

HAUSDORFFNESS AND VANISHING OF THE FIRST COHOMOLOGY GROUP $H^1(G, M)$

ANTHONY J. NARKAWICZ (VIRGINIA TECH)

Abstract. This paper characterizes the first cohomology group $H^1(G, M)$ where M is a Banach space (with norm $\| \cdot \|_M$) that is also a left $\mathbb{C}G$ module such that the elements of G act on M as continuous \mathbb{C} -linear transformations. We study this group for G an infinite, finitely generated group. Of particular interest are the implications of the vanishing of the group $H^1(G, M)$. The first result is that $H^1(G, \mathbb{C}G)$ imbeds in $H^1(G, M)$ whenever $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \in \mathbb{N}$. This is an unpublished result and shows immediately that if $H^1(G, M) = 0$, then G can have only 1 end. Secondly (also a new result), we show that $H^1(G, M)$ is not Hausdorff if and only if there exist $f_i \in M$ with norm 1 ($\|f_i\|_M = 1$) for all i with the property that $\|gf_i - f_i\|_M \rightarrow 0$ as $i \rightarrow \infty$ for every $g \in G$. This is then used to show that if M and $\| \cdot \|_M$ satisfy certain properties and if G satisfies a “strong Følner condition,” then $H^1(G, M)$ is not Hausdorff. For the second half of this paper, we give several applications of these last two theorems focusing on the group $G = \mathbb{Z}^n$. These applications were the main source of motivation for this project.

1. Introduction

Motivation for this paper comes from two papers, one by Mohammed E.B. Bekka and Alain Valette ([3]) and the other, an expository paper, by Edward G. Effros ([2]). The first paper examines the group $H^1(G, L^2(G))$ and focuses on the implications of the vanishing of this group. It shows the following:

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- (1) $H^1(G, L^2(G))$ is Hausdorff if and only if G is non-amenable.
- (2) The G -module imbedding $\mathbb{C}G \rightarrow L^p(G)$ induces an imbedding of $H^1(G, \mathbb{C}G)$ into $H^1(G, L^p(G))$, $p \geq 1$.
- (3) If $H^1(G, L^2(G)) = 0$, then G is non-amenable with just one end.

The first of these results is due in part to the following result by A. Guichardet ([1], Corollary 2.3 of Chapter III): $H^1(G, L^p(G))$ is not Hausdorff if and only if there exists a sequence e_n in $L^p(G)$ such that $\|e_n\|_p = 1$ for all n with the property that $\|ge_n - e_n\|_p \rightarrow 0$ for all $g \in G$. We show that $L^p(G)$ (with norm $\|\cdot\|_p$) may be replaced by any Banach space M (with norm $\|\cdot\|_M$) that is a $\mathbb{C}G$ module and has the property that the elements of G act on M as continuous \mathbb{C} -linear transformations. This is not trivial from Guichardet's theorem. In fact, the topology on $H^1(G, L^p(G))$ (induced by the L^p -norm topology on $L^p(G)$) is entirely different from the topology on $H^1(G, M)$ which is induced by the norm topology on M . This can be used to show that if G satisfies a "strong Følner condition," then $H^1(G, M)$ is not Hausdorff.

As for the second result, we also show that when $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \in \mathbb{N}$, the G -module imbedding $\mathbb{C}G \rightarrow M$ induces an imbedding of groups $H^1(G, \mathbb{C}G) \rightarrow H^1(G, M)$. This (along with a minor result in [3]) shows that if $H^1(G, M) = 0$, then G can have only one end.

The second paper (by E.G. Effros) characterizes the \mathbb{C} algebra $C_{red}^*(\mathbb{Z}^n)$ and provides a very explicit description of this algebra. He shows first that $C_{red}^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ as \mathbb{C} algebras (where \mathbb{T}^n is the n -torus). Showing that $C_{red}^*(\mathbb{Z}^n)$ has no non-trivial idempotents of course then shows that neither does $C(\mathbb{T}^n)$, which shows that \mathbb{T}^n is connected. Though this result is completely trivial, it ensues some interesting mathematics and gives rise to the paper's title: *Why the Circle is Connected: An Introduction to Quantized Topology*. What Effros's paper does mainly for ours is give us an explicit isomorphism from $C(\mathbb{T}^n)$ onto $C_{red}^*(\mathbb{Z}^n)$.

Our first criterion for the Hausdorffness of $H^1(G, M)$ (Theorem 3) and the isomorphism between $C(\mathbb{T}^n)$ and $C_{red}^*(\mathbb{Z}^n)$ allow us to show that for $n \geq 1$, $H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$ is not Hausdorff and so $\dim_{\mathbb{C}} H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n)) = \infty$.

Secondly, the isomorphism from $C(\mathbb{T}^n)$ onto $C_{red}^*(\mathbb{Z}^n)$ allows us to describe the groups $H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$ explicitly. This in turn shows that there is a natural sequence of imbeddings

$$0 \rightarrow H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z})) \rightarrow H^2(\mathbb{Z}^2, C_{red}^*(\mathbb{Z}^2)) \rightarrow H^3(\mathbb{Z}^3, C_{red}^*(\mathbb{Z}^3)) \rightarrow \dots$$

This and the fact that $H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z}))$ is not Hausdorff allows us to show that for each $n \geq 1$, $\dim_{\mathbb{C}} H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n)) = \infty$.

Finally, our second criterion for the Hausdorffness of $H^1(G, M)$ (Theorem 4) gives another (more simple) proof that $H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z}))$ is not Hausdorff.

I would like to give special thanks to my advisor, Dr. Peter Linnell. His accessibility and the opinions that he gave me about the problems I was working on during this project were absolutely crucial to this paper. I consider myself lucky to have had the opportunity to work with him and to know him.

2. Preliminaries and Definitions

Let G be an infinite, finitely generated group.

Definition 1. *By a G module M , we will mean a \mathbb{C} vector space M along with a homomorphism of G into $\text{Aut}(M)$.*

Definition 2. *G satisfies the strong Følner condition means that for every finite subset S of G and every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $M \in \mathbb{N}$ there exists a finite subset X of G such that $|X| > M$ and $|X - g \cdot X| < N$ for every $g \in S$.*

Because this definition may seem tedious, we should note that if G satisfies this condition, then G must be amenable. Also, a simple calculation shows that \mathbb{Z} satisfies the strong Følner condition.

Definition 3. $\mathbb{C}G$ is the free \mathbb{C} vector space with basis G . In other words, $\mathbb{C}G$ is the set of all finite formal sums of the form $\sum_g a_g g$ where the a_g are in \mathbb{C} and the g are in G .

