{if } g(0) \geq 0,$$

$$W_g = \bigcup_{i\in\mathbb{Z}} [gz^i(0), z^i(0)) \quad \text{if } g(0) \leq 0.$$

Using the $G$-invariance of $\mu$, it is easily checked that $\mu(W_{gh}) \leq \mu(W_g) + \mu(W_h)$ and that $\mu(W_{g^{-1}}) = \mu(W_g)$ for all $g, h \in G$. We claim that $g \in F$ if and only if $\mu(W_g) = 0$. To establish this we may assume that $g(0) \geq 0$. If $g \in F$, then $g(0)$ is bounded above by $s$ where $1 > s \in \mathbb{R}$ and we see that $W_{g^n} = \bigcup_{i=0}^{n-1} g^i W_g$ is a disjoint union for $n \in \mathbb{N}$. Therefore

$$n\mu(W_g) = \mu(W_{g^n}) \leq \mu(W_s) \leq 1$$

for all $n \in \mathbb{N}$ and we deduce that $\mu(W_g) = 0$. Suppose now that $\mathrm{Fix}_W(g) = \emptyset$. Then $\{g^i(0) \mid i \in \mathbb{N}\}$ is not bounded above, consequently there exists $n \in \mathbb{N}$ such that $g^n(0) \geq 1$ which means that $W_{g^n} = W$ and we deduce that $\mu(W_{g^n}) = 1$. But $\mu(W_{g^n}) \leq n\mu(W_g)$, hence $\mu(W_g) \neq 0$ and the claim is established.

Finally we show that $F \lhd G$. Clearly $f^{-1} \in F$ and $f^g \in F$ for all $f \in F$ and $g \in G$. If $f, h \in F$, then $\mu(W_f) = \mu(W_h) = 0$, hence $\mu(W_{fh}) = 0$ because $\mu(W_{fh}) \leq \mu(W_f) + \mu(W_h)$ and we deduce that $fh \in F$ as required. ☐

**Lemma 4.4.** *Let $W$ be a nonempty closed subset of $\mathbb{R}$ and let $H \lhd G \leqslant \mathrm{Aut}(W)$. Suppose $H$ is solvable and $G/H$ is supramenable. Then either $G'' = 1$, or there exists $F \lhd G$ such that $F \neq 1$ and $\mathrm{Fix}_W(E) \neq \emptyset$ whenever $E$ is a finitely generated subgroup of $F$. Furthermore in the former case there exists $A \lhd G$ such that $A$ and $G/A$ are torsion free abelian, and $\boldsymbol{C}_G(A) = A$.*

*Proof.* Let $\mathcal{H}$ denote the Hirsch-Plotkin radical [11, p. 58] of $H$, and let $F = \{h \in \mathcal{H} \mid \mathrm{Fix}(h) \neq \emptyset\}$. Then $\mathcal{H}$ is a locally nilpotent normal subgroup of $G$, hence supramenable [14, p. 198] and we conclude from Lemma 4.2 that $\mathrm{Fix}_W(E) \neq \emptyset$ whenever $E$ is a finitely generated subgroup contained in $F$. It follows that $F \lhd G$ and so the result is proven if $F \neq 1$, thus we may suppose that $F = 1$. Using Lemma 4.1 we see that $\mathcal{H}$ is abelian, and we deduce from [11, lemma 2.32] that $\mathbf{C}_H(\mathcal{H}) = \mathcal{H}$. Let $C = \mathbf{C}_G(\mathcal{H})$ and $D = \{c \in C \mid \mathrm{Fix}_W(c) \neq \emptyset\}$.

Suppose $\mathcal{H} = 1$. Then $H = 1$ and hence $G$ is supramenable. Thus in the case $D \neq 1$, we may let $F = D$ in the statement of the lemma and the result follows from Lemma 4.2. On the other hand if $D = 1$, then $G$ is abelian by Lemma 4.1 and we are finished. Therefore we may assume that $\mathcal{H} \neq 1$, in particular $C$ has a central element $z$ such that $\mathrm{Fix}_W(z) = \emptyset$.

