An Introduction to Self-Similarity by Way of the Cantor Sets

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Introduction

Roughly speaking a self-similar object is one that is similar, or approximately similar, to part of itself. That is, that the object can be scaled and translated on top of a portion of the original object. Though self-similarity is difficult to define generally, much study has been done on self-similar objects. The most common approach to constructing such a shape is taking the set of formed through an iterative process of construction. A few examples of such processes follow below. A fractal is defined loosely as a geometric shape that is highly self-similar.

A classical example of a fractal is the Cantor middle thirds set. The Cantor set is constructed by beginning with the unit interval [0,1] and at each step removing the middle third of the remaining intervals. Below are the first few iterations of the process used to create the Cantor middle third set. We study the Cantor sets more generally later as constructions outside of Euclidean space.

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0                1/3               2/3               1
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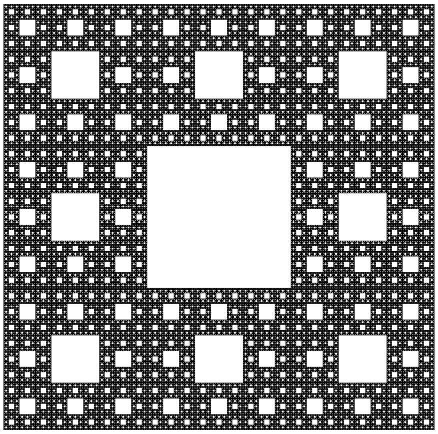
To move off of the real line we discuss the Sierpinski carpet, a comparable example in the plane. The Sierpinski carpet is formed by starting with a square, then dividing the square into 9 equal size squares and removing the middle square, then repeating the process on the eight remaining squares and so on. Likewise, if we remove any other combination of squares at each step we get fractals in the plane. Below are the Sierpinski carpet, shown after 6 iterations, and another common ‘box’ fractal with the first few iterations illustrated.



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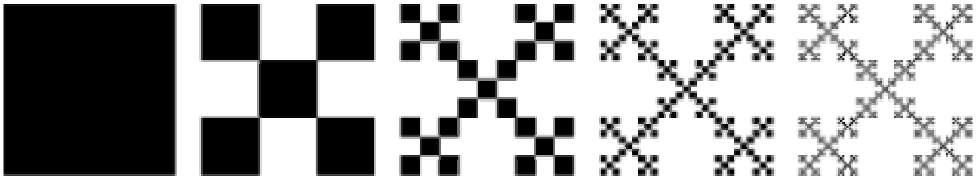
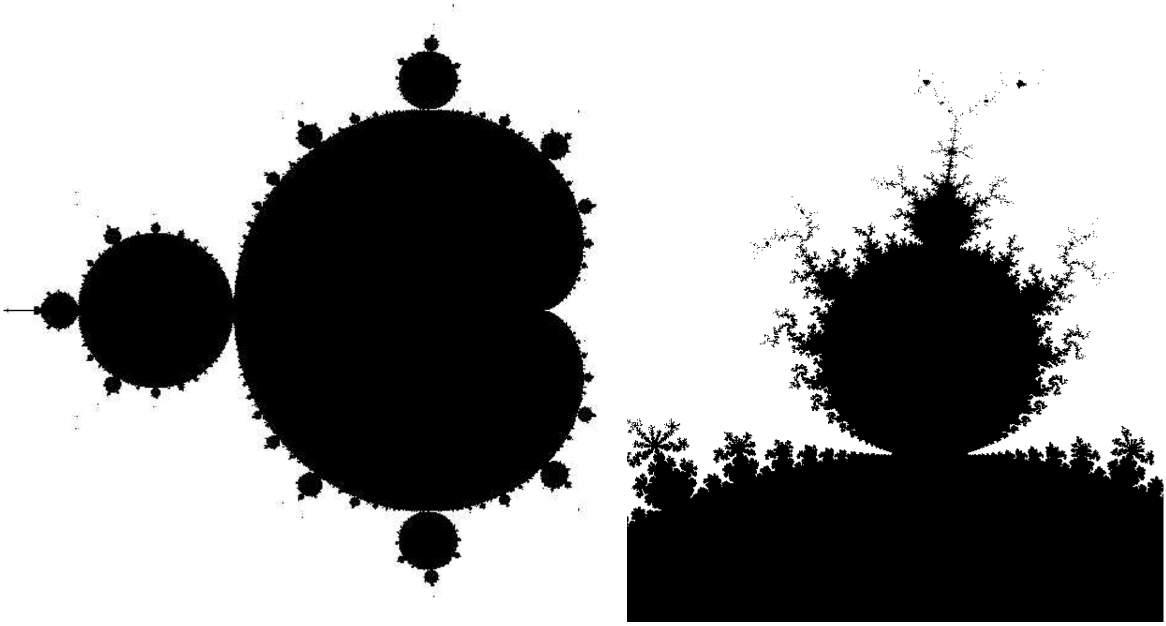


