Option Valuation with Sinusoidal Heteroskedasticity

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1 Black-Scholes-Merton Option Pricing

Ito drift-diffusion process (1) can be used to derive the Black Scholes formula (2). [1]

$$dS = \sigma S dX + \mu dt \tag{1}$$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0 \tag{2}$$

By applying boundaries conditions $V(S,T) = max\{S-K,0\}$ and $V(S,T) = max\{0,S-K\}$ to (2) we can solve the PDE to find its closed form solution for European calls and puts, the Black-Scholes Model. [7]

$$\begin{aligned} C(s,t) &= SN(d_1) - Ke^{r(T-t)}N(d_2) \\ P(s,t) &= Ke^{r(T-t)}N(-d_2) - SN(-d_1) \\ d_1 &= \frac{ln(\frac{S}{K}) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned} \tag{3}$$

The limitation of the above method for derivative valuation is that it makes a variety of assumptions that may or may not be appropriate. These assumptions are as follows:

1. Volatility of the underlying asset class is constant over the entire time period, also known as homoskedasticity.
2. The risk-free interest rate will remain constant over time.
3. Stock behavior follows a normally distributed Brownian motion as described by dX in formula (1).

By relaxing the first of these assumptions we demonstrate significant improvement in the appropriateness of our options model.

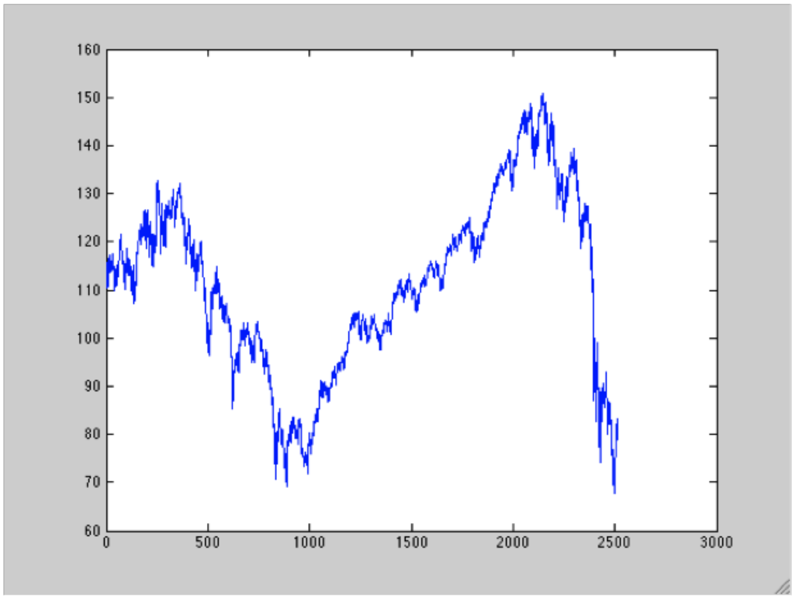

Figure 1: ETF, ticker: SPY. Data taken from daily market close [4/1/99, 3/31/09].

2 Time-Dependent Volatility

Despite the assumptions in the Black-Scholes-Merton framework, one may assume that volatility does change with time. As a case study this paper analyzes an ETF fund, ticker SPY, that tracks the S&P 500 index. Inspection of Figure 1 shows that market prices fluctuate between periods of high and low volatility.

Relaxing the homoskedasticity constraint we will define σ_t, the instantaneous volatility at time t, and $\bar{\sigma} = \sqrt{\frac{1}{T-t}\int_t^T \sigma_\tau^2 d\tau}$, the average volatility over a specific time period $[t_0, t_1]$ [1].

$$dS_t = \sigma_t S_t dX + \mu dt \tag{4}$$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + rS_t \frac{\partial f}{\partial S_t} - rf = 0 \tag{5}$$

3 Historical Data Analysis

Rolling historical volatility of the data set in Figure 1 is calculated in (6). [8] Historical volatility calculation in Figure 2 reveals that volatility of asset prices can fluctuate on an order of magnitude.

