LATIN SQUARE THUE-MORSE SEQUENCES ARE OVERLAP-FREE

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Abstract. We define a morphism based upon a Latin square that generalizes the Thue-Morse morphism. We prove that fixed points of this morphism are overlap-free sequences, generalizing results of Allouche - Shallit and Frid.

1. Introduction

In his 1912 paper, Axel Thue introduced the first binary sequence that does not contain an overlap [6]. It is now called the Thue-Morse sequence:

$$0110100110010110100101100110100 1\ldots.$$

An overlap is a string of terms in the form $c\mathbf{x}c\mathbf{x}c$ where c is a single term and $\mathbf{x}$ is finite string that is potentially empty. Overlaps begin with a square, namely $\mathbf{ww}$ where $\mathbf{w} = c\mathbf{x}$ as given above. It is easy to observe, as Thue did, that any binary string of four or more terms must contain a square.

There are several ways to define the Thue-Morse sequence [2]. We will derive it as a fixed point of a morphism. Let Σ be an alphabet and let Σ^* be the collection of all strings over Σ, including infinite strings or sequences. A morphism is a mapping

$$h : \Sigma^* \to \Sigma^*$$

that obeys the identity $h(xy) = h(x)h(y)$, for x a finite string and $y \in \Sigma^*$ [1, p. 8].

By [1, p. 16], define the Thue-Morse morphism on $\Sigma = \{0, 1\}$ as

$$\mu(t) = \begin{cases} 01, & \text{for } t = 0 \\ 10, & \text{for } t = 1 \end{cases}. \tag{1}$$

The Thue-Morse sequence is then $\mu^\omega(0)$, a countably infinite number of compositions of μ on 0. In particular,

$$\begin{aligned}
\mu(0) &= 01 \\
\mu^2(0) &= \mu(\mu(0)) = \mu(01) = \mu(0)\mu(1) = 0110 \\
\mu^3(0) &= \mu(\mu^2(0)) = \mu(0110) = 01101001 \\
&\vdots \\
\mu^\omega(0) &= 01101001100101101001011001101001\ldots.
\end{aligned}$$

Notice that $\mu^\omega(\mu(0)) = \mu^\omega(0)$ and $\mu(\mu^\omega(0)) = \mu^\omega(0)$. This second observation says that the Thue-Morse sequence is a fixed point of μ [1, p. 10].

1991 *Mathematics Subject Classification.* 11B85.

We can identify the binary alphabet of the Thue-Morse sequence with $\mathbb{Z}_2$ the integers modulo 2. It is natural to then generalize it to $\mathbb{Z}_n$, by considering the alphabet $\Sigma = \{0, 1, \ldots, n-1\}$, and for $i \in \Sigma$, defining the morphism

$$\phi_n(i) = \overline{i+0}\,\overline{i+1}\ldots\overline{i+(n-1)},$$

where $\bar{i}$ is the residue modulo n. Notice that for $\Sigma = \{0, 1\}$, $\phi_2(i) = \mu(i)$. In 2000, Allouche and Shallit proved that ϕ_n^ω is overlap-free [3].

In this paper, we generalize ϕ_n, which is based on the Cayley table of $\mathbb{Z}_n$, to Latin squares of arbitrary finite size n. We define our morphism based the Latin square, and prove that the fixed point of the Latin square morphism is an overlap-free sequence. Note that the Cayley table for $\mathbb{Z}_n$ is a Latin square, but not every Latin square is a Cayley table.

2. Latin Square Morphisms produce Tilings

Allouche and Shallit's morphism can be seen as a mapping of i to the i^{th} row (that begins with i) of the Cayley table for $\mathbb{Z}_n$. For example when $n = 3$, we have

$$\begin{array}{ccccc} & \phi_3 & & & \\ 0 & \to & 0 & 1 & 2 \\ 1 & \to & 1 & 2 & 0 \\ 2 & \to & 2 & 0 & 1 \end{array}$$

This suggests a natural generalization to any Latin square.