$\mathbb{C}G$ is a G module in the obvious way (G acts on an element of $\mathbb{C}G$ by multiplication).

Definition 4. For $1 \leq p < \infty$, $L^p(G)$ is the set of all formal sums (not necessarily finite) of the form $\sum_g a_g g$ with the property that $\sum_g |a_g|^p < \infty$.

Definition 5. The norm $\| \cdot \|_p$ on $L^p(G)$ is given by

$$\| \sum_g a_g g \|_p = [\sum_g |a_g|^p]^{1/p}.$$

$L^p(G)$ is complete in the topology induced by the norm $\| \cdot \|_p$, so $L^p(G)$ is always a Banach space. There is a multiplication defined on $\mathbb{C}G$ in the natural way ($a_i g_i \cdot a_j g_j = (a_i a_j)(g_i g_j)$). While $\mathbb{C}G$ is a ring (under componentwise addition and this multiplication as its ring multiplication), $L^2(G)$ is not necessarily a ring. One can have $\alpha, \beta \in L^2(G)$ with $\alpha \cdot \beta \notin L^2(G)$. However, it is well known that if $\alpha \in \mathbb{C}G$ and $\beta \in L^2(G)$, then $\alpha \cdot \beta \in L^2(G)$. $L^2(G)$ is certainly a Banach space over \mathbb{C} and via this multiplication, $\mathbb{C}G$ may be considered a subset of $B(L^2(G))$ (the bounded linear operators on $L^2(G)$).

Definition 6. The operator norm of an element $\alpha \in \mathbb{C}G$ is given by

$$\| \alpha \|_{op} = \text{Sup}\{ \| \alpha \cdot \beta \|_2 : \beta \in L^2(G), \| \beta \|_2 = 1 \}.$$

Definition 7. $C_{red}^*(G)$ is the metric space completion of $\mathbb{C}G$ under the operator norm $\| \cdot \|_{op}$.

We immediately have that $C_{red}^*(G)$ is composed entirely of bounded linear operators on $L^2(G)$ and that $C_{red}^*(G)$ is complete and therefore a Banach space. In

addition, $\mathbb{C}G \subset C_{red}^*(G) \subset L^2(G)$. We also know that G acts on $C_{red}^*(G)$ (in the obvious way) as continuous \mathbb{C} -linear transformations.

Now we turn our attention to the groups $H^1(G, M)$. We will view these groups in two different (though of course equivalent) ways.

First, let A_G be the set of set maps $f: G \rightarrow M$ with the property that for all a and b in G , $a \cdot f(b) - f(ab) + f(a) = 0$. Let B_G be the set of all such maps given by $f(b) = b \cdot \alpha - \alpha$ for some fixed $\alpha \in M$. Then we have the following first definition of $H^1(G, M)$.

Definition 8. $H^1(G, M)$ is the quotient group A_G/B_G .

Now consider the ring $\mathbb{Z}G$ of all finite formal sums of the form $\sum_g z_g g$ with $z_g \in \mathbb{Z}$ for all $g \in G$. Recall that for an arbitrary ring R , an R module is projective if and only if it is a direct summand of a free R module (there are several definitions). We say that an infinite exact sequence of $\mathbb{Z}G$ modules $\cdots \rightarrow P_1 \rightarrow P_0$ is a projective resolution of \mathbb{Z} as $\mathbb{Z}G$ modules if it extends to an exact sequence $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$ of $\mathbb{Z}G$ modules. The $\mathbb{Z}G$ module structure on \mathbb{Z} is given by $g \cdot z_i = z_i$ for $g \in G, z_i \in \mathbb{Z}$. Let $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \rightarrow 0$ be an exact sequence of $\mathbb{Z}G$ modules with $\cdots \rightarrow P_1 \rightarrow P_0$ a projective resolution of \mathbb{Z} . This first sequence induces another sequence

$$0 \rightarrow \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \xrightarrow{d_0^*} \text{Hom}_{\mathbb{Z}G}(P_0, M) \xrightarrow{d_1^*} \text{Hom}_{\mathbb{Z}G}(P_1, M) \xrightarrow{d_2^*} \cdots$$

which gives us our second definition of $H^1(G, M)$.

Definition 9. $H^1(G, M) = \ker(d_1^*)/\text{im}(d_0^*)$.

As it turns out, this definition is independent of the choice of projective resolution $\cdots \rightarrow P_1 \rightarrow P_0$ of \mathbb{Z} as $\mathbb{Z}G$ modules. For the final part of our paper, we will also need the following definition of $H^n(G, M)$.

Definition 10. For $n \geq 1$, $H^n(G, M) = \ker(d_n^*)/\text{im}(d_{n-1}^*)$.

Note that the two definitions of $H^1(G, M)$ are equivalent.

To define the topology on $H^1(G, M)$, we employ the first definition of this group, A_G/B_G . The topology is induced by the topology of point-wise convergence on A_G . That is for $f_n \in A_G, f_n \rightarrow 0$ means that $f_n(g) \in M$ converges to 0 in the norm $\| \cdot \|_M$ on M for every $g \in G$. We should note that this is where the topologies on $H^1(G, L^2(G))$ and on $H^1(G, M)$ differ. The basic open sets in A_G are the $f \in A_G$ such that $\|f(g_1) - a_1\|_M, \dots, \|f(g_n) - a_n\|_M < \varepsilon$ for some choice of fixed $g_i \in G, a_i \in M, \varepsilon > 0$ and $n \in \mathbb{N}$. In other words, the set of all such sets forms a basis for the topology on A_G .

Finally, we will need to view $C(\mathbb{T}^n)$ as a subset of $C(\mathbb{T}^{n+1})$ (especially in the proof of Theorem 8). We do this as follows. For $f \in C(\mathbb{T}^n)$ and $(z_1, \dots, z_{n+1}) \in \mathbb{T}^{n+1}$, define $f(z_1, \dots, z_{n+1}) = f(z_1, \dots, z_n)$.

3. The group $H^1(G, M)$

Keep supposing that G is infinite and finitely generated. In addition, suppose that M is a Banach space with norm $\| \cdot \|_M$ that is a left $\mathbb{C}G$ module and satisfies the property that G acts on M as continuous \mathbb{C} -linear transformations. Note that this implies that M is a G module.