Another famous, and much more complicated, fractal is the Mandelbrot set. The Mandelbrot set is the set of values, c, in the complex plane for which the sequence defined recursively by $z_{n+1} = (z_n)^2 + c$, $z_0 = 0$, does not tend to infinity. The set is named for Benoit Mandelbrot.



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Fractal geometry and self-similar objects are important to many fields of science. Fractals occur naturally in clouds and coastlines, biologically in the circulatory system and lung surface, in economics, telecommunications, traffic studies, and other fields. Fractals have become an increasingly accessible field of interest since the rise of the computer as a computational tool. Amongst other things the computer has made the visualization of many complicated sets possible and has made it easier to study iterative functions which were otherwise too complicated to calculate by hand.

In their book, Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure, Guy David and Stephen Semmes[1] propose to classify self-similar geometries into families so that they can be analyzed systematically by examining the different families to which they belong. In this paper we will discuss the family of Cantor sets with varying metrics, call them ‘Cantor spaces’, and try to establish relationships among them. David and Semmes discuss these spaces but leave many of the proofs to the reader. We will prove the statements that they leave to the reader. Going beyond the authors’ work, we introduce a distance function between the spaces known as the Gromov-Hausdorff distance. The final focus of this paper will be to show that there exists a continuous relationship between our choice of metrics and the family of bilipschitz inequivalent Cantor spaces. We begin by introducing some key ideas to aid in our analysis.

Definitions and Key Concepts

In order to discuss the relationships between certain self-similar spaces we will first need to introduce metric spaces, as these will be our primary focus. A metric space is a set with a nonnegative distance function defined on it. This distance function, called a metric, must satisfy three properties. Let $d(x,y)$ denote the distance between x and y. In order to be a valid metric:

i) $d(x,y)=0$ if and only if $x=y$.
ii) For all x and y in the space X, $d(x,y)=d(y,x)$.
iii) For all x, y, and z in the space X, $d(x,z)\le d(x,y)+d(y,z)$.

A set with such a distance function $d(x,y)$ is called a metric space.

If we strengthen the third property of the metric we obtain an *ultrametric*. For an ultrametric $d(x,y)$ we have:

iii) For all x, y, and z in the space X, $d(x,z)\le \max\left\{d(x,y), d(y,z)\right\}$.

Note that properties (i) and (ii) must still hold in order for $\left(X, d(x,y)\right)$ to be a valid ultrametric space.

Examples:

(1) Euclidean space $\mathbb{R}^n$ with the standard Euclidean distance function is a metric space. Write $\mathrm{x}=(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_\mathrm{n})$ and $\mathrm{y}=(\mathrm{y}_1, y_2, ..., y_\mathrm{n})$,

$$d(x,y)=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\ .$$

(2) A set X with the discrete metric $d(x,y)=\begin{cases}0 & x=y\\ 1 & x\neq y\end{cases}$ is an ultrametric space.

There are several types of equivalences between two metric spaces; we now define a few of them for future use. The simplest such equivalence is a bijection. We say that X is bijectively equivalent to Y provided there exists a one-to-one and onto map from X to Y. It follows that there exists a one-to-one and onto map from Y to X since any such map is invertible. This equivalence is not influenced by the metric imposed on the two spaces, merely by the sets X and Y. The next type of equivalence that we consider is a distance preserving bijection. A distance preserving bijection is a bijective map $f: X\to Y$ such that $d_Y\left(f(x), f(y)\right)=d_X\left(x,y\right)$ where d_X is the metric on X, similarly d_Y is the metric on Y, and x and y are arbitrary elements of X. Again, this implies the inverse map $f^{-1}: Y\to X$ is also distance preserving.

