Left-ordered groups

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arXiv http://arxiv.org/archive/math
MathSciNet http://www.ams.org/mathscinet/search.html

Definitions

Definition

A set S is totally ordered if it has a relation $\leq$ such that for all $x, y, z \in S$,

- $x \leq x$
- $x \leq y$ or $y \leq x$
- $x \leq y$ and $y \leq x$ implies $x = y$
- $x \leq y$ and $y \leq z$ implies $x \leq z$

Definition

A group G is left-ordered means that it has a total order $\leq$ such that for all $g, x, y \in G$ with $x \leq y$, then $gx \leq gy$.

Examples

$\mathbb{Z}$, $\mathbb{R}$

Properties

Proposition

A left-ordered group is torsion free.

Proof.

If $g^n = 1$ where n is a positive integer and $1 < g$, then $1 < g < \cdots < g^n = 1$ and hence $1 < 1$, a contradiction. □

Remarks

- Torsion-free abelian groups are left orderable.
- Not every torsion-free group is left orderable.
- If G is a nontrivial finitely generated solvable left ordered group, then it can be shown that G/G' is infinite.

Properties

Theorem

Let G be a group.

- *If $H \leq G$ and G is left ordered, then H is left ordered.*
- *If $H \triangleleft G$, then G is left orderable if H and G/H are left orderable.*
- *If $G_1 \leq G_2 \leq \cdots \leq G$ with G_i is left orderable for all i and $G = \bigcup_{i=1}^{\infty} G_i$, then G is left orderable.*

Theorem

Free abelian groups, free groups and Braid groups are left orderable.

Personal Motivation

Theorem

If G is left orderable, then $\mathbb{C}G$ satisfies the zero divisor conjecture: if $0 \neq \alpha, \beta \in \mathbb{C}G$, then $\alpha\beta \neq 0$.

Recall that $\mathbb{C}G$ is the *group ring* of G over $\mathbb{C}$. These rings are like polynomial rings.

If G is a free abelian group of rank n, then $\mathbb{C}G \cong \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the Laurent polynomial ring in n indeterminates.

Let $\ell^2(G)$ denote the Hilbert space with basis the elements of G.

Theorem

If G is left orderable, $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in \ell^2(G)$, then $\alpha\beta \neq 0$.

Definition

If $\alpha = \sum_{g \in G} a_g g \in \mathbb{C}G$ ($a_g \in \mathbb{C}$, a finite sum) and $\beta = \sum_{g \in G} b_g g \in \ell^2(G)$ (so $\sum_g |b_g|^2 < \infty$), then $\alpha\beta = \sum_{h,g} a_h b_g hg =$

$$\sum_{g \in G} \left(\sum_{x \in G} a_{gx^{-1}} b_x \right) g.$$

Left ordered groups and groups of homeomorphisms

Let $\mathrm{Aut}(\mathbb{R})$ denote the orientation preserving homeomorphisms of the real line. Then under composition of functions (so $(fg)(x) = f(g(x))$ for $f, g \in \mathrm{Aut}(\mathbb{R})$), $\mathrm{Aut}(\mathbb{R})$ becomes a left-ordered group.

Let $\{q_1, q_2, \dots\}$ be a countable dense subset of $\mathbb{R}$. Then for $f, g \in \mathrm{Aut}(\mathbb{R})$, choose the least positive integer i such that $f(q_i) \neq g(q_i)$. Then $f < g$ if and only if $f(q_i) < g(q_i)$ defines a left order on $\mathrm{Aut}(\mathbb{R})$.

Theorem

Let G be a countable left-ordered group. Then G is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{R})$.

Further Motivation

Often one is interested in the group of homeomorphisms of an n-manifold (e.g. the surface of a sphere is a 2-manifold, $\mathbb{R}^3$ is a 3-manifold). Also one is interested in the subgroup of diffeomorphisms.
The 1-manifolds are precisely the circle S^1 and the real line $\mathbb{R}$. Thus determining $\mathrm{Aut}(\mathbb{R})$ is a first step in this problem.

$\mathrm{PSL}_2(\mathbb{R}) < \mathrm{Aut}(S^1)$ by Möbius transformation: if $S^1 = [-\infty, \infty]$ and $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then define $f \in \mathrm{Aut}(S^1)$ by $f(x) = \frac{ax+b}{cx+d}$.

Though $\mathrm{PSL}_2(\mathbb{R})$ is not left orderable, by covering theory, there exists a group left-orderable group G with a central infinite cyclic subgroup Z such that $G/Z \cong \mathrm{PSL}_2(\mathbb{R})$.

The space of left orders

Definition

Let $\mathcal{O}_G$ denote the set of left orders on the group G and for each $g \in G$, let $V_g = \{< \in \mathcal{O}_G \mid 1 < g\}$. Then the $\mathcal{O}_G$ becomes a topological space with subbase $\{V_g \mid g \in G\}$.

Theorem

Let G be a group. Then $\mathcal{O}_G$ is a totally disconnected compact Hausdorff space. Furthermore if G is countable, then G is metrizable.

Theorem

Let G be a group. Then $\mathcal{O}_G$ is either finite or uncountable.