Bond Percolation: Critical Probability of the Inhomogeneous Case on the Square Lattice

Brad Shapiro
May 3, 2007
Advised by Dr. William Floyd, Department of Mathematics

Abstract

In this paper, the study of bond percolation on the square lattice is explored. In particular, the probability at which there exist infinite clusters on an inhomogeneous model of the square lattice is proven from the basics used to prove the model with equal probabilities.

Introduction

What is bond percolation? What is a critical probability? The idea behind percolation may be illustrated by the problem of water on a porous rock. Consider a piece of rock that has water poured on it. Will the center of the rock be wet? We simplify the rock to a square graph of the integers, $\mathbf{L}^2$. Every time water gets to a vertex, it must figure out whether there is somewhere for it to go. In order for the water to continue, there must be another edge which is open for travel. For the graph, we consider the probability $p \varepsilon [0, 1]$, that each edge is open. Therefore each edge is closed with probability $1 - p$. What conditions are sufficient to nearly guarantee that the center of the rock is wet? It turns out that if there is an infinite cluster of open edges, the center of the rock will almost surely be wet. We consider p_c to be the "critical probability," or the probability above which there is almost surely an infinite open cluster.

The inhomogeneous case of bond percolation assigns separate probabilities for horizontal and vertical edges, p_h and p_v, respectively. The idea of a critical probability is now to find a critical surface, as we are now working with probabilities in [0 , 1] x [0 , 1]. Known results give us three points on the surface; however the entirety of the surface is not clear based on results from the basic model. Intuition would suggest that symmetry might play a role in determining the connection. In section IV, the critical surface is shown to be a line segment connecting the three known points.

The paper will be organized as follows. Section I presents some basic notation and results from the simplest case of bond percolation are explained. Section II explains the problem of the inhomogeneous model and some important preliminary results. Section III contains a proof of the critical probability for the inhomogeneous case. Section IV introduces a discussion of possible applications to bond percolation. Section V offers some concluding remarks.

I. Results from the Basic Theory

First, a quantity of great interest in percolation theory is the *percolation probability*.

Definition: The *percolation probability* $\theta(p)$, is the probability that a given vertex belongs to an infinite open cluster.

$$\theta(p) = P_p(|C| = \infty)$$

Without loss of generality, we can let this point be the origin by the translation invariance of the lattice and probability measure. It is also fundamental to percolation that there exists a critical probability, p_c, such that:

$$\theta(p) = 0 \text{ if } p < p_c$$
$$\theta(p) > 0 \text{ if } p > p_c$$

Now, for a more formal definition of p_c,

Definition: *The critical probability* $p_c = \sup\{p : \theta(p) = 0\}$.

Here, now, are some interesting results that have been shown for bond percolation on a square lattice:

Proposition 0: *Let G be a finite, connected subgraph of* $\mathbf{L}^2$. *There exists a unique circuit,* $\sum(G)$ *of* $\mathbf{L_d}^2$ *containing G in its interior and with the property that every edge of* $\sum(G)$ *crosses an edge of the boundary of G.*

There is some relatively interesting topology that goes on here, but for our purposes we'll call this obvious based on the geometry of the situation.

Theorem 1: *The probability* $\psi(p)$ *that there exists an infinite open cluster satisfies*

$$\psi(p) = 0 \text{ if } \theta(p) = 0$$
$$\psi(p) = 1 \text{ if } \theta(p) > 0$$

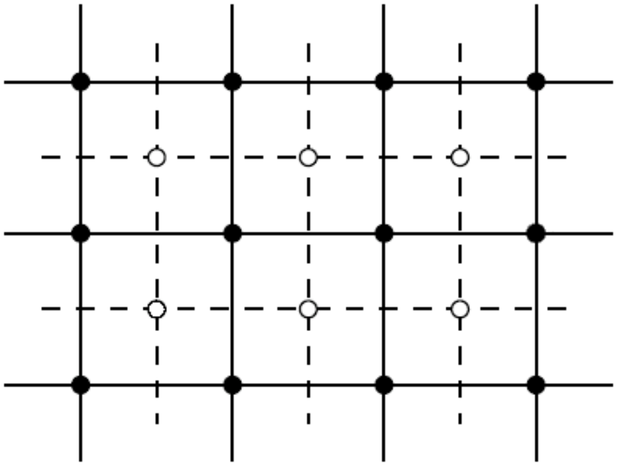
The proof of this is presented on page 14 of Grimmett.