Theorem 1. *Suppose that $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \in \mathbb{N}$. Then the G -module imbedding $\mathbb{C}G \rightarrow M$ induces an imbedding of groups $H^1(G, \mathbb{C}G) \rightarrow H^1(G, M)$.*

Proof: Our result stated above has not been published to date and could prove to be useful. We will, however, follow the proof of Z. Q. Chen as described in [3] (Proposition 1), which shows that the imbedding $\mathbb{C}G \rightarrow L^2(G)$ induces an imbedding $H^1(G, \mathbb{C}G) \rightarrow H^1(G, L^2(G))$. Since G is finitely generated, suppose that S is a finite generating set for G . For an arbitrary G -module N , we define $C^n(G, N)$ to be the set of all set maps from G^n to N . In the case $n = 0$, we set $C^0(G, N) = N$. We

have maps

$$d_0: C^0(G, \mathbb{C}G) \rightarrow C^1(G, \mathbb{C}G), \delta_0: C^0(G, M) \rightarrow C^1(G, M) \text{ and}$$

$$d_1: C^1(G, \mathbb{C}G) \rightarrow C^2(G, \mathbb{C}G), \delta_1: C^1(G, M) \rightarrow C^2(G, M)$$

defined by $[d_0(a)](g) = g \cdot a - a$, $a \in \mathbb{C}G$, $[\delta_0(f)](g) = g \cdot f - f$, $f \in M$, $[d_1(h)](g_1, g_2) = g_1 h(g_2) - h(g_1 g_2) + h(g_1)$, $[\delta_1(f)](g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$. Then

$$H^1(G, \mathbb{C}G) = \frac{\ker(d_1)}{\text{im}(d_0)} \text{ and } H^1(G, M) = \frac{\ker(\delta_1)}{\text{im}(\delta_0)}.$$

The proposed imbedding is of course $a + (\text{im}(d_0)) \mapsto a + (\text{im}(\delta_0))$, so define the natural homomorphism

$$\theta: \ker(d_1) \rightarrow \frac{\ker(\delta_1)}{\text{im}(\delta_0)}$$

via $a \mapsto a + (\text{im}(\delta_0))$. Since $\mathbb{C}G \subset M$, we have that $\text{im}(d_0) \subset \text{im}(\delta_0)$, so $\text{im}(d_0) \subset \ker(\theta)$.

Let $b \in \ker(\theta)$. Then $\text{im}(b) \subset \mathbb{C}G$ and $b \in \text{im}(\delta_0)$. So there exists $f \in M$ such that for all $g \in G$, $b(g) = g \cdot f - f \in \mathbb{C}G$. We aim to show that f must lie in $\mathbb{C}G$ and thus $b \in \text{im}(d_0)$. Thus, we want to show that f has finite support. Suppose $f = \sum a_g g$. Note that for all $h \in G$, $h \sum a_g g - \sum a_g g = \sum (a_g - a_{hg}) hg \in \mathbb{C}G$. Thus, for for all $h \in G$, $\varphi(h) = \{g \in G : a_g - a_{hg} \neq 0\}$ is finite. Since S is a finite set, it follows that

$$F(G) = \bigcup_{s \in S} \varphi(s)$$

is finite as well. We may assume that for all $s \in S$, $s^{-1} \in S$. Let X be the Cayley graph of G with vertex set G and edge set $\{(g, sg) : s \in S\}$. By assuming that S is closed under inverses, it follows that X can be viewed as an undirected graph. Thus, if r and q (elements of G) are connected by an edge, it follows that $r = sq$ for some $s \in S$. Now consider the graph $X - F(G)$. $F(G)$ is finite, so there are finitely many connected components of $X - F(G)$. Then there exists a component of $X - F(G)$ that is infinite (since G is infinite). Let q and r be in this connected component. So there exist $s_1, \dots, s_n \in S$ such that $s_1 \cdots s_n q = r$, and this path

cannot pass through $F(G)$, so we have the property that for all i , $s_i \cdots s_n q \notin \varphi(s)$ for any $s \in S$. Since $s_{i-1} \in S$ for every i , it follows that $a_{s_{i-1}s_i \cdots s_n q} = a_{s_i \cdots s_n q}$ for every i and so $a_q = a_{s_n q} = a_{s_{n-1}s_n q} = \cdots = a_r$. Thus, all of the a_g 's are equal for all the g 's in $X - F(G)$. By virtue of there being infinitely g 's in $X - F(G)$ and since $f = \sum_g a_g g \in M$ satisfies $\sum_g |a_g|^p < \infty$, it follows that for all g in this connected component of $X - F(G)$, $a_g = 0$. Thus, if $a_t \neq 0$, it follows that t lies in one of the finite connected components of this graph, and there are only finitely many such components (Since $F(G)$ is finite), so there are only finitely many such t . i.e. $a_i = 0$ for all but finitely many i , and thus $f \in \mathbb{C}G$. \square

Lemma 1. $\dim_{\mathbb{C}} H^1(G, \mathbb{C}G) = |b(G)| - 1$ where $|b(G)|$ is the number of ends of G .

Proof: For a proof of this, see [3](Lemma 2). \square

Theorem 2. Suppose that $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \in \mathbb{N}$. If $H^1(G, M) = 0$, then G has exactly 1 end.

Proof: If $H^1(G, M) = 0$, then by Theorem 1, $H^1(G, \mathbb{C}G) = 0$. The previous lemma then tells us that $|b(G)| = 1$ (G has exactly 1 end). \square

Theorem 3. $H^1(G, M)$ is not Hausdorff if and only if there exists $f_i \in M$ with norm 1 ($\|f_i\|_M = 1$) for all i with the property that $\|gf_i - f_i\|_M \rightarrow 0$ as $i \rightarrow \infty$ for every $g \in G$.

Proof: For this proof, we think of $H^1(G, M)$ as the set A_G of set maps f from G to M satisfying $a \cdot f(b) - f(ab) + f(a) = 0$ for all $a, b \in G$ modulo the set B_G of maps h of the form $h(g) = g \cdot e - e$ for some $e \in M$. Saying that $H^1(G, M)$ is Hausdorff is equivalent to saying that 0 is closed in $H^1(G, M)$ which is of course equivalent to saying that B_G is closed in A_G . This is equivalent to saying that B_G is complete and is thus a Frechet space. Remember that the topology on A_G is that of point-wise

convergence, that is $f_n: G \rightarrow M$ tends to zero if and only if $f_n(g) \rightarrow 0$ for every $g \in G$ in the norm $\|\cdot\|_M$ on M .

