Sudokus or Pseudo-Clues?

Jessica Kline
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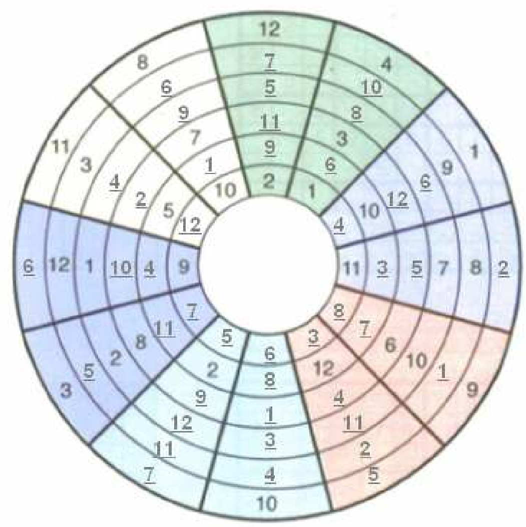
"Pseudo…what?"
"No, *Sudoku*!"

Everyone has surely been a part of this conversation at one point in time. These popular puzzles have swept across the nation fascinating more and more mathematical minds as the fad grows. They were first published in Japan in 1984 under the title *suji wa dokushin ni kagiru* translating to "only single numbers allowed". This soon shortened to *sudoku*, concatenating the *su* "number" and *doku* "single"[1]. Seen in Figure 1(a), the standard puzzle consists of a 9x9 grid divided into smaller 3x3 grids. Below is an example of a solved Sudoku puzzle where the numbers in red were filled in from the clues provided in black. Though the game has proven to be complex, the rules are quite simple. Each number 1-9 must be used exactly once in every row, every column, and every 3x3 square. Thus, a Sudoku is a specialized Latin Square.

The Clock Sudoku, created by Laura Taalman and Philip Riley, is a variation of the traditional Sudoku puzzle which instead uses 12 numbers total, providing a clock-like image. There are 12 wedges, one for each "o-clock" position and each wedge contains 6 numbers. Figure 1(b) shows an example of the Clock Sudoku grid that was solved by filling in the underlined numbers. The rules for this puzzle denote that each number from 1-12 must be used exactly once in each ring, each pair of double wedges, and each pair of antipodal wedges, which will be later explained in detail. In this paper, a proof of the total number of valid Clock Sudoku grids will be given.

(a)[2]



(b)

Figure 1

Theorem: There are 2882552743578688039157760000 possible Clock Sudoku grids.

The Clock Sudoku rules are as follows:

- Fill each of the 6 double wedges with the numbers 1-12
- Fill each of the 6 antipodal wedges with the numbers 1-12
- Fill each of the 6 rings with the numbers 1-12
- No number can be repeated in any of the above groupings

Figure 2 shows a Clock Sudoku grid with the above 3 pieces outlined and labeled.