Theorem 2: *The critical probability of bond percolation on the square lattice equals ½.*

The proof of this theorem is the main result, and probably the most important one in percolation theory. The first half of Chapter 11 in Grimmett's book is devoted to the proof of this theorem. It will not be proved here, but it does use the concept of duality, which will be described here.

The planar dual of a graph *G* is obtained by first placing a vertex in each face of *G*. We then connect each pair of neighboring vertices by an edge. We define neighboring by requiring that the edge connecting the vertices only cross one edge of *G*. We call each edge of the dual open (respectively closed) if it crosses an open (respectively closed) edge of *G*. In the case of the square lattice, $\mathbf{L}^2$, the dual, $\mathbf{L}_d^2$ is clearly isomorphic to $\mathbf{L}^2$. In this case we define the vertices of the dual to be the set $\{x + (\frac{1}{2}, \frac{1}{2}) : x \,\varepsilon\, \mathbf{Z}^2\}$. This also implies that there is a bond percolation process on the dual that has the same edge probabilities as $\mathbf{L}^2$. This concept is employed to obtain the result of Theorem 2, and it will be used to gain the result of the inhomogeneous case. See Figure 1 below.

Figure 1



Note that the dark circles represent vertices in the graph and open circles represent vertices in the dual

Theorem 3: *It is the case that* $\theta(\frac{1}{2}) = 0$.

This is actually used as a Lemma by Grimmett in order to prove Theorem 2. With Theorem 2 established, this implies that $\theta(p_c) = 0$, almost surely, there does not exist an infinite open cluster at the critical point. Grimmett's proof of this is on page 288, and similar techniques are used in proving Theorem 6.

II. The Inhomogeneous Problem and Preliminaries

In the inhomogeneous case of bond percolation, we again consider the square lattice. This time, instead of assigning a probability p that applies to every edge, we assign probabilities p_h and p_v to horizontal and vertical edges, respectively. Now, the basic problem is to find some condition on p_h and p_v that will guarantee the existence of an infinite open cluster. Consider what we already know:

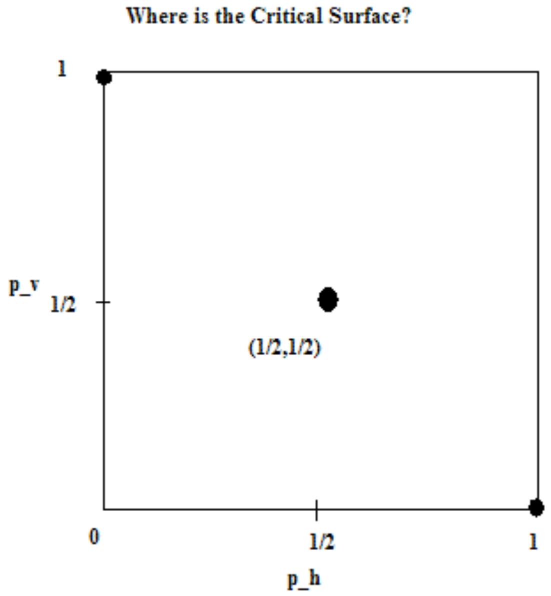
i.) If $p_h = p_v$, we have exactly the same thing as our basic model, with critical probability ½. Note also that in this case $p_h + p_v = 1$

ii.) If $(p_h, p_v) = (1,a)$ or $(a,1)$, with $a \,\varepsilon\, [0,1]$, then trivially, there will be an infinite open cluster.

iii.) By symmetry of the square graph, if (p_h , p_v) results in an infinite open cluster, (p_v , p_h) will as well.

See Figure 2 below.

Figure 2



Critical Surface of Inhomogeneous Model

Knowing all of these things, one may conjecture that an the critical probability p = (p_h , p_v) occurs when $p_h + p_v = 1$. This is precisely what will be proven in the next section. First, some preliminaries are necessary for that proof.

Theorem 4: Uniqueness of the infinite open cluster. *If p is such that* $\theta(p) > 0$*, then* P_p (there exists exactly one infinite open cluster) = 1.

This is proven on page 198 of Grimmett. This turns out to be especially useful for contradiction arguments.

