RIGHT ORDERABLE RESIDUALLY FINITE p-GROUPS AND A KOUROVKA NOTEBOOK PROBLEM

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Abstract. A. H. Rhemtulla proved that if a group is a residually finite p-group for infinitely many primes p, then it is two-sided orderable. In problem 10.30 of the Kourovka notebook 14 th. edition, N. Ya. Medvedev asked if there is a non-right-orderable group which is a residually finite p-group for at least two different primes p. Using a result of Dave Witte, we will show that many subgroups of finite index in $\mathrm{GL}_3(\mathbb{Z})$ give examples of such groups. On the other hand we will show that no such example can exist among solvable by finite groups.

A group G is right orderable if it has a total ordering $\leq$ such that $x \leq y \Rightarrow xg \leq yg$ whenever $g, x, y \in G$. It is *two-sided orderable* if in addition $x \leq y \Rightarrow gx \leq gy$ whenever $g, x, y \in G$. For much information on right ordered groups, see the books [1, 3]. It was proved in [4] that a group which is a residually finite p-group for infinitely many primes p is two-sided orderable. In problem 10.30 of the Kourovka notebook 14 th. edition [2], N. Ya. Medvedev asks if there is a non-right-orderable group which is a residually finite p-group for at least two different primes p. We shall prove

Theorem 1. *Let $\mathcal{P}$ be a finite set of primes. Then there exists a non-right-orderable group which is a residually finite p-group for all $p \in \mathcal{P}$.*

The proof depends on a theorem of Witte [5], and we will see that the groups in Theorem 1 can be taken to be subgroups of finite index in $\mathrm{GL}_3(\mathbb{Z})$, the group of 3 by 3 invertible matrices with integer entries.

Going in the opposite direction, we shall prove the following result.

Theorem 2. *Let p, q be distinct primes and let G be a solvable by finite group. If G is a residually finite p-group and a residually finite q-group, then G has a series*

$$1 = G_0 \lhd \cdots \lhd G_n = G$$

with $G_i \lhd G$ and G_{i+1}/G_i torsion free abelian for all i. In particular G is right orderable.

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Proof of Theorem 1. For each prime p, let $G_p = \{A \in \mathrm{GL}_3(\mathbb{Z}) \mid A \equiv I \mod p\}$, the congruence subgroup of level p (here I denotes the identity matrix of $\mathrm{GL}_3(\mathbb{Z})$). Then G_p is a residually finite p-group. Set $G = \bigcap_{p \in \mathcal{P}} G_p$. Then G is a residually

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finite p-group for all $p \in \mathcal{P}$. Furthermore G has finite index in $\mathrm{GL}_3(\mathbb{Z})$, so by [5, Proposition 3.3] we see that G is not right orderable. This completes the proof of Theorem 1. □

Proof of Theorem 2. Observe that G is torsion free. Let H be a normal solvable subgroup of finite index in G with minimal derived length. We shall prove the result by induction on the derived length of H, the result being obvious when this is zero because then $H = 1$ and thus G will be finite. We now assume that the derived length of H is at least one.

Let A_0 denote the penultimate term of the derived series of H. Then A_0 is a normal abelian subgroup of G and H/A_0 has strictly smaller derived length than that of H. Let A be a maximal normal abelian subgroup of G containing A_0.

We shall let $\hat{K}$ denote the pro-p completion of a group K. Then the exact sequence $1 \to A \to G \to G/A \to 1$ yields an exact sequence

$$\hat{A} \longrightarrow \hat{G} \longrightarrow \widehat{G/A} \longrightarrow 1.$$

Let B denote the image of $\hat{A}$ in $\hat{G}$. Since A is abelian, we see that $\hat{A}$ is abelian and we deduce that B is abelian. Also we may view G as a subgroup of $\hat{G}$ because G is a residually finite p-group. Therefore $B \cap G$ is an abelian normal subgroup of G containing A and we conclude that $B \cap G = A$. But $B \cap G$ can also be described as the kernel of the natural map $G \to \widehat{G/A}$. Therefore G/A is isomorphic to a subgroup of $\widehat{G/A}$ and we deduce that G/A is a residually finite p-group. Similarly G/A is a residually finite q-group. Since the derived length of HA/A is strictly less than the derived length of H, induction shows that G has a series

$$A = G_1 \lhd \cdots \lhd G_n = G$$

with $G_i \lhd G$ and G_{i+1}/G_i torsion free abelian for all i. By setting $G_0 = 1$, we obtain the required series for the first part of Theorem 2.

The assertion that G is right orderable now follows from [3, Theorem 7.3.2] and the fact that torsion free abelian groups are right orderable. □

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