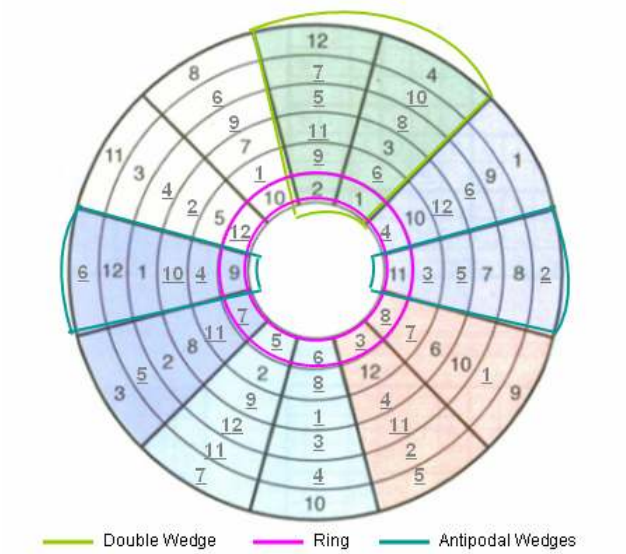


Figure 2

When counting the total number of grids it would be advantageous to have a flat grid layout, similar to the tradition Sudoku grid. This can be done by transforming each of the six pairs of antipodal wedges into a horizontal row, placed one on top of the other. To determine their order of placement, the top two rows, middle two rows, and bottom two rows should form the double wedges shown in Figure 2 by combining their right and left halves. Figure 3 demonstrates this idea and shows the breakdown of antipodal wedges and double wedges for the first two rows of the block.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
R1	4	10	8	3	6	1	5	2	9	12	11	7
R2	12	7	5	11	9	2	6	8	1	3	4	10

Double Wedge   Antipodal Wedges

Figure 3

Finally, if each of the rings from Figure 2 is cut in half vertically and then flattened, it becomes two columns of six numbers. We can also rotate the right half by 180º.

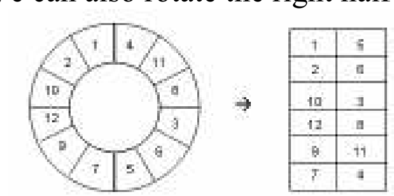


Figure 4

Each of the double wedges in the Clock Sudoku grid contains two numbers side by side from every ring. The antipodal wedges also contain two numbers from every ring, but they are mirrored across the center of the circle. The double wedges in the block grid can then be arranged so that all numbers in the far left column (C1) and far right column (C12) of the grid make up the outermost ring in Figure 2 and all numbers in the middle columns (C6 and C7) of the grid make up the innermost ring. Once this arrangement is completed, each ring is now represented in the block as the pair of columns reflected across the y-axis. Figure 5 shows a complete representation of the Block Clock Sudoku grid and its equivalent representations of antipodal wedges, double wedges, and rings.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
R1	4	10	8	3	6	1	5	2	9	12	11	7
R2	12	7	5	11	9	2	6	8	1	3	4	10
R3	8	6	9	7	1	10	3	12	4	11	2	5
R4	11	3	4	2	5	12	8	7	6	10	1	9
R5	6	12	1	10	4	9	11	3	5	7	8	2
R6	3	5	2	8	11	7	4	10	12	6	9	1

y-axis

Double Wedges — Ring — Antipodal Wedges

Figure 5

The rules for this grid, as altered by our geometric change are as follows:

- Fill each of the 3 left and 3 right half-row pairs with the numbers 1-12
- Fill each of the 6 rows with the numbers 1-12
- Fill each of the 6 pairs of columns mirrored across the y-axis with numbers 1-12
- No number can be repeated in any of the above groupings

The algorithm used for counting the total number of valid Clock Sudoku grids will be to count one pair of antipodal wedges at a time, which is equivalent to counting one row of the block at a time. Notice that completing two rows will also complete two sets of double wedges, leaving minimal restrictions on the odd numbered rows.

Proof: Show that there are 288255274357868803915776000 possible Clock Sudoku grids by counting the permutations by row of the equivalent Block Clock Sudoku.

ROW 1:

The grid is blank so the only restriction is to fill the row's twelve positions with the numbers 1-12 without repetition. This gives 12! possible permutations of the first row.

ROW 2:

Let the left half of Row 1 (R1) be labeled *A* and the right half *B*. Similarly, label the left half of R2 *C* and the right half *D* (see figure below).