Theorem 5: Exponential tail decay of the radius of an open cluster. *Let S(n) be the ball of radius n centered at the origin(the distance from the origin to a point in S(n) is always less than or equal to n). Let* A_n *be the event that there exists an open path joining the origin to some vertex in the surface of S(n). If* $p < p_c$*, there exists* $\psi(p) > 0$ *such that* $P_p(A_n) < e^{-n\psi(p)}$.

This implies that the expected size of the open cluster that contains the origin is finite. It also provides an estimate on the tail probabilities of that cluster. Theorem 4 is proven on pgs 88-89 of Grimmett.

Square Root Trick: *If* $A_1, A_2, \ldots, A_m$ *are increasing events having equal probability, then* $P_p(A_1) \geq 1 - \{1 - P_p(\square A_i)\}^{1/m}$

This is valuable for more than just our application, and it is a simple proof. Grimmet shows it on page 289.

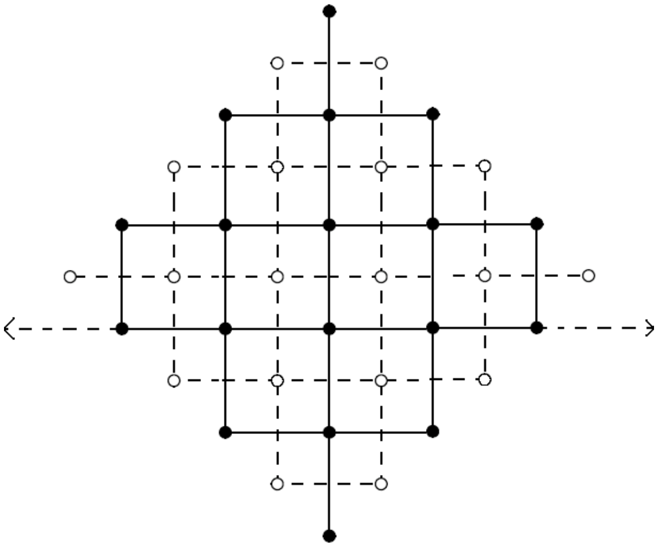
We now have the necessary tools to begin proving the critical surface of the inhomogeneous case.

III. Proof of Inhomogeneous Case on Square Lattice

Theorem 6: The Critical Surface of the Inhomogeneous Square Lattice. *Assume that* $\mathbf{p} = (p_h, p_v)$ *such that* $p_h, p_v < 1$. *Now define* $\varphi(\mathbf{p}) = p_h + p_v$. *Then* $\theta(\mathbf{p}) = 0$ *if* $\varphi(\mathbf{p}) \leq 1$, and $\theta(\mathbf{p}) > 0$ if $\varphi(\mathbf{p}) > 1$.

Proof: This proof relies heavily on the self duality property. Now, call each edge of the dual to be open (respectively closed) if it crosses an open edges (respectively closed edge) of $\mathbf{L}^2$. So every horizontal (respectively vertical) edge of $\mathbf{L_d}^2$ is open with probability p_h(respectively p_v). Let n be a positive integer, and let $D(n)$ be the subgraph of $\mathbf{L}^2$ consisting of the vertices within an offset diamond defined by $\{(x_1,x_2)\, \varepsilon\, \mathbf{R}^2 : \ |x_1| + |x_2 - ½\,| \leq n + ½\,\}$. See $D(n)$ and a subset of the dual lattice $D(n)_d$ in Figure 3 below.

Figure 3



The solid edges represent $D(n)$ and the dotted edges represent $D(n)_d$

Note that $D(n)$ and $D(n)_d$ are isomorphic graphs. Now, we employ theorem 4 to get:

(6.1) Either $P_p(\ I = 1\ ) = 0$ or $P_p(\ I = 1\ ) = 1$, when $0 < p_h, p_v < 1$, where I is the number of infinite open clusters in $\mathbf{L}^2$

Now, we move on to an intermediate result, call it

Lemma 1: $\theta(\boldsymbol{p}) = 0$ *if* $\varphi(\boldsymbol{p}) = 1$ and p_h , $p_v < 1$.

Assume to the contrary – that $\theta(\boldsymbol{p}) > 0$ when$\varphi(\boldsymbol{p}) = 1$. Consider the dual $D(n)_{\mathrm{d}}$. Note that each horizontal edge is closed with probability $(1 - \mathrm{p_v})$ and each vertical edge is closed with probability $(1 - \mathrm{p_h})$. Now, since $(1 - \mathrm{p_v}) + (1 - \mathrm{p_h}) = 2 - \mathrm{p_v} - \mathrm{p_h}$,

(6.2) $\varphi(1 - \boldsymbol{p}) = 1$.