Since $G$ is amenable [14, theorem 10.4], we see from Lemma 4.3 that $D \leqslant C$ and thus clearly $D \lhd G$. It follows that $D$ is a normal supramenable subgroup of

$G$ because $D \cap H = 1$ and $G/H$ is supramenable, so in view of Lemma 4.2 we may assume that $D = 1$. Therefore $\operatorname{Fix}_W(c) = \emptyset$ for all $c \in C \backslash 1$, hence $C$ is abelian by Lemma 4.1 and we deduce that $\mathbf{C}_G(C) = C$. We now set $A = C$, and then another application of Lemma 4.1 completes the proof. □

**Lemma 4.5.** *Let $H \lhd G \leqslant F$ be groups such that $F$ is finitely generated and $G/H$ is solvable-by-supramenable. Suppose $F$ is left orderable and there exists a $b$-subgroup $B$ of $F$ such that $H \subseteq B \neq F$. Then there exists a $b$-subgroup $B_1$ of $F$ such that $B_1 \neq F$, $B_1 \cap G \lhd G$, and $G/B_1 \cap G$ has a self centralizing torsion free abelian normal subgroup $B_2/B_1 \cap G$ such that $G/B_2$ is torsion free abelian.*

*Proof.* For each $X \subseteq F$, let $\mathfrak{b}X$ denote the smallest $b$-subgroup of $F$ containing $X$, and let $\mathcal{S} = \{I \lhd G \mid H \subseteq I \text{ and } \mathfrak{b}I \neq F\}$. Then $\mathcal{S}$ is partially ordered by inclusion. Suppose $\mathcal{T}$ is a nonempty chain in $\mathcal{S}$. Then $\bigcup_{I \in \mathcal{T}} \mathfrak{b}I$ is a $b$-subgroup of $F$ by Lemma 2.2(ii), which is not the whole of $F$ because $F$ is finitely generated and $\mathfrak{b}I \neq F$ for all $I \in \mathcal{T}$, consequently $\mathcal{T}$ is bounded above by $\bigcup_{I \in \mathcal{T}} I$. But $H \in \mathcal{S}$ because $\mathfrak{b}H \subseteq B \neq F$, hence $S \neq \emptyset$ and we may apply Zorn's lemma to deduce that $\mathcal{S}$ has a maximal element $E$ say. Set $B_1 = \bigcap_{g \in G} (\mathfrak{b}E)^g$, which by Lemma 2.2(i) is a $b$-subgroup of $F$, so using the maximality of $E$ we see that $B_1 \cap G = E$ (thus $B_1 = \mathfrak{b}E$). If $G/E$ has a self centralizing torsion free abelian normal subgroup $B_2/E$ such that $G/B_2$ is torsion free abelian, then we are finished so we assume that this is not the case.

By Lemmas 2.4 and 2.3, there is an order preserving action of $F$ on $\mathbb{R}$ with kernel $\operatorname{core}_F(\mathfrak{b}E)$ such that $\operatorname{Stab}_F(0) = \mathfrak{b}E$, and $\operatorname{Stab}_F(v) \neq F$ for all $v \in \mathbb{R}$. Replacing $F$ with $F/\operatorname{core}_F(\mathfrak{b}E)$ and using Lemmas 2.1 and 2.2(i), we may assume that $\operatorname{core}_F \mathfrak{b}E = 1$.

Let $W = \operatorname{Fix}_{\mathbb{R}}(E)$. Then $W$ is a nonempty closed subset of $\mathbb{R}$, and $G/E$ is naturally a subgroup of $\operatorname{Aut}(W)$. By Lemma 4.4 there is a nontrivial normal subgroup $A/E$ of $G/E$ such that $\operatorname{Fix}_W(C) \neq \emptyset$ whenever $C$ is a finitely generated subgroup of $A/E$. Write $A = \bigcup_{i \in \mathbb{N}} A_i$ where $E \leqslant A_1 \leqslant A_2 \leqslant \cdots$ and $A_i/E$ is finitely generated for all $i$ (if $A/E$ is finitely generated, we may choose $A_i = A$ for all $i$), and $X_i = \operatorname{Fix}_{\mathbb{R}}(A_i)$. Then $X_i \neq \emptyset$ for all $i$, and $\operatorname{Stab}_F(X_1) \leqslant \operatorname{Stab}_F(X_2) \leqslant \cdots$ is an ascending chain of $b$-subgroups of $F$ with the property that $A_i \subseteq \operatorname{Stab}_F(X_i) \neq F$ for all $i$. Furthermore $\bigcup_{i \in \mathbb{N}} \operatorname{Stab}_F(X_i)$ is a $b$-subgroup of $F$ by Lemma 2.2(ii), which cannot be $F$ itself because $F$ is finitely generated. We deduce that $\mathfrak{b}A \neq F$ which contradicts the maximality of $E$, and the result follows. □

**Theorem 4.6.** *Let $G \leqslant F \neq 1$ be groups such that $G \in \mathcal{C}$ and $F$ is finitely generated and left orderable. Then there exists a $b$-subgroup $B$ of $F$ such that $B \neq F$, $B \cap G \lhd G$, and $G/B \cap G$ has a self centralizing torsion free abelian normal subgroup $A/B \cap G$ such that $G/A$ is torsion free abelian.*