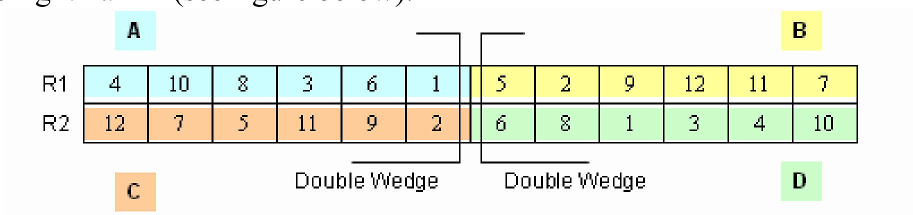


Figure 6

Given that a double wedge and a row must contain the numbers 1-12 without repeats, the six numbers used in *A* cannot be repeated in *C* or *B*. Similarly, the six numbers used in *B* cannot be repeated in *D* or *A*. Therefore, *D* must contain all of the numbers in *A* and *C* must contain all of the numbers in *B*; so there are six numbers restricted to each half of R2. According to the ring rule, a number in R2 cannot repeat any number in R1 that lies in its direct column or mirrored column. Because the double wedges do not contain repeated numbers, mirrored columns are the only factor to take into account for this row. Each position in R2 has one mirrored column, so there is exactly one position where the six number choices for each half cannot be placed. Because these restrictions on *C* and *D* are equivalent, the total row count will be the square of the number of permutations found for one of the halves. A summary for one half of R2 is as follows:

- 6 positions and 6 numbers
- 5 possible positions per number and 5 possible numbers per position

All variables are then evenly distributed, so the number of permutations with any of the five variable choices for Position 1 will be the same. Counting by hand, there are 53 arrangements found for any chosen first variable, so there are $53 \cdot 5 = 265$ total permutations of *C* resulting in $265^2 = 70225$ possible permutations of R2.

ROW 3:

The block now has two complete double wedges, leaving only the antipodal wedge and ring rules to be considered. Each position in R3 has four different numbers filling its direct and mirrored columns and each number from 1-12 has been used twice resulting in four total positions a number cannot be placed. The summary is:

- 12 positions and 12 numbers
- 8 possible positions per number and 8 possible numbers per position

Let the twelve variables *a* through *l* represent the number choices for R3. The permutations for this row are to be maximized, so the number of variable overlaps between adjacent positions must be minimized. The smallest overlap possible is four variables, since each of twelve variables must be a choice for eight positions, so groups of four variables can be represented as a set. Let **A** = {*a, b, c, d*}; **B** = {*e, f, g, h*}; **C** = {*i, j, k, l*}. The choices per position are shown in Figure 7 below. Since each ring has the same variable choices, the choices for positions 7-12 are known to mirror those of 1-6.

Pos 1:	Pos 2:	Pos 3:	Pos 4:	Pos 5:	Pos6:	Pos 7:	Pos 8:	Pos 9:	Pos 10:	Pos 11:	Pos 12:
A	C	B	A	C	B	B	C	A	B	C	A
B	A	C	B	A	C	C	A	B	C	A	B

Figure 7

Let a *choice pattern* be defined as a permutation of the set letters. The total number of permutations for this row is found by multiplying the number of choice patterns by the number of permutations per choice pattern. Because the numbers 1-12 cannot repeat in a row of the grid, the variables cannot repeat in our choice pattern, so the sets **A**, **B**, and **C** appear exactly four times in each pattern. Consider the pattern AAAABBBBCCCC. Finding the number of variable choices per position and placing those positions with the same number of choices together it is seen that the permutations on every choice pattern is $4 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 13824$. The only task left is to find how many valid choice patterns exist with the restrictions listed above. To do this, the number of permutations on AAAABBBBCCCC with A, B, and C having eight valid placements in the twelve total placement options must be found. Rearrange the positions of Figure 8 so that all positions with the same set choices are adjacent, then re-label the position numbers so that positions 1-4 now have choices A and B, positions 5-8 have choices B and C, and positions 9-12 have choices C and A. This rearrangement is displayed below:

AAAA BBBB CCCC
BBBB CCCC AAAA

It must be noted that the choices listed above are not valid for a row in the Clock Sudoku since the columns are not mirrored, however, any pattern found to be valid in the above representation can be rearranged to a valid choice pattern from Figure 7 simply by setting the positions back to their original labelings. Based on the above arrangement, any number of A's and B's can be chosen from the first set of four and will dictate the results of the following two sets to maintain a choice pattern of exactly 4 A's, 4 B's and 4 C's. The choices and number of arrangements of each choice are as follows:

(0A's & 4B's) (0B's & 4C's) (0C's & 4A's) = (1)(1)(1) = 1
(1A & 3B's) (1B & 3C's) (1C & 3A's) = (4)(4)(4) = 64
(2A's & 2B's) (2B's & 2C's) (2C's & 2A's) = (6)(6)(6) = 216
(3A's & 1B) (3B's & 1C) (3C's & 1A) = (4)(4)(4) = 64
(4A's & 0B's) (4B's & 0C's) (4C's & 0A's) = (1)(1)(1) = 1

Notice that the numbers cubed above are those in Row 4 of Pascal's Triangle. This is because we are choosing 0, 1, 2, 3, and 4 "items" from our total of 4 possible items in each set. The total number of valid choice patterns is then 1 + 64 + 216 + 64 + 1 = 346, so there is a total of $346 \cdot 13824 = 4783104$ possible permutations for the third row.

ROW 4:

Half of a double wedge remains to be filled, so six numbers are again restricted to the left and right halves of R4. Each of these six numbers already appears three times in the grid, creating a total of six blocked spots per number. Taking the ring properties into account, these six possibilities per half each have three positions to be placed in the half. The summary for the left half of R4 follows:

- 6 positions and 6 numbers
- 3 possible positions per number and 3 possible numbers per position

Again, there must be minimal variable overlap between adjacent positions to maximize the permutation count. Let the number choices be represented by the variables *a, b, c, d, e, f*. The optimal layout for the variables is as follows:

Position 1:	Position 2:	Position 3:	Position 4:	Position 5:	Position 6:
a	d	a	d	a	d
b	e	b	e	b	e
c	f	c	f	c	f

Figure 8

There is no overlap of choices between consecutive positions so the only restriction is that all variables must be used. Therefore, total number of possible permutations for the half is $3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 36$. There are then $36^2 = 1296$ permutations for the fourth row.

ROW 5:

Four double wedges are complete so that rule can once again be ignored. Each of the numbers 1-12 has been used four times in the grid and each position of R5 has eight different numbers in its direct and mirrored columns. The summary is:

- 12 positions and 12 numbers
- 4 possible positions per number and 4 possible numbers per position

Just like all other rows, a minimal overlap of variables between adjacent positions is to be found. Let the number choices be represented by the variables *a, b, c, d, e, f, g, h, i, j, k, l* and follow the procedure from R4:

Pos 1:	Pos 2:	Pos 3:	Pos 4:	Pos 5:	Pos6:	Pos 7:	Pos 8:	Pos 9:	Pos 10:	Pos 11:	Pos 12:
a	e	i	a	e	i	a	e	i	a	e	i
b	f	j	b	f	j	b	f	j	b	f	j
c	g	k	c	g	k	c	g	k	c	g	k
d	h	l	d	h	l	d	h	l	d	h	l

Figure 9

There is no overlap of choices between adjacent positions once again, so there are $4 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 13824$ possible permutations of R5.

ROW 6:

One row now remains to fill a complete grid. Each number from 1-12 occurs in five previous rows resulting in two possible positions left per number. These two spots must be in mirrored columns, so there is one possibility in the left half and one in the right half of the row for each number. Also, half of the last two double wedges must still be filled by R5, eliminating one of our two possibilities for each number. With one option remaining per number, there is only one possible permutation for R6.

Now that each row has been computed individually, the total number of valid grids is the quotient of all row permutations since every row is independent of the others. This results in $12! \cdot 70225 \cdot 4783104 \cdot 1296 \cdot 13824 \cdot 1 = 2882552743578688039157760000$ possible Clock Sudoku grids, as claimed.

References

1. Information obtained from “Japanese Translation and Typesetting”. See page 24 for source information.
2. Image from LIFE Magazine online Sudoku game.

Works Cited

Japanese Translator.co.uk: Japanese Translation and Typesetting. Philip Ronan. 2004. 12 November 2006. <http://japanesetranslator.co.uk/portfolio/sudoku/>.

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