Now, let $\mathrm{A}^{\mathrm{tr}}(n)$ (respectively $\mathrm{A}^{\mathrm{tl}}(n)$, $\mathrm{A}^{\mathrm{br}}(n)$, and $\mathrm{A}^{\mathrm{bl}}(n)$) be the event that some vertex on the top right (respectively top left, bottom right, and bottom left) side of $D(n)$ lies in an infinite open path of $\mathbf{L}^2$ which uses no other vertex of $D(n)$. These are clearly non-decreasing events in n and to see they have equal probability note that the model is symmetric about the vertical axis and the line $\{(\mathrm{x}_1,\mathrm{x}_2)\ \varepsilon \mathbf{R}^2 :\ \mathrm{x}_2 = ½\ \}$. The symmetry about the vertical line gives us that $\mathrm{A}^{\mathrm{tr}}(n) = \mathrm{A}^{\mathrm{tl}}(n)$ and $\mathrm{A}^{\mathrm{br}}(n) = \mathrm{A}^{\mathrm{bl}}(n)$. The symmetry about the horizontal line gives us that $\mathrm{A}^{\mathrm{tr}}(n) = \mathrm{A}^{\mathrm{br}}(n)$ and $\mathrm{A}^{\mathrm{tl}}(n) = \mathrm{A}^{\mathrm{bl}}(n)$. Now the union of these events is the event that some vertex in $D(n)$ lies in an infinite open cluster. Since there exists an infinite open cluster with probability 1,

$P_{\varphi(\boldsymbol{p}) = 1}(\mathrm{A}^{\mathrm{tr}}(n) \ \square\ \mathrm{A}^{\mathrm{tl}}(n) \ \square\ \mathrm{A}^{\mathrm{br}}(n) \ \square\ \mathrm{A}^{\mathrm{bl}}(n)) \rightarrow 1$ as $n \rightarrow \infty$

By the Square Root Trick above, it follows that

(6.3) $P_{\varphi(\boldsymbol{p}) = 1}(\mathrm{A}^{\mathrm{u}}(n)) \rightarrow 1$ as $n \rightarrow \infty$ for u = tr, tl, br, bl.

Since this is the case, choose integer N such that

(6.4) $P_{\varphi(\boldsymbol{p}) = 1}(\mathrm{A}^{\mathrm{u}}(N)) > \square$.

Now move to the dual lattice, $D(n)_d$ and let $\mathrm{A_d}^{\mathrm{tr}}(n)$ (respectively $\mathrm{A_d}^{\mathrm{tl}}(n)$, $\mathrm{A_d}^{\mathrm{br}}(n)$, and $\mathrm{A_d}^{\mathrm{bl}}(n)$) be the event that a vertex on the top right (respectively top left, bottom right, and bottom left) side of $D(n)_d$ lies in an infinite closed path of $\mathbf{L}_{\mathrm{d}}^2$ which uses no other vertex of $D(n)_d$. Now each vertical edge of $D(n)_d$ is closed with probability $1 - p_h$ and each horizontal edge of $D(n)_d$ is closed with probability $1 - p_v$. So the probability governing the dual lattice is $1 - \mathbf{p} = (1 - p_v, 1 - p_h)$. By (6.1) and (6.4), we have that

(6.5) $P_{\varphi(1 - \boldsymbol{p}) = 1}(\mathrm{A_d}^{\mathrm{u}}(N)) = P_{\varphi(\boldsymbol{p}) = 1}(\mathrm{A}^{\mathrm{u}}(N)) > \square$ for u = tr, tl, br, bl.

Now define an event $\mathrm{A} = \mathrm{A}^{\mathrm{tl}}(N) \ \square\ \mathrm{A}^{\mathrm{br}}(N) \ \square\ \mathrm{A_d}^{\mathrm{tr}}(N) \ \square\ \mathrm{A_d}^{\mathrm{bl}}(N)$, that there exist infinite open paths of $\mathbf{L}^2 \backslash D(n)$ touching the top left and bottom right sides of $D(n)$, and there exists infinite closed paths of $\mathbf{L}^2_d \backslash D(n)_d$ touching the top right and bottom left sides of $D(n)_d$. Now the probability that the complement of A occurs satisfies