We have a continuous one-to-one map $h: M \rightarrow A_G$ whose image is B_G (the obvious map). Since we are no longer talking about a topology on B_G induced by $\|\cdot\|_p$, saying that this map is continuous needs justification. Suppose that $(e_n) \in M$ satisfies $\|e_n\|_M \rightarrow 0$. Fix $g \in G$. Then $\|[h(e_n)](g)\|_M = \|ge_n - e_n\|_M \leq \|g-1\|_M \cdot \|e_n\|_M \rightarrow 0$ as $n \rightarrow \infty$, since $\|g-1\|_M$ is fixed. Since the topology on B_G is that of point-wise convergence, it follows that $h(e_n) \rightarrow 0$, so h is continuous.

Now M is a Banach space, so M is certainly complete and a Frechet space. Since a continuous, bijective map between Frechet spaces has a continuous inverse, saying that B_G is a Frechet space is equivalent to saying that the inverse map from B_G to M is continuous. We claim that this is equivalent to saying that there does not exist a sequence $e_n \in M$ such that $\|e_n\|_M = 1$ for all n and $\|ge_n - e_n\|_M \rightarrow 0$ for all $g \in G$.

The inverse map from B_G to M is given by the following. Say $f \in B$. Then f is given by $f(g) = g \cdot e - e$ for some $e \in M$. This inverse map sends f to e . Suppose that there exists such a sequence $e_n \in M$ with $\|e_n\|_M = 1$ and $\|ge_n - e_n\|_M \rightarrow 0$ for all $g \in G$. Thus, the maps in B_G determined by e_n converge to 0 (point-wise). However, their image under this inverse map does not, which means that this map cannot be continuous.

Conversely, suppose that this map is NOT continuous. Noting that the topology on B_G is that of point-wise convergence, this means that there exists a sequence $a_n(g) = ge_n - e_n \in B_G$ such that $\|a_n(g)\|_M \rightarrow 0$ for all $g \in G$, but e_n does not converge to 0 in M . Then there exists $\varepsilon > 0$ such that for all $N \geq 1$, there exists $m \geq N$ such that $\|e_m\|_M \geq \varepsilon$. Multiply e_n by $1/\varepsilon$ and call the new sequence e_n . Note that since $\varepsilon > 0$ is fixed, the sequence of pre-images of the new e_n under this map still converge to 0 for each $g \in G$. Now we can choose a subsequence b_n of e_n such that $\|b_n\|_M \geq 1$ for all n . For each m , multiply b_m by the unique real number $0 < y_m \leq 1$

such that the new element of M (rename it X_m) has norm 1. Since $y_m \leq 1$ for all m , it follows that for all m , $\|X_m\|_M = 1$ and $\|gX_m - X_m\|_M \leq \|gb_m - b_m\|_M \rightarrow 0$ for all $g \in G$. \square

Theorem 4. *Keep the same assumptions as above about G and M and suppose that $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \geq 2$. Suppose further that for every $\alpha \in \mathbb{C}G$, $\|\alpha\|_1 \geq \|\alpha\|_M \geq \|\alpha\|_p$. If G satisfies the strong Følner condition, then $H^1(G, M)$ is not Hausdorff.*

Proof: Since G is finitely generated, we may let $G = \{g_1, g_2, \dots\}$. For each $n \in \mathbb{N}$, let $G_n = \{g_1, \dots, g_n\}$. Because G satisfies the strong Følner condition, for each $n \in \mathbb{N}$, we may pick $N_n \in \mathbb{N}$ such that for every $M \in \mathbb{N}$, there exists a finite subset X of G such that $|X| > M$ and $|X - g_i \cdot X| < N_n$ for every $g_i \in S_n$. Given any such n , choose a finite subset X_n of G such that $|X_n| > (n \cdot N_n)^p$ and $|X_n - g_i \cdot X_n| < N_n$ for every $g_i \in S_n$. Let

$$\beta_n = \frac{\sum_{x \in X_n} x}{\|\sum_{x \in X_n} x\|_M}.$$

Note that for each n , $\|\beta_n\|_M = 1$. Fix any $g_i \in G$. Then for $n \geq i$, we have

$$\begin{aligned} \|g \cdot \beta_n - \beta_n\|_M &= \frac{\|g \sum_{x \in X_n} x - \sum_{x \in X_n} x\|_M}{\|\sum_{x \in X_n} x\|_M} \leq \\ &= \frac{\|g \sum_{x \in X_n} x - \sum_{x \in X_n} x\|_1}{\|\sum_{x \in X_n} x\|_p} \leq \frac{\|g \sum_{x \in X_n} x - \sum_{x \in X_n} x\|_1}{\|\sum_{x \in X_n} x\|_p} = \\ &= \frac{2 \cdot |X_n - g_i \cdot X_n|}{\|\sum_{x \in X_n} x\|_p} < \frac{2 \cdot N_n}{\|\sum_{x \in X_n} x\|_p} = \frac{2 \cdot N_n}{(|X_n|)^{\frac{1}{p}}} < \frac{2 \cdot N_n}{((n \cdot N_n)^p)^{\frac{1}{p}}} = \frac{2}{n} \rightarrow 0. \end{aligned}$$

Then $\|\beta_n\|_M = 1$ for all n and $\|g \cdot \beta_n - \beta_n\|_M \rightarrow 0$ for every $g \in G$. By the previous theorem, $H^1(G, M)$ is not Hausdorff. \square

Corollary 1. *Suppose that G satisfies the strong Følner condition. Then $H^1(G, C_{red}^*(G))$ is not Hausdorff and $\dim_{\mathbb{C}}(H^1(G, M)) = \infty$. \square*

4. An Application: What Do the Groups $H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$ and $H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$ look like to \mathbb{C} ?

Let $S(\mathbb{T}^n)$ be the square integrable functions on \mathbb{T}^n . In other words, $S(\mathbb{T}^n)$ is the set of functions $f: \mathbb{T}^n \rightarrow \mathbb{C}$ such that the following integral exists and is finite:

$$\int_{|z_1|=1} \cdots \int_{|z_n|=1} |f(z_1, \dots, z_n)|^2 dz_n \cdots dz_1.$$

Effros's paper ([2]) shows us that we have isomorphisms $\theta_n: S(\mathbb{T}^n) \rightarrow L^2(\mathbb{Z}^n)$ and $\phi_n: C(\mathbb{T}^n) \rightarrow C_{red}^*(\mathbb{Z}^n)$ both given by

$$f \mapsto \sum_{m_1, \dots, m_n} a_{m_1, \dots, m_n} Z_1^{m_1} \cdots Z_n^{m_n} \quad \text{where}$$

$$a_{m_1, \dots, m_n} = \int_{|z_1|=1} \cdots \int_{|z_n|=1} Z_1^{-1-m_1} \cdots Z_n^{-1-m_n} \cdot f(z_1, \dots, z_n) \cdot \frac{1}{(2\pi i)^n} \cdot dz_n \cdots dz_1$$

For $f \in S(\mathbb{T}^n)$, Effros's paper ([2]) shows us that we also have the following.

$$\|\theta_n(f)\|_2 = \left[\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |f(z_1, \dots, z_n)|^2 dz_n \cdots dz_1 \right]^{1/2}.$$

Lemma 2. For $f \in C(\mathbb{T}^n)$, $\|\phi_n(f)\|_{op} = \sup_{z \in \mathbb{T}^n} |f(z)|$.