If we lessen our restriction on the distance preserving features of $f: X\to Y$ we can define a homeomorphism. We say that two spaces X and Y are homeomorphically equivalent if there exists a continuous bijection $f: X\to Y$ such that the inverse map $f^{-1}: Y\to X$ is also continuous. A stronger statement of equivalence is bilipschitz equivalence. We say that a map $f: X\to Y$ is lipschitz provided for some real number C and for x and y in X, $d_Y\left(f(x), f(y)\right)\le C\, d_X\left(x,y\right)$. A map is said to be bilipschitz if for x and y in X, $C^{-1}d_X\left(x,y\right)\leq d_Y\left(f(x),f(y)\right)\leq C\,d_X\left(x,y\right)$ for some C that is independent of x and y. If there exists a bilipschitz map that is onto, then X and Y are said to be bilipschitz equivalent.

Proposition: If two metric spaces are bilipschitz equivalent, then they are homeomorphically equivalent.

Proof:

Let X and Y be our spaces. Assume they are bilipschitz equivalent. The bilipschitz inequalities guarantee that the map is one-to-one. Then since the bilipschitz map is by definition onto tells us $f:X\to Y$ is invertible. The fact that $d_Y\left(f(x),f(y)\right)\leq C\,d_X\left(x,y\right)$, x and y in X, tells us that f is continuous, uniformly continuous actually. Similarly, since $C^{-1}d_X\left(x,y\right)\leq d_Y\left(f(x),f(y)\right)$ we know $d_X\left(f^{-1}(w),f^{-1}(z)\right)\leq C\,d_Y\left(w,z\right)$ where w and z are in Y so then we have that the inverse $f^{-1}:Y\to X$ is continuous as well. Then we have a homeomorphism $f:X\to Y$.

It can be difficult to determine whether or not two spaces are bilipschitz equivalent, so we now introduce a bilipschitz invariant to help us verify this equivalence. Define the *Hausdorff Dimension* of a space X as follows [1]. For a positive real number δ choose a countable cover for X, $\left\{\mathrm{E_j}\right\}$, such that for each j, $diam(E_j)<\delta$. For positive real number s, consider the sum $\sum_{\mathrm{j}=1}^{\infty}\left(\mathrm{diam(E_j)}\right)^s$.

Define $S_\delta(s)=\inf_{\mathrm{covers}\left\{\mathrm{E_j}\right\}}\left\{\sum_{j=1}^{\infty}\left(diam(E_j)\right)^s\right\}$. As we let δ tend to 0, there exists a unique value s such that for all $\mathrm{x}<\mathrm{s}$ we have $\lim_{\delta\to0}\mathrm{S}_\delta(\mathrm{x})\to\infty$ and for all $\mathrm{x}>\mathrm{s}$ we have $\lim_{\delta\to0}S_\delta(x)=0$.

Definition: We define the *Hausdorff Dimension* of the space X to be the value s such that for all $\mathrm{x}<\mathrm{s}$, $\lim_{\delta\to0}\mathrm{S}_\delta(\mathrm{x})\to\infty$, and for all $\mathrm{x}>\mathrm{s}$, $\lim_{\delta\to0}S_\delta(x)=0$.

Proposition: The Hausdorff dimension of a space is bilipschitz invariant.

Proof:

Let X and Y be our spaces. Assume they are bilipschitz equivalent. Let $f:X\to Y$ be our bilipschitz map and C be such that for x and y in X, $C^{-1}d_X\left(x,y\right)\leq d_Y\left(f(x),f(y)\right)\leq C\,d_X\left(x,y\right)$ *.Let s be the Hausdorff dimension of X. Let x be greater than s. We can see that there is a correspondence between the covers $\left\{\mathrm{E_j}\right\}$ for X with $diam(E_j)<\delta$ and the covers $\left\{A_\mathrm{j}\right\}$ for Y with $diam(E_j)<C\delta$. Consider the sums $S_\delta(s)=\inf_{\mathrm{covers}\left\{A_\mathrm{j}\right\}}\left\{\sum_{j=1}^{\infty}\left(diam(E_j)\right)^s\right\}$ on Y. As δ tends to 0, so does $C\delta$. For $x>s$,*