*Proof.* We shall prove the result by transfinite induction on $G$, so by Lemma 3.1(ii) choose the least ordinal $\alpha$ such that $G \in \mathcal{X}_\alpha$ and assume that the result is true whenever $H \in \mathcal{X}_\beta$ and $\beta < \alpha$. Now $\alpha$ cannot be a limit ordinal, and the result is clearly true if $\alpha = 0$. Therefore we may assume that $\alpha = \gamma + 1$ for some ordinal $\gamma$, and then there exists $H \lhd G$ such that $G/H$ is supramenable and $H \in \mathrm{L}\,\mathcal{X}_\gamma$. Using Lemma 3.1(i), we may write $H = \bigcup_{i \in \mathbb{N}} H_i$ where $H_1 \leqslant H_2 \leqslant \cdots \leqslant H$ and every subgroup of $H_i$ is in $\mathcal{X}_\gamma$ for all $i$. For each $X \subseteq F$, let $\mathfrak{b}X$ denote the smallest $b$-subgroup of $F$ containing $X$.

We now have an ascending chain of $b$-subgroups $\mathfrak{b}(H_1'') \leqslant \mathfrak{b}(H_2'') \leqslant \cdots$, so their union is also a $b$-subgroup by Lemma 2.2(ii) which contains $H''$. The inductive hypothesis shows that $\mathfrak{b}(H_i'') \neq F$ for all $i$ and since $F$ is finitely generated, we deduce that $\mathfrak{b}(H'') \neq F$. But $G/H''$ is solvable-by-supramenable, so an application of Lemma 4.5 completes the proof. □

## 5. Infinitely Generated Left Ordered Groups

In this section we give an example to show that for nonfinitely generated left ordered groups $G$, one cannot in general say much about $G/G'$.

**Example 5.1.** *There is a countable left ordered group $G \neq 1$ such that $G/G'$ is a group of exponent 2 and $G'' = 1$.*

*Proof.* We shall define an ascending sequence of subgroups $G_n$ for $n = 0, 1, 2, \dots$ and elements $x_n \in G_n$ for $n \geq 1$ with the properties that

(i) $G_0 = 1$ and $G_1 = \langle x_1 \rangle$, where $x_1$ has infinite order.
(ii) If $n > 1$, then $G_n = G_{n-1} \rtimes \langle x_n \rangle$ where $x_n$ has infinite order. Furthermore $x_n x_{n-1} x_n^{-1} = x_{n-1}^{-1}$ and $x_n$ centralizes $G_{n-2}$.

We shall establish this by induction on $n$, the cases $n = 0$ and $n = 1$ being clear. If $n \geq 1$, then we will be finished if we can show that the map $\theta \colon G_n \to G_n$ determined by

$$\theta(g x_n^r) = g x_n^{-r} \quad \text{where } g \in G_{n-1} \text{ and } r \in \mathbb{Z}$$

is a homomorphism, because then we will be able to define the conjugation action of $x_{n+1}$ on $G_n$ to be the homomorphism $\theta$. So we let $g, h \in G_{n-1}$, $r, s \in \mathbb{Z}$, and then we have

$$\begin{aligned}\theta(g x_n^r\, h x_n^s) &= \theta(g x_n^r h x_n^{-r}\, x_n^{r+s}) = g(x_n^r h x_n^{-r})\, x_n^{-r-s} \\ &= g x_n^{-r} h x_n^r\, x_n^{-r-s} \quad \text{because conjugation by } x_n \text{ has order } \leq 2, \\ &= g x_n^{-r}\, h x_n^{-s} = \theta(g x_n^r)\theta(h x_n^s)\end{aligned}$$

as required.

Now let $G$ be the union of the $G_n$, and let $A = \langle x_1^2, x_2^2, \dots \rangle$. Since $x_i x_j = x_j x_i$ if $j \geq i+2$ and $x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} = x_i^{-2}$, we see that $A$ is abelian, $A \subseteq G'$, and $G/A$ is an abelian group of exponent 2. Finally $G_n$ is left orderable for all $n$ and hence so is $G$, by two applications of [10, theorem 7.3.2]. This establishes the result. □

*Note added in proof.* Most of Lemma 2.4 is contained in Lemma 2.2 of 'Arithmetic groups of higher $\mathbb{Q}$-rank cannot act on 1-manifolds' by Dave Witte, *Proc. Amer. Math. Soc.* 122 (1994) 333-340.

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