Begin with a generic alphabet of n characters, which we may assume to be $\{1, 2, \ldots, n\}$. Recall that a Latin square $\mathcal{L}$ is an $n \times n$ table with n different elements such that each symbol occurs only once in each column and only once in each row. We will concern ourself with the Latin squares in which the first column retains the natural order of our alphabet $(1, 2, \ldots, n)$. For $n = 3$, there are two such Latin squares. The one that does not come from $\mathbb{Z}_3$ directly is

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

Let $\mathcal{L}_t$ denote the t^{th} row of our Latin square $\mathcal{L}$. For some $t \in \Sigma$ we define the Latin square morphism by $\ell(t) = \mathcal{L}_t$. For example we can use the above Latin square for $n = 3$ to define the following morphism,

$$\ell(t) = \begin{cases} 132, & \text{for } t = 1 \\ 213, & \text{for } t = 2 \\ 321, & \text{for } t = 3 \end{cases}$$

Given any $t \in \Sigma$, $\ell(t), \ell^2(t), \ell^3(t), \ldots$ converges to a sequence $\ell^\omega(t)$, which is a fixed point of the morphism ℓ. So,

$$\ell(\ell^\omega(t)) = \ell^\omega(t) \tag{2}$$

In fact every fixed point of ℓ is of the form $\ell^\omega(t)$ for some $t \in \Sigma$ [1, p. 10].

Express the sequence as $\ell^\omega(t_1) = t_1 t_2 t_3 \ldots$, so

$$\ell^\omega(t_1) = \ell(\ell^\omega(t_1)) = \ell(t_1 t_2 t_3 \ldots) = \ell(t_1)\ell(t_2)\ell(t_3)\ldots$$

Thus, we have a tiling of our sequence (and of the natural numbers) by the rows of our Latin square $\mathcal{L}$. Again, in terms of our example where $n = 3$ we have three tiles 132, 213, and 321 and so

$$\ell^\omega(1) = 132321213321213132\ldots = |132|321|213|321|213|132|\ldots.$$

Now, consider the subsequence created by taking the first term of each tile. Notice that this sequence is in fact our original sequence. Thus our sequence contains itself as a subsequence. These two observations, our sequence as a tiling and our sequence equaling a subsequence of itself, will be critical for the proof of our main result.

3. Overlap-Free Latin Square Sequences

In this section we prove our main result.

Theorem 3.1. *Let $\Sigma = \{1, 2, \ldots, n\}$, and let $\mathcal{L}$ be an $n \times n$ Latin square using the symbols from Σ, with the first column in its natural order. For an arbitrary $t \in \Sigma$, let $\mathcal{L}_t$ denote the row of $\mathcal{L}$ corresponding to t in the first column. If we define the Latin square morphism as*

$$\ell(t) = \mathcal{L}_t,$$

then we have that for any $t \in \Sigma$, $\ell^\omega(t)$ is an overlap-free sequence.

Remark. The Latin square for $n = 3$ above can be seen to be the Cayley table for $\mathbb{Z}_3$ with the last two columns transposed. Frid has shown that all morphisms based upon such Latin squares for $\mathbb{Z}_n$ produce overlap-free sequences as their fixed points [5]. Of course not every Latin square comes from a group Cayley table. For an example of a Latin square that is not a group Cayley table see [4, p. 27].

Proof. Let $\ell^\omega(t_1) = t_1t_2t_3\ldots$ so the j^{th} term in the sequence is t_j. Similarly, the m^{th} tile in the sequence is T_m. We will be also using the notion of length of a string of terms, meaning the number of terms in a string. For an arbitrary string w the length of w will be denoted $\|w\|$. Use r to denote the location of t_j on its tile T_m, so $j = (m-1)n + r$ with $\|T_m\| = n$ and $r \in \{1, 2, \ldots, n\}$.