$$\lim_{\delta\to 0} S_\delta(x) = \lim_{C\delta\to 0} \inf_{\text{covers}\{A_j\}} \left\{ \sum_{j=1}^{\infty} \left(diam(A_j) \right)^s \right\} = \lim_{\delta\to 0} \inf_{\text{covers}\{E_j\}} \left\{ \sum_{j=1}^{\infty} \left(diam(f(E_j)) \right)^s \right\}$$

$$\le C^s \lim_{\delta\to 0} \inf_{\text{covers}\{E_j\}} \left\{ \sum_{j=1}^{\infty} \left(diam(E_j) \right)^s \right\} = 0.$$ *So*

$$\lim_{\delta\to 0} S_\delta(x) = \lim_{C\delta\to 0} \inf_{\text{covers}\{A_j\}} \left\{ \sum_{j=1}^{\infty} \left(diam(A_j) \right)^s \right\} = 0.$$ *Next, for* $x < s$ *:*

$$\lim_{\delta\to 0} S_\delta(x) = \lim_{C\delta\to 0} \inf_{\text{covers}\{A_j\}} \left\{ \sum_{j=1}^{\infty} \left(diam(A_j) \right)^s \right\} = \lim_{\delta\to 0} \inf_{\text{covers}\{E_j\}} \left\{ \sum_{j=1}^{\infty} \left(diam(f(E_j)) \right)^s \right\}$$

$$\ge C^{-s} \lim_{\delta\to 0} \inf_{\text{covers}\{E_j\}} \left\{ \sum_{j=1}^{\infty} \left(diam(E_j) \right)^s \right\} = \infty.$$ *So then* s *is the Hausdorff dimension for* $Y = f(X)$ *as well.*

We next define a distance function between compact metric spaces, the definition is attributed to M. Gromov[4] and can be found as stated below in N. Hitchin's[5] "Global Differential Geometry." Consider functions $f : X \to Y$ and $g : Y \to X$ where (X, d_X) and (Y, d_Y) are compact metric spaces. If there exists positive ε such that for all x, x_1, and x_2 in X we have $\left| d_X(x_1, x_2) - d_Y\left(f(x_1), f(x_2)\right) \right| < \varepsilon$ and $d_X\left(x, g(f(x))\right) < \varepsilon$, and the analogous inequalities hold for points in Y, then we say the *Gromov-Hausdorff* distance $d_{GH}(X, Y)$ is less than or equal to ε.

Definition: We define the Gromov-Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) to be the infimum of such ε.

Cantor Sets

We define the Cantor sets as sets of sequences with values from a finite set. Let F be a set with k ≥ 2 elements. To indicate the number of elements in F we will denote the set as $\mathrm{F_k}$. Let $\mathrm{F_k^\infty}$ be the set of infinite sequences with elements from $\mathrm{F_k}$. We call $\mathrm{F_k^\infty}$ the k-Cantor set. In order to consider the k-Cantor set as a metric space we define a metric d as follows. First, let us write $x \in \mathrm{F_k^\infty}$ as $\{x_i\}$ where x_i is the i^th^ element of the sequence x. Define a function $L : \mathrm{F_k^\infty} \times \mathrm{F_k^\infty} \mapsto \mathbb{Z}_{\geq 0}$ as follows:

$$L(x,y) = \begin{cases} 0 & x_1 \neq y_1 \\ n & \text{for all } i \leq n, x_i = y_i \text{ and } x_{n+1} \neq y_{n+1} \\ \infty & x = y, \text{i.e. for all } i \ \ x_i = y_i \end{cases}.$$

Then for any element $a \in (0,1)$, we can define a metric $d_a(x,y) = a^{L(x,y)}$.

Proposition: d_a is an ultrametric on the k-Cantor set.

Proof:

Choose $a \in (0,1)$. *Assume* $d_a(x,y) = 0$*, then* $L(x,y) = \infty$*, so* $x = y$.

Let $x = y$*, then* $L(x,y) = \infty$*, so for any* $a \in (0,1)$*,* $d_a(x,y) = a^{L(x,y)} = a^\infty = 0$.