Proof: Let d be the usual distance metric on \mathbb{C}^n . First, suppose that $g \in S(\mathbb{T}^n)$ satisfies $\|\theta_n(g)\|_2 = 1$.

$$\begin{aligned} & \implies \|\phi_n(f)\theta_n(g)\|_2 = \\ & \left[\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |g(z_1, \dots, z_n)|^2 |f(z_1, \dots, z_n)|^2 dz_n \cdots dz_1 \right]^{1/2} = \\ & \leq \sup_{z \in \mathbb{T}^n} |f(z)| \cdot \left[\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |g(z_1, \dots, z_n)|^2 dz_n \cdots dz_1 \right]^{1/2} = \\ & = \sup_{z \in \mathbb{T}^n} |f(z)|. \\ & \implies \|\phi_n(f)\|_{op} \leq \sup_{z \in \mathbb{T}^n} |f(z)|. \end{aligned}$$

Now, without loss of generality, $\|\phi_n(f)\|_{op} = 1$. Suppose that $\sup_{z \in \mathbb{T}^n} |f(z)| > 1$. So there is an $\varepsilon > 0$ such that $\sup_{z \in \mathbb{T}^n} |f(z)| > 1 + \varepsilon$. Then there is a $w \in \mathbb{T}^n$ and $1 > \delta > 0$ such that $|f(z)| > 1 + \varepsilon$ for $d(z, w) \leq \delta$ where d is the usual distance

metric on \mathbb{C}^n (This is due to the continuity of $f(z)$). By $\int \cdots \int_{d(z,w) \leq \delta}$, we will mean $\int \cdots \int_{\{d(z,w) \leq \delta\} \cap \mathbb{T}^n}$, the integral over the intersection of the δ disk around w (in \mathbb{C}^n) with \mathbb{T}^n . Since

$$\left[\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} \cdot 1 \cdot dz_n \cdots dz_1 \right]^{1/2} = 1,$$

and since $\delta < 1$, $\{z \in \mathbb{T}^n : d(z,w) \leq \delta\}$ is properly contained in \mathbb{T}^n . Thus, there exists a $k \in \mathbb{R}^+$ such that $k > 1$ and a $g \in S(\mathbb{T}^n)$ defined by $g(z) = 0$ for $d(z,w) > \delta$ and $g(z) = k$ for $d(z,w) \leq \delta$ with the property that

$$\|\theta(g)\|_2^2 = \frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |g(z_1, \dots, z_n)|^2 dz_n \cdots dz_1 = 1.$$

Also, by the definition of g ,

$$\begin{aligned} & \int \cdots \int_{d(z,w) \leq \delta} |g(z_1, \dots, z_n)|^2 dz_n \cdots dz_1 = \\ & = \int_{|z_1|=1} \cdots \int_{|z_n|=1} |g(z_1, \dots, z_n)|^2 dz_n \cdots dz_1. \end{aligned}$$

Thus,

$$\begin{aligned} 1 &= \|\phi_n(f)\|_{op} \geq \|\theta_n(g)\phi_n(f)\|_2 = \\ &= \left[\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |g|^2 |f|^2 dz_n \cdots dz_1 \right]^{1/2} = \\ &= \left[\frac{1}{(2\pi)^n} \int \cdots \int_{d(z,w) \leq \delta} |f|^2 |g|^2 dz_n \cdots dz_1 \right]^{1/2} = \\ &\geq \left[\frac{1}{(2\pi)^n} \int \cdots \int_{d(z,w) \leq \delta} (1 + \varepsilon)^2 |g|^2 dz_n \cdots dz_1 \right]^{1/2} = \\ &= (1 + \varepsilon) \left[\frac{1}{(2\pi)^n} \int \cdots \int_{d(z,w) \leq \delta} |g|^2 dz_n \cdots dz_1 \right]^{1/2} = \\ &= (1 + \varepsilon) \left[\frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} |g|^2 dz_n \cdots dz_1 \right]^{1/2} = 1 + \varepsilon > 1, \end{aligned}$$

A contradiction. Thus, $\|\phi_n(f)\|_{op} = \sup_{z \in \mathbb{T}^n} |f(z)|$. \square

Theorem 5. *Let $n \geq 1$. Then $H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$ is not Hausdorff and so $\dim_{\mathbb{C}} H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n)) = \infty$.*

Proof: Again, let d be the usual distance metric on \mathbb{C}^n . Suppose that $H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$ is Hausdorff. By Theorem 3, there cannot exist $(f_i) \in C_{red}^*(\mathbb{Z}^n) = C(\mathbb{T}^n)$ such that $\|\phi_n(f_i)\|_{op} = 1$ for all i with the property that $\|\phi_n(gf_i - f_i)\|_{op} \rightarrow 0$ as $i \rightarrow \infty$ for every $g \in \mathbb{Z}^n$. We will exhibit such an (f_i) .

Since given an i , $\{(z_1, \dots, z_n) \in \mathbb{T}^n : d((z_1, \dots, z_n), (1, \dots, 1)) > 1/i\}$ and $\{(z_1, \dots, z_n) : d((z_1, \dots, z_n), (1, \dots, 1)) < 1/(2i) \text{ for all } j\}$ are disjoint open subsets of \mathbb{T}^n with positive distance between them (this distance is $1/(2i)$), for each i we have a function $f_i \in C(\mathbb{T}^n)$ with image contained in the closed unit disk that is identically 0 on the first set and identically 1 on the second. By the previous lemma, $\|\phi_n(f_i)\|_{op} = \sup_{z \in \mathbb{T}^n} |f_i(z)| = 1$ for all i . Choose $(m_1, \dots, m_n) \in \mathbb{Z}^n$. Let $B_i = \{(z_1, \dots, z_n) \in \mathbb{T}^n : d((z_1, \dots, z_n), (1, \dots, 1)) \leq 1/i\}$. Then since each f_i is zero on the set $A_i = \{(z_1, \dots, z_n) \in \mathbb{T}^n : d((z_1, \dots, z_n), (1, \dots, 1)) > 1/i\}$,