Assume for a contradiction that $\ell^\omega(t_1)$ contains an overlap; moreover that $c\mathbf{x}c\mathbf{x}c$ is the shortest overlap in $\ell^\omega(t_1)$. Write $\ell^\omega(t_1) = Ac\mathbf{x}c\mathbf{x}cB$, where c is a single term, $\mathbf{x}$ is a finite string with $\|c\mathbf{x}\| \geq n$, A is a finite string, and B is the infinite tail of our sequence. We have that $\|c\mathbf{x}\| \geq n$ (bound by the length of the tiles) because each tile is a permutation of $1, 2, \ldots, n$, and we cannot have two of the three copies of c contained in one tile. Our subscripts place this overlap in our sequence. For $i \in \{1, 2, 3\}$, let j_i denote the subscript of the i^{th} c. Thus,

$$\begin{aligned} A &= t_1 \cdots t_{j_1-1} \\ c &= t_{j_1} = t_{j_2} = t_{j_3} \\ \mathbf{x} &= t_{j_1+1} \cdots t_{j_2-1} = t_{j_2+1} \cdots t_{j_3-1} \\ B &= t_{j_3+1}t_{j_3+2}t_{j_3+3} \cdots , \end{aligned} \tag{3}$$

Our argument proceeds as follows: there are two cases $\|c\mathbf{x}\| \not\equiv 0 \pmod{n}$ and $\|c\mathbf{x}\| \equiv 0 \pmod{n}$. In the first case we use the fact that we have a tiling of $\ell^\omega(t_1)$ by the rows of a Latin square, to show that the overlap $c\mathbf{x}c\mathbf{x}c$ is not possible. In the second case, when $\|c\mathbf{x}\| \equiv 0 \pmod{n}$, we argue based upon the fact that $\ell^\omega(t_1)$ contains itself as a subsequence that the existence of the overlap $c\mathbf{x}c\mathbf{x}c$ leads to to the existence of a shorter overlap, and thus a contradiction.

3.1. **Case 1:** $\|c\mathbf{x}\| \not\equiv 0 \pmod{n}$

For each $i \in \{1,2,3\}$, let $r_i \in \{1,2,\ldots,n\}$ such that $r_i \equiv j_i \pmod{n}$. In other words t_{j_i} is the ${r_i}^{\text{th}}$ term in its tile in $\ell^\omega(t_1)$. Also, we will refer to the tile containing t_{j_i} as T_{m_i}. It is now possible to write the length of $c\mathbf{x}$ as $\|c\mathbf{x}\| \equiv r_2 - r_1 \equiv r_3 - r_2 \pmod{n}$. So,

$$r_3 \equiv 2r_2 - r_1 \pmod{n}. \tag{4}$$

3.1.1. *Six Cases.* Since $r_2 - r_1 \equiv \|c\mathbf{x}\| \not\equiv 0 \pmod{n}$ there are two main cases that we will first consider: $r_1 < r_2$ and $r_2 < r_1$. However, for the explicit details of our conclusions we will consider all six of the following possibilities depending on the value of r_3,

$$r_3 = 2r_2 - r_1 \longleftrightarrow \begin{cases} r_1 < r_2 < r_3 \\ \\ r_3 < r_2 < r_1 \end{cases}$$

$$r_3 = 2r_2 - r_1 - n \longleftrightarrow \begin{cases} r_1 \leq r_3 < r_2 \\ \\ r_3 < r_1 < r_2 \end{cases}$$

$$r_3 = 2r_2 - r_1 + n \longleftrightarrow \begin{cases} r_2 < r_1 \leq r_3 \\ \\ r_2 < r_3 < r_1 \end{cases}$$

The equalities on the left arise out of equation (4) and the fact that the integer $2r_2 - r_1$ satisfies, $-n \leq 2r_2 - r_1 \leq 2n$. This means that r_3 is the element in the set $\{2r_2 - r_1 + n, 2r_2 - r_1, 2r_2 - r_1 - n\}$ that lies in the interval $0 < r_3 \leq n$. Notice that $r_3 = 2r_2 - r_1$ in both cases when $r_1 < r_2$ and $r_2 < r_1$.