(6.6) $P_{\varphi(\boldsymbol{p}) = 1}(\square) \leq P_{\varphi(\boldsymbol{p}) = 1}(\square^{\mathrm{tl}}(N)) + P_{\varphi(\boldsymbol{p}) = 1}(\square^{\mathrm{br}}(N)) + P_{\varphi(\boldsymbol{p}) = 1}(\square_{\mathrm{d}}{}^{\mathrm{tr}}(N)) + P_{\varphi(\boldsymbol{p}) = 1}(\square_{\mathrm{d}}{}^{\mathrm{bl}}(N))$

$\leq \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$ by (6.4) and (6.5)

This conveniently gives us that $P_{\varphi(\boldsymbol{p}) = 1}(\mathrm{A}) > \tfrac{1}{2}$. If A occurs, then we have two infinite open paths in $\mathbf{L}^2 \backslash D(n)$. These paths must be disjoint because they are separated by two infinite closed paths of the dual. Similarly, we have two infinite closed paths in $\mathbf{L}_{\mathrm{d}}^2 \backslash D(n)_{\mathrm{d}}$ which must be disjoint because they are separated by infinite open paths of $\mathbf{L}^2 \backslash D(n)$. Now, by our initial assumption that $\theta(\boldsymbol{p}) = 1$ *if* $\varphi(\boldsymbol{p}) = 1$ together with Theorem 4, we have almost surely a unique infinite open cluster. This means that there is almost surely an open connection between the two infinite paths described above. Any such connection would form a barrier to potential closed connections of the dual that join the two infinite closed clusters. So, almost surely on A the dual lattice contains *at least* two infinite closed clusters. That event has probability 0, we get that $P_{\varphi(\boldsymbol{p}) = 1}(\mathrm{A}) = 0$, which is a contradiction of (6.6). So our initial hypothesis that $\theta(\boldsymbol{p}) = 1$ *if* $\varphi(\boldsymbol{p}) = 1$ must be false, and we have the desired result of Lemma 1, that $\theta(\boldsymbol{p}) = 0$ *if* $\varphi(\boldsymbol{p}) = 1$ and $p_h, p_v < 1$.□

Lemma 2: $\theta(\boldsymbol{p}) > 0$ *if* $\varphi(\boldsymbol{p}) > 1$

For contradiction, assume that $\theta(\boldsymbol{r}) = 0$ for some $\boldsymbol{r} = (r_h, r_v)$ satisfying $\varphi(\boldsymbol{r}) > 1$. Let

$\mathbf{p} = (p_h, p_v) = (r_h / (r_h + r_v), r_v / (r_h + r_v))$

so that $\varphi(\boldsymbol{p}) = 1$ and $p_h < r_h$, $p_v < r_v$. Since $\theta(\boldsymbol{r}) = 0$, the point $\mathbf{p} = (p_h, p_v)$ is strictly in the subcritical phase of percolation, satisfying the hypothesis of Theorem 5. Define $S(n) = \{x \,\varepsilon\, \mathbf{Z}^2 : \delta(0, x) \leq n\}$. So by Theorem 5, there exists a $\rho = \rho(\mathbf{p})$ so that

(6.7) $P_{\mathbf{p}}(0$ □ $\partial S(n)) \leq e^{-n\rho}$ for all $n \geq 1$, where 0 □ $\partial S(n)$ is the event that there is an open path from the origin to the surface of $S(n)$

Now, define B_n (respectively C_n) as the event that $D(n)$ (respectively $D(n)_{\mathrm{d}}$) contains an open path(respectively closed path) joining some vertex on the upper left side to some vertex on the lower right side (respectively some vertex on its lower left side to some vertex on its upper right side). We now argue that $P_{\mathbf{p}}(B_n) + P_{\mathbf{p}}(C_n) = 1$. It is clear that B_n □ $C_n = \phi$ since if both B_n and C_n occur then there exists an open path in $D(n)$ which crosses an closed path in $D(n)_{\mathrm{d}}$. To see this point, look at the point of intersection between the two paths. At that point, an open edge of $D(n)$ must cross a closed edge of $D(n)_{\mathrm{d}}$ which is impossible. Also, we argue that either B_n or C_n *must* occur. Suppose that B_n does not occur, and let E be the set of all vertices of $D(n)$ which may be reached from the left side of $D(n)$ along open paths; make E a graph by adding all open edges of $D(n)$ joining pairs of vertices in E. By Proposition 0 (and clearly from the geometry), there exists a closed path of $\mathbf{L}^2_{\mathrm{d}}$ crossing $D(n)_{\mathrm{d}}$ and which crosses only edges of $D(n)$ contained in the edge boundary of E. Thus C_n occurs whenever B_n does not. So we have our desired result.