$$\begin{aligned} & \|\phi_n(Z_1^{m_1} \cdots Z_n^{m_n} f_i(z_1, \dots, z_n) - f_i(z_1, \dots, z_n))\|_{op} = \\ &= \sup_{(z_1, \dots, z_n) \in \mathbb{T}^n} |Z_1^{m_1} \cdots Z_n^{m_n} f_i(z_1, \dots, z_n) - f_i(z_1, \dots, z_n)| = \\ &= \sup_{B_i} |Z_1^{m_1} \cdots Z_n^{m_n} f_i(z_1, \dots, z_n) - f_i(z_1, \dots, z_n)| \leq \\ & \leq \sup_{B_i} |Z_1^{m_1} \cdots Z_n^{m_n} - 1| \cdot \text{Sup}_{B_i} |f_i| = \\ &= \sup_{B_i} |Z_1^{m_1} \cdots Z_n^{m_n} - 1| \rightarrow 0 \text{ as } i \rightarrow \infty. \end{aligned}$$

This is due to the fact that on B_i , $d((Z_1, \dots, Z_n), (1, \dots, 1)) \leq 1/i$ and the fact that the m_1, \dots, m_n are fixed. In other words, $\|gf_i - f_i\|_{op} \rightarrow 0$ as $i \rightarrow \infty$ for every $g \in \mathbb{Z}^n$. \square

Now we turn our attention to the groups $H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$. We view \mathbb{T}^n as $\{(z_1, \dots, z_n) : |z_i| = 1 \text{ for all } i\}$. Again from Effros's paper ([2]), we know that for all n , $C_{red}^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ as \mathbb{Z}^n modules. But what is the \mathbb{Z}^n module structure on $C(\mathbb{T}^n)$? We view \mathbb{Z}^n as the free abelian group on the generators x_1, \dots, x_n . Then the action of an element $x_1^{e_1} \cdots x_n^{e_n}$ on an element $f(z_1, \dots, z_n)$ of $C(\mathbb{T}^n)$ is given trivially

by $(x_1^{e_1} \cdots x_n^{e_n}) \cdot f(z_1, \dots, z_n) = (z_1^{e_1} \cdots z_n^{e_n}) f(z_1, \dots, z_n)$. First, we must recall a case of the Kunneth theorem for projective resolutions.

Theorem 6. (Künneth Theorem) *Let G be a group and let $P: \cdots \rightarrow P_1 \xrightarrow{d_1} P_0, Q: \cdots \rightarrow Q_1 \xrightarrow{\delta_1} Q_0$ be projective resolutions of \mathbb{Z} as $\mathbb{Z}G$ modules. Then $P \otimes Q$ is a projective resolution of \mathbb{Z} as $\mathbb{Z}G$ modules where $[P \otimes Q]_n = \bigoplus_{i+j=n} P_i \otimes Q_j$. The boundary maps ∂_n are given by $\partial_n(p_i \otimes q_j) = d_i(p_i) \otimes q_j + (-1)^j p_i \otimes \delta_j(q_j)$ on simple tensors $p_i \otimes q_j$.*

Proof: This is Theorem V.2.1 of [4]. \square

We aim to give an explicit description of the groups $H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$ and to show that there is a natural sequence of imbeddings

$$0 \rightarrow H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z})) \rightarrow H^2(\mathbb{Z}^2, C_{red}^*(\mathbb{Z}^2)) \rightarrow H^3(\mathbb{Z}^3, C_{red}^*(\mathbb{Z}^3)) \rightarrow \dots$$

In order to do so, we must prove the following lemma.

Lemma 3. *Let $G = \mathbb{Z}^n$ be the free abelian group on X_1, \dots, X_n . Then there exists a projective resolution of \mathbb{Z} as $\mathbb{Z}G$ modules with exactly $n+1$ nonzero modules (including \mathbb{Z}): $0 \rightarrow \mathbb{Z}G \xrightarrow{\rho_n} (\mathbb{Z}G)^n \rightarrow \cdots \rightarrow \mathbb{Z} \rightarrow 0$ where the map ρ_n is given by $\rho_n(1) = [(X_1 - 1), -(X_2 - 1), \dots, -(X_n - 1)]$.*

Proof: Note that for all n , $\mathbb{Z}G = \mathbb{Z}[X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}]$. For $n = 1$, we know that there is a projective resolution of \mathbb{Z} : $0 \rightarrow \mathbb{Z}G \xrightarrow{d} \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0$, where d is given by $d(f(X_1)) = (X_1 - 1)f(X_1)$. Suppose that n is at least 2.

Next we must note that for any n , $\mathbb{Z}G \cong \mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z}$ where there are n terms, and the isomorphism is an isomorphism of $\mathbb{Z}G$ modules. The isomorphism $\theta: \mathbb{Z}G \rightarrow \mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z}$ is given explicitly by $1 \mapsto 1 \otimes \cdots \otimes 1$. The action of G (the free abelian group on X_1, \dots, X_n) on $\mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z}$ is as follows:

$$X_1^{r_1} \cdots X_n^{r_n} * a_1 \otimes \cdots \otimes a_n = X_1^{r_1} a_1 \otimes \cdots \otimes X_n^{r_n} a_n.$$

Let us look again at the projective resolution of \mathbb{Z} as a $\mathbb{Z}G$ module for $n = 1$ given by $0 \rightarrow \mathbb{Z}G \xrightarrow{d} \mathbb{Z}G \xrightarrow{aug} \mathbb{Z} \rightarrow 0$. In order to properly view this resolution as a resolution, we will call the first $\mathbb{Z}G$ P_1 and the second P_0 . Thus, we have a resolution $0 \rightarrow P_1 \xrightarrow{d} P_0 \xrightarrow{aug} \mathbb{Z} \rightarrow 0$, where $P_1 = P_0 = \mathbb{Z}\mathbb{Z}$.