3.1.2. *G and the beginning of each $c\mathbf{x}$.* When $r_1 < r_2$, we pick $G \subset \Sigma$ to be the last $r_2 - r_1$ terms in T_{m_1} such that G has no specific order and $G \neq \emptyset$. Of course, the remainder of the terms in T_{m_1} are in $\overline{G}$, the compliment of G. Notice that this puts $c = t_{j_1} \in \overline{G}$. By equating the terms in T_{m_1} with the corresponding terms in $t_{j_2}\mathbf{x}t_{j_3}$, we find that the last $n - r_2 + 1$ terms of T_{m_2} (starting with $c = t_{j_2}$) are in $\overline{G}$. Also, we find that the first $r_2 - r_1$ terms of T_{m_2+1} are G.

When $r_2 < r_1$, we pick $G \subset \Sigma$ to be the last $r_1 - r_2$ terms in T_{m_2} such that G has no specific order and $G \neq \emptyset$. Obviously, the remainder of terms in T_{m_2} must be those that make up $\overline{G}$ again placing $c = t_{j_2} \in \overline{G}$. By equating the terms in T_{m_2} with the corresponding terms in $t_{j_1}\mathbf{x}t_{j_2}$ we find that the last $n - r_1 + 1$ terms of T_{m_1} (starting with $c = t_{j_1}$) are in $\overline{G}$. Also, we find that the first $r_1 - r_2$ terms of T_{m_1+1} are G.

We have discussed the appearance of G and its complement $\overline{G}$ in the beginning of each $c\mathbf{x}$. So, we set forth to describe G and $\overline{G}$ at the end of each $c\mathbf{x}$.

3.1.3. *Following G through the overlap.* It is a basic observation that because each tile is a permutation of the elements in Σ, each tile can be partitioned into G and its complement $\overline{G}$. It is fundamental to our argument that because of the equality $t_{j_1}\mathbf{x}t_{j_2} = c\mathbf{x}c = t_{j_2}\mathbf{x}t_{j_3}$, the elements of G form a contiguous collection of elements in each tile involved in our overlap excluding T_{m_i} (each of which will need further description), either the beginning or the ending of each tile. The idea involved in following G through the overlap is quite simple, we illustrate it in one particular case $r_1 < r_2 < r_3$.

We have explicitly described the location of G at the beginning of each $c\mathbf{x}$. We will now use our example $r_1 < r_2 < r_3$ to show to the reader how the tiling of our sequence can be used to find the location of G at the end of each $c\mathbf{x}$. In doing so, we will refer to Figure 1.

In Figure 1, we have displaced the overlap from our sequence (represented by the continuous solid horizontal line). We have also split our overlap in half leaving T_{m_2} intact for equality purposes. We have placed $t_{j_1}\mathbf{x}t_{j_2}$ over $t_{j_2}\mathbf{x}t_{j_3}$ with t_{j_1} directly over t_{j_2} and t_{j_2} directly over t_{j_3} so that we can see equality of terms simply by looking straight up or straight down (displayed by vertical arrows). The set of terms G is represented by a horizontal solid line above and below our sequence line, and the set of terms $\overline{G}$ is represented by horizontal dotted lines above and below the sequence line. Also, notice that we have drawn in the edges of the tiles with smaller vertical black lines.

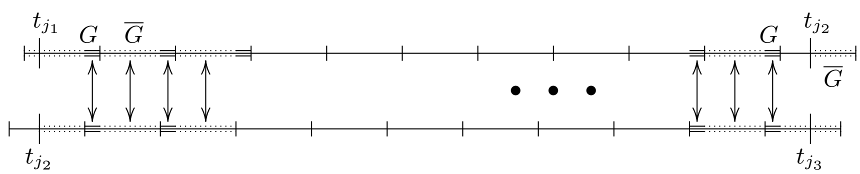

Figure 1: The situation when $r_1 < r_2 < r_3$.