Thus $d_a(x,y) = 0 \Leftrightarrow x = y$.

Since $L(x,y) = L(y,x), d_a(x,y) = d_a(y,x)$.

Choose x, y, z *in the k-Cantor set. We want* $d_a(x,z) \leq \max\{d_a(x,y), d_a(y,z)\}$. *This is equivalent to the statement that* $L(x,z) \geq \min\{L(x,y), L(y,z)\}$ *since* $a \in (0,1)$. *Since for all* $\mathrm{n} \leq \min\{L(x,y), L(y,z)\}$ *we have that* $x_n = y_n$ and $y_n = z_n$, so then $x_n = z_n$*, it follows that* $L(x,z) \geq \min\{L(x,y), L(y,z)\}$ *and hence* $d_a(x,z) \leq \max\{d_a(x,y), d_a(y,z)\}$. *Therefore* d *is an ultrametric on the k-Cantor set.*

We note for later use that the open balls in this metric are precisely the sets that consist of all possible sequences with a finite number of their first terms fixed. We can say that this is a ball centered at any sequence in the set since all of the sets share these first elements. By definition all distances in this metric must be of the form a^n for some nonnegative integer *n.* Then all open balls can be identified as a^n balls, i.e. balls that fix the first *n* elements of the sequences contained in the ball. It follows easily from the metric definition that any sequences *x* and *y* with the same first *n* terms have $d_a(x,y) = a^{L(x,y)} \leq a^n$. Also, if *x* and *y* differ in the first *n* terms then $d_a(x,y) = a^{L(x,y)} \geq a^n$ since $L(x,y) \leq n$.

This construction of the Cantor sets allows us to see their self-similarity characteristics. Consider any subset C of a k-Cantor set so that C contains a set with all possible m-tails of a sequence with its first m-1 terms fixed, label this set C^m. We can obtain the entire Cantor set from C^m by re-indexing each element in C^m, viewed as a sequence of elements from F_{k}, to begin with its m^th^ term. This process works for any n larger than m as well, thus we can take an arbitrarily small portion of the k-Cantor set and rescale it so that it recovers the entire set. For $\varepsilon > 0$ choose N so that we have $a^N < \varepsilon$ where $a \in (0,1)$ as above. Choose $C^N \subset C^m$, then C^N can be scaled to equal the entire Cantor set and C^N has diameter less than ε.

The k-Cantor sets as described above are related to the classical self-similar Cantor sets in Euclidean Space. The middle-thirds set can be viewed as the 2-Cantor set with $a = \frac{1}{3},\ i.e.\, d_a(x,y) = \left(\frac{1}{3}\right)^{L(x,y)}$. This space and the Cantor middle-thirds set are bilipschitz equivalent.

We consider the relationships between the k-Cantor sets as metric spaces. When we vary the value of k we do not get any interesting relationships since we are working over considerably different sets. We restrict ourselves to a single value of k and vary only the metric on the space, say (X, d_1) and (Y, d_2) where X and Y are the same Cantor set, then we have an obvious bijection between the spaces using the identity map. We will show that we also have a homeomorphism using the identity map. The two spaces, however, will not have a distance-preserving bijection between them. This would require each space to have the same metric and hence the two spaces would be equivalent.

Proposition: (X, d_1) and (Y, d_2) are homeomorphic, where X and Y are the same Cantor set, $d_1(x,y) = a_1^{L(x,y)}$ and $d_2(x,y) = a_2^{L(x,y)}$ with $a_1, a_2 \in (0,1)$.

Proof:

Let f : Cantor set $\rightarrow$ Cantor set be the identity map. Fix x_0 in the Cantor set. Choose $\varepsilon > 0$. Pick N such that $a_2^N < \varepsilon$. Let $\delta = a_1^N$. Then for all x so that $d_1(x, x_0) < \delta$ we have $L(x, x_0) \geq N$. So $d_2(f(x), f(x_0)) = d_2(x, x_0) \leq a_2^{L(x,x_0)} \leq a_2^N < \varepsilon$, and thus the identity map f : Cantor set $\rightarrow$ Cantor set is continuous. An analogous proof tells us that f^{-1} : Cantor set $\rightarrow$ Cantor set is also continuous. Therefore the two spaces are homeomorphically equivalent.