The Künneth theorem tells us that the tensor product of resolutions of \mathbb{Z} is again a resolution, so by induction on n , for all n at least 2, there is a resolution of \mathbb{Z} :

$$0 \rightarrow (P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_1) \xrightarrow{d_n} (P_1 \otimes_{\mathbb{Z}} P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_0) \oplus \cdots \oplus (P_0 \otimes_{\mathbb{Z}} P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_1) \rightarrow \cdots \rightarrow \mathbb{Z} \rightarrow 0$$

where everywhere \dots appears implies that there are exactly n objects being either \oplus ed or \otimes ed together, except the last \dots which implies that there are exactly $n + 1$ (including \mathbb{Z}) nonzero terms in the sequence. The map d_n is given explicitly from the definition of tensor products of resolutions by

$$d_n(a_1 \otimes \cdots \otimes a_n) = [(X_1 - 1)a_n \otimes a_2 \otimes \cdots \otimes a_n, -(a_1 \otimes (X_2 - 1)a_2 \otimes \cdots \otimes a_n), \dots, -(a_1 \otimes \cdots \otimes a_{n-1} \otimes (X_n - 1)a_n)].$$

Note that the map d_n is just a generalization of our original map $d : P_1 \rightarrow P_0$. We know that

$$(P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_1 \otimes_{\mathbb{Z}} P_0) \oplus \cdots \oplus (P_0 \otimes_{\mathbb{Z}} P_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P_1) \cong (\mathbb{Z}G)^n$$

via a map ζ where the isomorphism is as $\mathbb{Z}G$ modules (and where G is still the free abelian group on X_1, \dots, X_n). The map ζ sends $d_n(1 \otimes \cdots \otimes 1) = [((X_1 - 1) \otimes 1 \otimes \cdots \otimes 1), -(1 \otimes (X_2 - 1) \otimes \cdots \otimes 1), -\dots, -(1 \otimes \cdots \otimes 1 \otimes (X_n - 1))]$ to $[X_1 - 1, -(X_2 - 1), \dots, -(X_n - 1)]$. Translating back into $\mathbb{Z}G$ lingo, it follows directly from the Künneth Theorem that we have a projective resolution of \mathbb{Z} as $\mathbb{Z}G$ modules with exactly $n + 1$ (including \mathbb{Z}) nonzero modules

$$0 \rightarrow \mathbb{Z}G \xrightarrow{\rho_n} (\mathbb{Z}G)^n \rightarrow \cdots \rightarrow \mathbb{Z} \rightarrow 0.$$

The map $\rho_n : \mathbb{Z}G \rightarrow (\mathbb{Z}G)^n$ is then of course given by a composition of maps:

$$\rho_n(1) = \zeta(d_n(\theta(1))) = \zeta(d_n(1 \otimes \cdots \otimes 1)) = [X_1 - 1, -(X_2 - 1), \dots, -(X_n - 1)]. \quad \square$$

Now we are able to give an explicit description of the groups $H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$.

Theorem 7. *Let $G = \mathbb{Z}^n$ be the free abelian group on X_1, \dots, X_n . Then we have that*

$$H^n(G, C_{red}^*(G)) \cong \frac{C(\mathbb{T}^n)}{(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)}.$$

Proof: Since for all n , $C_{red}^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$, it suffices to prove the statement for $H^n(G, C(\mathbb{T}^n))$ in the place of $H^n(G, C_{red}^*(G))$. From the previous lemma, we have a projective resolution of \mathbb{Z} with $n + 1$ nonzero terms (including \mathbb{Z}):

$$0 \rightarrow \mathbb{Z}G \xrightarrow{\rho_n} (\mathbb{Z}G)^n \rightarrow \cdots \rightarrow \mathbb{Z} \rightarrow 0.$$

From this, we get a new sequence B :

$$0 \rightarrow Hom_{\mathbb{Z}G}(\mathbb{Z}, C(\mathbb{T}^n)) \rightarrow \cdots \rightarrow Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n)) \xrightarrow{\rho_n^*} Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n)) \rightarrow 0.$$

Using this sequence, we may compute the cohomology group

$$H^n(G, C_{red}^*(G)) = H^n(G, C(\mathbb{T}^n)) = \frac{Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n))}{\rho_n^*(Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n)))}.$$

Define $\theta: Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n)) \rightarrow C(\mathbb{T}^n)/[(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)]$ by $\phi \mapsto \phi(1) + [(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)]$. This map is certainly onto, because if $F \in C(\mathbb{T}^n)$, we may just define $\sigma \in Hom_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n))$ by $\sigma(1) = F$. Now we can compute $ker(\theta)$.

Suppose that $\phi \in Hom_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n))$. Then

$$\begin{aligned} \phi(\rho_n(1)) &= \phi(X_1 - 1, -(X_2 - 1), \dots, -(X_n - 1)) = \\ &= (X_1 - 1) * \phi(1, 0, \dots, 0) + \cdots + (X_n - 1) * \phi(0, \dots, 0, -1) \in \\ &\in [(Z_1 - 1)C(\mathbb{T}^n) + \cdots + (Z_n - 1)C(\mathbb{T}^n)]. \end{aligned}$$

$$\implies \rho_n^*(\text{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n))) \subset \ker(\theta).$$

Now suppose that $\phi \in \ker(\theta)$. So $\phi(1) = (Z_1 - 1)f_1 + \dots + (Z_n - 1)f_n$ for some $f_i \in C(\mathbb{T}^n)$. Define $\pi \in \text{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n))$ by $\pi(1, 0, \dots, 0) = f_1, \pi(0, 1, 0, \dots, 0) = -f_2, \dots, \pi(0, \dots, 0, 1) = -f_n$. Then $\pi(\rho_n(1)) = \phi(1)$, and therefore we have that $\phi = \pi(\rho_n) \in \rho_n^*(\text{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n)))$.

$$\begin{aligned} \implies H^n(G, C_{red}^*(G)) &= H^n(G, C(\mathbb{T}^n)) = \\ &= \frac{\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, C(\mathbb{T}^n))}{\rho_n^*(\text{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^n, C(\mathbb{T}^n)))} \cong \frac{C(\mathbb{T}^n)}{(Z_1 - 1)C(\mathbb{T}^n) + \dots + (Z_n - 1)C(\mathbb{T}^n)}. \quad \square \end{aligned}$$

Theorem 8. *There is a sequence of group (and \mathbb{C} algebra) imbeddings*

$$0 \rightarrow H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z})) \xrightarrow{d_1} H^2(\mathbb{Z}^2, C_{red}^*(\mathbb{Z}^2)) \xrightarrow{d_2} H^3(\mathbb{Z}^3, C_{red}^*(\mathbb{Z}^3)) \rightarrow \dots$$

Proof: By the previous theorem, for all $n \geq 1$,

$$H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n)) = \frac{C(\mathbb{T}^n)}{(z_1 - 1)C(\mathbb{T}^n) + \dots + (z_n - 1)C(\mathbb{T}^n)}.$$