Now notice that by using the tiles we can equate terms in $t_{j_1}\mathbf{x}t_{j_2}$ with $t_{j_2}\mathbf{x}t_{j_3}$ all the way through the overlap. Since we know that G occurs in the first $r_2 - r_1$ terms of T_{m_2+1}, then $\overline{G}$ is the last $n - (r_2 - r_1)$ terms of $T_{m_2} + 1$. This causes $\overline{G}$ to be the first $n - (r_2 - r_1)$ terms of T_{m_1+1}, and thus G appears in the last $r_2 - r_1$ terms of T_{m_1+1}. Thus we can conclude that G occurs in the last $r_2 - r_1$ terms of all the tiles in $t_{j_1}\mathbf{x}t_{j_2}$ except for T_{m_2}. We can also conclude that G occurs in the first $r_2 - r_1$ terms of all the tiles in $t_{j_2}\mathbf{x}t_{j_3}$ up through T_{m_3-1}. We can approach every case by the same process.

3.1.4. *G and how each $c\mathbf{x}$ ends.* We now will explain the conclusions for the six possible cases that we defined earlier, leaving the actual drawing to the reader.

Case $r_1 < r_2 < r_3$ (as seen in Figure 1). After we follow G through the overlap, we find that G occurs in the first $r_2 - r_1$ terms of T_{m_3}. Recall $r_3 = 2r_2 - r_1$. So, we have that the next $r_3 - (r_2 - r_1) = r_2$ terms of T_{m_3} are not in G. Notice that the size of G, $r_2 - r_1$, added to r_2 make up all of r_3. This places the boundary between T_{m_2-1} and T_{m_2} exactly in line with the end of G in T_{m_3} and the beginning of $\overline{G}$. We then equate the first terms in T_{m_3} with those in T_{m_2} to find that G occurs nowhere in T_{m_2}. So now, we have described T_{m_2} fully. Earlier we defined G such that $\overline{G}$ occurred from t_{j_2} to the end of the tile, and we have just shown that the first r_2 terms of T_{m_2} (which includes t_{j_2}) must be in $\overline{G}$. So G does not appear in anywhere in T_{m_2}, and since $G \neq \emptyset$, we must have a contradiction.

Cases $r_1 \leq r_3 < r_2$ and $r_3 < r_1 < r_2$. After we follow G through the overlap, we find that G occurs in the first $r_2 - r_1$ terms of T_{m_3-1}. So, $\overline{G}$ occurs in the final $n - (r_2 - r_1)$ terms of T_{m_3-1} causing the first $n - (r_2 - r_1)$ terms of T_{m_2} to be $\overline{G}$. Notice that $r_2 = [n - (r_2 - r_1)] + r_3$. So the boundary between $\overline{G}$ and G in T_{m_2}

coincides with the boundary between T_{m_3-1} and T_{m_3}. This means that $t_{j_2} \in G$, but we assumed that $c \notin G$ earlier which is a contradiction.

Case $r_3 < r_2 < r_1$. After we follow G through the overlap, we find that G occurs in the last $r_1 - r_2$ terms of T_{m_3-1}. This causes G to occur in the first $r_1 - r_2$ terms of T_{m_2} by equality of $t_{j_1}\mathbf{x}t_{j_2}$ and $t_{j_2}\mathbf{x}t_{j_3}$. To describe the remaining terms of T_{m_2} up to and including t_{j_2} consider $r_2 - (r_1 - r_2) = r_3$. So $\overline{G}$ occurs in the next r_3 terms after G. Thus we have that G is repeated twice in T_{m_2} so we have our contradiction.

Cases $r_2 < r_1 \le r_3$ and $r_2 < r_3 < r_1$. After we follow G through the overlap we find that G occurs in the first $r_1 - r_2$ terms of T_{m_2-1}. This causes $\overline{G}$ to occur in the final $n - (r_1 - r_2)$ terms of T_{m_2-1} and thus the first $n - (r_1 - r_2)$ terms of T_{m_3}. Since $r_2 = r_3 - [n - (r_1 - r_2)]$, we see that the left boundary of T_{m_2} coincides with the right boundary of these first $n - (r_1 - r_2)$ terms of T_{m_3}. In particular, this means that the last $r_1 - r_2$ terms of T_{m_3}, which include c, are in G. But, this contradicts the fact that $c \notin G$.