$$\sigma = \sqrt{\frac{Z}{n-2}\sum_{i=1}^{n-1}(r_i - \bar{r})^2} \tag{6}$$

$$r_i = ln(\frac{C_{i+1}}{C_i})$$

$$\bar{r} = \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1}$$

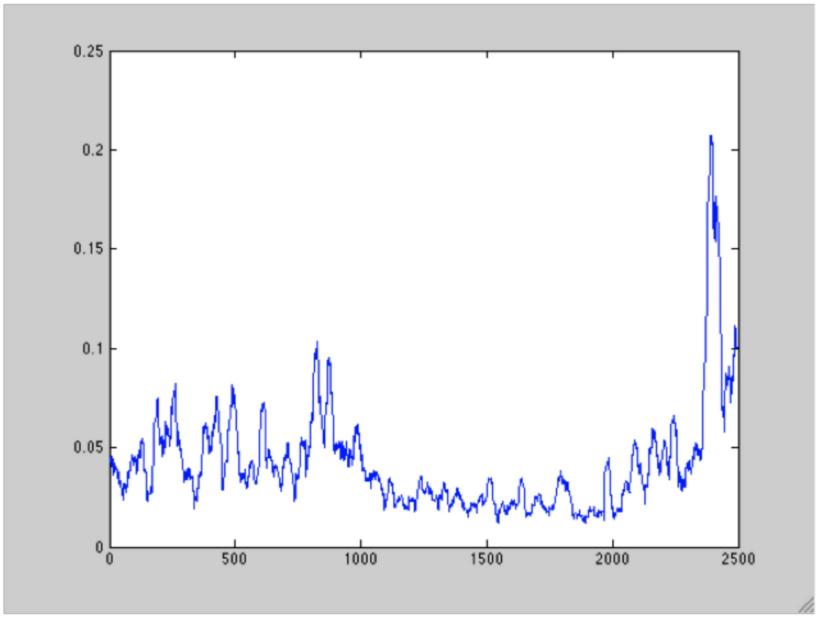

Figure 2: Data set: Rolling historical volatility of Figure 1, $n = 20$.

4 Forecasting Future Volatilities

Assumption: market volatility changes with time and varies as a finite summation of sinusoids.

$$s(t) = A_0 + \sum_{i=1}^{M} A_i cos(2\pi f_i + \phi_i) \tag{7}$$

4.1 Fourier Analysis

Let x_n be the 20-day rolling historical volatility signal plotted in Figure 2. We then let X_n be the discrete frequency domain representation of x_n according to (8).

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}kn} \\ k &= 0, \ldots, N-1 \end{aligned} \tag{8}$$

where $A_i = |X_i|$, $f_i = \frac{i}{N}$, $\phi_i = \angle X_i$ and $A_0 = \frac{1}{N}\sum_{n=0}^{N-1} x_n$. [4]

4.2 Reconstructing HV Signal

After calculating the A_i, f_i and ϕ_i series for the rolling historical volatility we can extract the M most significant frequencies according to the magnitude of X_i, $|X_i|$. Finally with those M most significant frequencies we can construct an approximation of x_n according to (7).

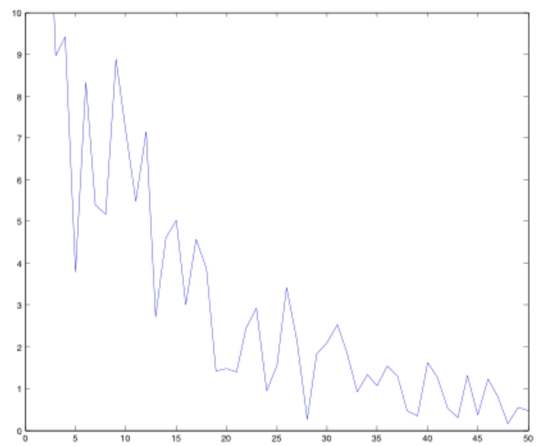

Figure 3: $|A_i|$ vs i, FFT of the historical volatility.