Fix an n and define

$$\begin{aligned} \phi: & \frac{C(\mathbb{T}^n)}{(z_1 - 1)C(\mathbb{T}^n) + \dots + (z_n - 1)C(\mathbb{T}^n)} \\ \rightarrow & \frac{C(\mathbb{T}^{n+1})}{(z_1 - 1)C(\mathbb{T}^{n+1}) + \dots + (z_{n+1} - 1)C(\mathbb{T}^{n+1})} \\ \text{via} & \quad f + [(z_1 - 1)C(\mathbb{T}^n) + \dots + (z_n - 1)C(\mathbb{T}^n)] \\ \mapsto & \quad f + [(z_1 - 1)C(\mathbb{T}^{n+1}) + \dots + (z_{n+1} - 1)C(\mathbb{T}^{n+1})]. \end{aligned}$$

Since $(z_1 - 1)C(\mathbb{T}^n) + \dots + (z_n - 1)C(\mathbb{T}^n) \subset (z_1 - 1)C(\mathbb{T}^{n+1}) + \dots + (z_{n+1} - 1)C(\mathbb{T}^{n+1})$, ϕ is certainly well defined. To see that ϕ is injective, we just let $n = 1$ (the other cases are identical). If $f \in C(\mathbb{T})$ and $f(z_1) = (z_1 - 1)f_1(z_1, z_2) + (z_2 - 1)f_2(z_1, z_2)$ with $f_1, f_2 \in C(\mathbb{T}^2)$, then since f doesn't vary at all with z_2 , it follows that we can set $z_2 = 1$ and then we have that $f(z_1) = (z_1 - 1)f(z_1, 1) \in (z_1 - 1)C(\mathbb{T})$. Thus, $\ker(\phi) = 0$ and so ϕ is injective. \square

Theorem 9. For $n \geq 1$,

$$\dim_{\mathbb{C}} H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n)) = \infty.$$

Proof: For all n , by the last theorem $H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z}))$ imbeds in $H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$. Since $\dim_{\mathbb{C}} H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z})) = \infty$, the result follows. \square

Finally, using Theorem 4, we are able give a more simple proof of the following result (one proof is given above in Theorem 5).

Theorem 10. $H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z}))$ is not Hausdorff and $\dim_{\mathbb{C}} H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z})) = \infty$.

Proof: We know that for $\alpha \in \mathbb{C}\mathbb{Z}$, $\|\alpha\|_1 \geq \|\alpha\|_{op} \geq \|\alpha\|_2$. Thus, Theorem 4 tells us that $H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z}))$ is not Hausdorff. \square

5. Our Results (Summarized)

Let G be an infinite, finitely generated group. Let M be a Banach space (with norm $\|\cdot\|_M$) that is also a left $\mathbb{C}G$ module such that G acts on M as continuous \mathbb{C} -linear transformations. In summary, the following useful results were proved in this paper and do not appear in any papers to date.

(1) Suppose that $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \in \mathbb{N}$. Then the G -module imbedding $\mathbb{C}G \rightarrow M$ induces an imbedding of groups $H^1(G, \mathbb{C}G) \rightarrow H^1(G, M)$.

(2) If $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \in \mathbb{N}$ and if $H^1(G, M) = 0$, then G has exactly 1 end.

(3) $H^1(G, M)$ is not Hausdorff if and only if there exists $f_i \in M$ with norm 1 ($\|f_i\|_M = 1$) for all i with the property that $\|gf_i - f_i\|_M \rightarrow 0$ as $i \rightarrow \infty$ for every $g \in G$.

(4) If $\mathbb{C}G \subset M \subset L^p(G)$ for some $p \geq 2$, $\|\alpha\|_1 \geq \|\alpha\|_M \geq \|\alpha\|_p$ for every $\alpha \in \mathbb{C}G$, and if G satisfies the strong Følner condition, then $H^1(G, M)$ is not Hausdorff and therefore $\dim_{\mathbb{C}} H^1(G, M) = \infty$.

We have used these results (with our motivation coming from Effros's paper ([2])) to show the following for $n \geq 1$.

(1) $H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n))$ is not Hausdorff and thus $\dim_{\mathbb{C}} H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n)) = \infty$ (with two proofs for $n = 1$).

(2) $\dim_{\mathbb{C}} H^n(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n)) = \infty$.

(3) There is a natural sequence of imbeddings

$$0 \rightarrow H^1(\mathbb{Z}, C_{red}^*(\mathbb{Z})) \rightarrow H^2(\mathbb{Z}^2, C_{red}^*(\mathbb{Z}^2)) \rightarrow H^3(\mathbb{Z}^3, C_{red}^*(\mathbb{Z}^3)) \rightarrow \dots$$

(4)

$$H^1(\mathbb{Z}^n, C_{red}^*(\mathbb{Z}^n)) = \frac{C(\mathbb{T}^n)}{(Z_1 - 1)C(\mathbb{T}^n) + \dots + (Z_n - 1)C(\mathbb{T}^n)}.$$

6. Conjectures and Future Work

Conjecture 1. *Let G be an infinite, finitely generated group. Let M be a Banach space (with norm $\| \cdot \|_M$) that is also a left $\mathbb{C}G$ module such that G acts on M as continuous \mathbb{C} -linear transformations. Then G is amenable if and only if $H^1(G, M)$ is not Hausdorff.*

We should note that the proof given in [3] of this with $L^2(G)$ is specific to that case.

Of course this would show that if M_1 and M_2 both have these properties, then the Hausdorffness of $H^1(G, M_1)$ is equivalent to the Hausdorffness of $H^1(G, M_2)$.

Motivation for this conjecture comes from several places. The paper by Mohammed Bekka and Alain Valette entitled *Group Cohomology, Harmonic Functions and the First L^2 -Betti Number* is one source of motivation for this conjecture. The paper shows that this conjecture holds for $M = L^2(G)$, and notes that the vanishing of $H^1(G, L^2(G))$ is remarkably applicable ([3]) in electrical engineering. Thus, it would be very helpful to have an easier way to compute groups of the form $H^1(G, L^2(G))$. Since for some $M \neq L^2(G)$, $H^1(G, M)$ may be more computable than

$H^1(G, L^2(G))$, it would be helpful to know that for any such M , $H^1(G, M)$ is Hausdorff if and only if $H^1(G, L^2(G))$ is as well.

Also due to the fact that $H^1(G, M)$ may be more computable for some $M \neq L^2(G)$, my conjecture would lead to another way (possibly faster than given in [3]) to decide the amenability of groups. Thus, a proof that this conjecture does hold would be very useful in several ways.

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502 I JACKSON STREET, BLACKSBURG, VA 24060, U.S.A

E-mail address: anarkawi@vt.edu