(6.8) $P_{\mathbf{p}}(B_n) + P_{\mathbf{p}}(C_n) = 1$.

Now the probability that an edge is closed in $\mathbf{L}^2_{\mathrm{d}}$ is $\mathbf{p}' = (1 - p_v, 1 - p_h) = (p_h, p_v) = \mathbf{p}$. So the set of closed edges in the dual is governed by the measure $P_{\mathbf{p}}$. Therefore, we get a final string:

(6.9) $P_{\mathbf{p}}(B_n), P_{\mathbf{p}}(C_n) \leq (n+1)\, P_{\mathbf{p}}\, (\, 0 \,□\, \partial\, S\,(n)\,) \leq (n+1)\, e^{-n\rho}$

Since both $P_{\mathbf{p}}(B_n)$ and $P_{\mathbf{p}}(C_n)$ are less than $(n+1)\, e^{-n\rho}$, we have a contradiction with (6.8) for sufficiently large n. We therefore have our desired result, that $\theta(\boldsymbol{p}) > 0$ *if* $\varphi(\boldsymbol{p}) > 1$. □

The combination of Lemma 1 and Lemma 2 give us our final conclusion, that $\theta(\boldsymbol{p}) = 0$ *if* $\varphi(\boldsymbol{p}) \leq 1$, and $\theta(\boldsymbol{p}) > 0$ if $\varphi(\boldsymbol{p}) > 1$. □

IV. Applications

As the introduction alluded, the origins of the applications to percolation theory were in geology. Important ideas and questions included: if I find oil in the rock, how likely is it that it will be a really large cache of oil? How large is that cache likely to be? Most of the application studies found seemed to be much more inspiration than anything else. They often find themselves retroactively modeling the past rather than predicting the future. There is a problem of determining what the critical probabilities are in rocks, especially considering that not only are they inhomogeneous, but they are likely to have many more than two probabilities governing the modes of travel.

Another application that I set out to find the merit in is the idea of diffusion of innovation across social networks in economics. The general idea is as follows. Define a network to be a countable number of actors. Each actor constitutes a vertex for the purposes of our graph. From each vertex, edges can be made to be different modes of communication. Critical probabilities would likely have to do with the usefulness of the invention, the expected popularity of the invention, or some other characteristics. Obviously, there is a massive oversimplification in this model. Every person in a network is not likely to even have approximately the same communication habits, consumption habits, number of other people they are connected to, or incomes. Also it is impossible to judge which characteristics will affect the critical probabilities. It isn't strictly usefulness. Take, for example, the Dvorak keyboard. Dvorak ran a study based on which keys on a keyboard were most often hit, and put those keys in the middle of the keyboard. He did a number of other studies to minimize the necessary finger movement for typing. In contrast, the QWERTY keyboard was created many years before in order to slow people down. Typists were typing so quickly as to break the hammers on the typewriters. With that problem gone, it seems as though the probability of an infinite spread of Dvorak would be imminent. However, as I sit here and type this on a QWERTY, we know that not to be true.

Most other applications seemed to be beyond the scope of my abilities, or purely inspiration.

V. Concluding Remarks

The bond percolation model on simple graphs does provide some interesting results. The questions that can be asked are numerous, but they are not easy to answer. Further knowledge in graph theory, probability theory, topology, and algebra will likely help the researcher to find more useful or interesting results. On a more basic level, most results have been proven before. I must note that my results are not original. Harry Kesten proved the critical surface of the inhomogeneous square lattice in 1982. When first approaching the problem, I was unaware of the existence of that proof, and went on without the help of Kesten's work. Grimmett provides a sketch of the proof with some ideas, but he leaves out many of the details. Most of the work on this project was concerned with filling in those details and finding meaningful applications. Unfortunately, the latter did not follow immediately from the theory.

References

Grimmett, Geoffrey. *Percolation*. New York: Springer-Verlag Berlin. 2nd Ed, 1999.

Kesten, Harry. *Percolation Theory for Mathematicians,* Boston: Birkhäuser, 1982.

Wikipedia, Wolfram MathWorld, and other informal online encyclopedias for background of thought, but nothing explicitly taken from them for the paper.