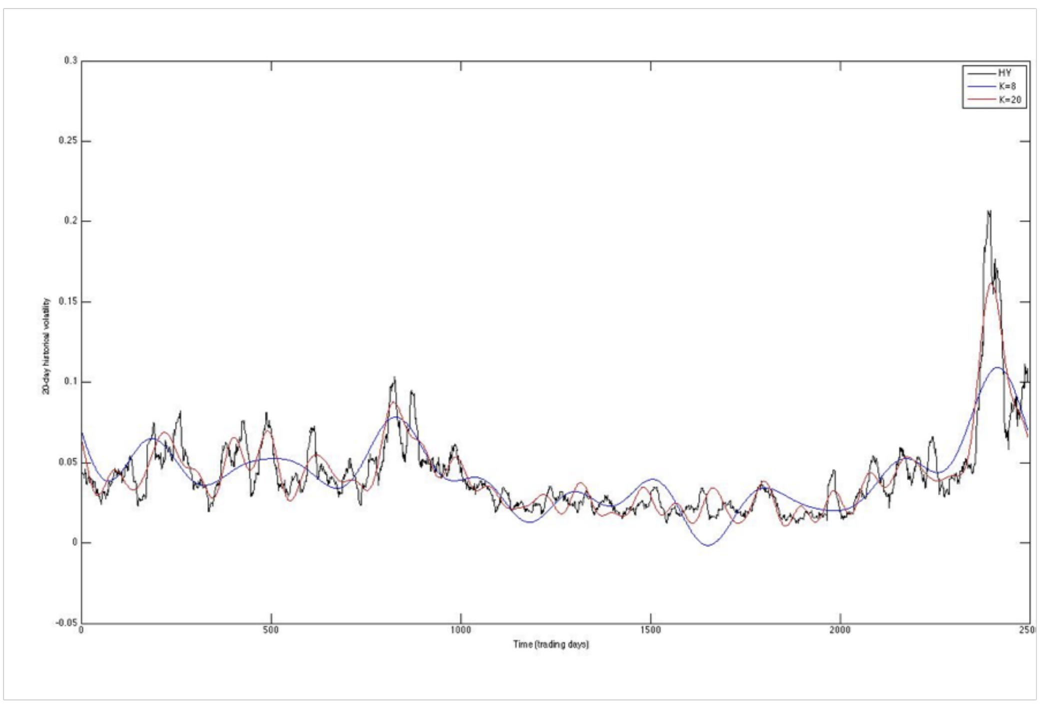

Figure 4: Reconstructing the historical volatility based on decreasing order of magnitude of frequencies according to (7). For $M = 8$, $r^2 = 0.7927$ and for $M = 20$, $r^2 = 0.9216$.

In conclusion, with relatively few sinusoids we can accurately approximate the trailing volatility over a period of time. Knowing this we can anticipate future volatility from recent trends by extrapolating the approximating function into the future. We will define this new estimated volatility as $\tilde{\sigma_t}$. We can calculate the average anticipated volatility, $\tilde{\tilde{\sigma}} = \sqrt{\frac{1}{T-t} \int_t^T \tilde{\sigma}_\tau^2 d\tau}$, and use this as the volatility parameter in the Black-Scholes Model, (3).

5 Case Study for Heteroskedastic Volatility Estimation

We will compare 3-month at-the-money options using three distinct volatility estimation techniques and compare them to a "realized" volatility that incorporates stock signal information unavailable at the time of investment. The three methods will be compared to the latter to empirically verify our volatility estimation over standard techniques. Dates were randomly chosen from year 2000 to 2008. The study will be performed on the S&P 500 Spider ETF, ticker: SPY.

Volatility Estimation Techniques:

1. Historical 1-month volatility
2. Historical 1-year volatility.
3. Historical 3-month heteroskedastic volatility.

Table 1: Empirical Volatility Comparison

Date (Y-M-D)	Method #1 (%)	Method #2 (%)	Method #3 (%)	Realized (%)
2001-08-22	14.84	22.36	16.52	24.43
2001-10-17	25.36	23.88	22.20	17.12
2003-05-13	17.52	27.12	23.62	15.76
2004-06-10	10.48	12.36	12.66	11.15
2005-01-05	8.66	11.18	9.54	10.03
2005-07-07	8.66	10.66	10.26	8.84
2005-07-26	8.19	10.45	8.26	10.86
2006-02-01	11.19	10.41	8.41	8.79
2008-06-13	18.65	19.50	16.14	20.99
2008-10-27	85.14	33.06	43.15	56.94
Average Error	5.743	5.177	4.589	0

Once the volatilites have been estimated for the 3 methods above and compared to the actual or "realized" volatility, we wish to see the improvement in pricing our method provides. To do this we make the following assumptions about Black-Scholes Model parameters: (1) the assets being evaluated are european call options, (2) the risk-free rate, r_f, is 5%, (3) the time to expiry, T, is 3 months, (4) the strike price, K, matches the stock price. The only independent variable in this case study is the volatility which we get from Table: 1 above. Our goal is to compare the pricing error produced by each of the three methods.

In summary, the sinusoidal heteroskedastic volatility estimation is preferable to the other methods as it improves pricing error by 20% over 1-month historical volatility estimation and 10% over 1-year historical volatility estimation.

Table 2: BSM Valuation Error From Realized

Date (Y-M-D)	Method #1 ($)	Method #2 ($)	Method #3 ($)
2001-08-22	1.87	0.40	1.54
2001-10-17	1.61	1.32	0.99
2003-05-13	0.34	2.22	1.53
2004-06-10	0.13	0.23	0.29
2005-01-05	0.26	0.22	0.09
2005-07-07	0.03	0.35	0.27
2005-07-26	0.51	0.08	0.49
2006-02-01	0.46	0.31	0.07
2008-06-13	0.46	0.29	0.95
2008-10-27	5.46	4.66	2.69
Average Error	1.11	1.02	0.89

6 Conclusion

In conclusion, improving methods for estimating volatility will improve option valuation techniques significantly. After discovering that stock market volatility can be modeled with Fourier analysis, we found that the same technique can be used to anticipate volatility in the future and realized dramatic improvement in option pricing error.

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