3.2. **Case 2:** $\|c\mathbf{x}\| \equiv 0 \pmod n$

We begin by considering some $\pi \in S_n$ the symmetric group on n terms. Note that we may apply π to any string by requiring π to act on each individual term, so $\pi(t_1t_2 \dots t_s) = \pi(t_1)\pi(t_2)\dots\pi(t_s)$. Thus π can be treated as a morphism. Moreover, $\pi : \Sigma^* \to \Sigma^*$ is an invertible map because $\pi \in S_n$. Thus $w \in \Sigma^*$ contains an overlap if and only if $\pi(w) \in \Sigma^*$ contains an overlap.

Define the function $d_{(a,n)} : \mathbb{N} \to \mathbb{N}$ by $d_{(a,n)}(m) = (m-1)n + a$. Now if we let $M = (t_s)$ be a sequence, then define the sequence given by the function $D_{(a,n)}(M)$ to be the subsequence $(t_{d_{(a,n)}(s)})$ of M. So for $i \in \{1, 2, \dots, n\}$ arbitrary we have that

$$D_{(i,n)}(\ell^\omega(t_1)) = t_i t_{i+n} t_{i+2n} \dots .$$

Define $\pi_i : \Sigma \to \Sigma$ with $\pi_i \in S_n$, such that if $\mathcal{L}_{t_1} = \{t_1, t_2, \dots, t_i, \dots, t_n\}$, $\pi_i(t_1) = t_i$. Recall that $\mathcal{L}_t$ refers to the t^{th} row of our Latin square $\mathcal{L}$. So we have that π_i maps each term in the first column of our Latin square, to the i^{th} entry of its corresponding row. Now, we want to show that $\pi_i(\ell^\omega(t)) = D_{(i,n)}(\ell^\omega(t))$ for all $t \in \Sigma$. So take

$$\begin{aligned} D_{(i,n)}(\ell^\omega(t_1)) &= D_{(i,n)}(\ell(\ell^\omega(t_1)) \\ &= D_{(i,n)}(\ell(t_1)\ell(t_2)\ell(t_3)\cdots) \\ &= \pi_i(t_1)\pi_i(t_2)\pi_i(t_3)\cdots \\ &= \pi_i(\ell^\omega(t_1)). \end{aligned}$$

Since $\pi_i \in S_n$ is invertible we can conclude that $D_{(i,n)}(\ell^\omega(t_1))$ contains an overlap if and only if $\ell^\omega(t_1)$ contains an overlap.

Since $\|c\mathbf{x}\| \equiv 0 \pmod n$ pick $i \equiv j_1 \equiv j_2 \equiv j_3 \pmod n$. By applying $D_{(i,n)}$ to (4) we obtain

$$D_{(i,n)}(\ell^\omega(t_1)) = A_i\ t_{j_1}\ \mathbf{x}_i\ t_{j_2}\ \mathbf{x}_i\ t_{j_3}\ B_i$$

where

$$\begin{aligned} A_i &= D_{(i,n)}(A) &= t_i t_{i+n} t_{i+2n} \ldots, \\ \mathbf{x}_i &= D_{(i,n)}(\mathbf{x}) &= t_{j_1+n} t_{j_1+2n} \ldots t_{j_1+(m-1)n} \\ & &= t_{j_2+n} t_{j_2+2n} \ldots t_{j_2+(m-1)n}, \\ B_i &= D_{(i,n)}(B) &= t_{j_3+n} t_{j_3+2n} t_{j_3+3n} \ldots, \end{aligned}$$

and $m = \|c\mathbf{x}\|/n$. Observe that $D_{(i,n)}(\ell^\omega(t_1))$ contains a shorter overlap which implies that $\ell^\omega(t_1)$ also contains a shorter overlap, a contradiction of our assumption. □

4. Acknowledgements

I would like to thank Dr. Griff Elder, my research advisor for his guidance. I would also like to thank Dr. Dan Farkas for introducing me to the Thue-Morse sequence at Virginia Tech's Undergraduate Research workshop in 2006 funded by the NSA and for the idea that lead to the argument for $\|c\mathbf{x}\| \equiv 0 \pmod{n}$.

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