Before we proceed, we want to show that our Cantor spaces, the sets together with their metric, are compact. To do so we first discuss their construction in the product topology and their obvious compactness. Then we will show that the product and metric topologies are equivalent in this setting.

Let $\prod_{j \in \mathbb{N}} X_j$ be our k-Cantor set where each X_j is the set, F_{k}, with k elements. Note that a finite set is compact in any topology since for any open cover it suffices to choose a subcover that consists of one set containing each point. In particular the all sets in F_{k} are compact in the topology that consists off all sets being open. Let $\pi_i : \prod_{j\in\mathbb{N}} X_j \to X_i$ be the projection mapping that gives the i^{th} term in the sequence from $\prod_{j\in\mathbb{N}} X_j$.

For simplicity we now restrict ourselves to the case where k=2, the other cases are analogous. In X_j U open implies $U \in \{\{0\},\{1\},\{0,1\},\varnothing\}$ where the two elements in F_{k} are 0 and 1. Note $\pi_i^{-1}(\{0\})$ is the set of sequences in $\prod_{j\in\mathbb{N}} X_j$ whose i^{th} term is 0. Similarly, $\pi_i^{-1}(\{1\})$ is the set of sequences in $\prod_{j\in\mathbb{N}} X_j$ whose i^{th} term is 1. $\pi_i^{-1}(\{0,1\})$ is the entire space $\prod_{j\in\mathbb{N}} X_j$ and $\pi_i^{-1}(\varnothing)$ is the null set. So then $S_i = \{\pi_i^{-1}(U) : U \in \{\{0\},\{1\},\{0,1\},\varnothing\}\}$ consists of the sequences with fixed i^{th} term and the whole space. Let $S = \bigcup_{j\in\mathbb{N}} S_j$.

Note that S is a subbasis for $\prod_{j\in\mathbb{N}} X_j$. So then by the Tychonoff Theorem[6] $\prod_{j\in\mathbb{N}} X_j$, the k-Cantor set, is compact in the product topology. Then to prove that the sets are compact in the metric topology it suffices to show that the two topologies are equivalent.

Proposition: The metric topology for d_a and product topology are equivalent on the k-Cantor set.

Proof:

We prove that the product topology is contained in the metric topology and that the reverse containment also holds. First, to see that the metric topology is contained in the product topology we show that every open ball in the metric topology is the finite intersection of subbasis elements in the product topology.

Every open ball in the metric topology consists of restrictions on a finite number of elements in a sequence. Hence each open ball is a finite intersection of subbasis elements, precisely the subbasis elements that fix those finite number of elements, for the product topology. Then since the subbasis generates the product topology, we have containment.

We now show the product topology is contained in the metric topology.

We consider any subbasis element C in the product topology as the set of sequences with a single, say the j^{th}, term restricted. Let $\varepsilon = a^j$. Then we can construct C as the union of k^{j-1} ε balls of that consist of all the k^{j-1} possible combinations of first j-1 terms with the j^{th} term restricted as in the C. Then since the metric topology is closed under unions and finite intersections it contains the product topology.

Then the two topologies are equivalent.

The question of whether or not two of these Cantor spaces are bilipschitz equivalent is complicated. To analyze this possibility we will show that two such spaces have different Hausdorff dimensions. Since the Hausdorff dimension of a space is bilipschitz invariant this means that the two spaces are not bilipschitz equivalent.

Theorem: The k-Cantor set with metric, $d_a(x,y) = a^{L(x,y)}$, $a \in (0,1)$ has Hausdorff dimension $\log_a(\frac{1}{\mathrm{k}})$.

Proof:

We restrict ourselves to the case where k = 2. The other cases are analogous. Label the 2-Cantor set X. Fix arbitrary $a \in (0,1)$. *Consider an arbitrary set* $\mathrm{E} \subset \mathrm{X}$. *The diameter of E is* a^m *where* $m = \inf\{L(x,y) : x, y \in X\}$. *If E has only one element, its diameter is 0, so assume E has at least two elements. Choose two distinct elements* x_0 *and* y_0 *in E.*

Let $m_0 = L(x_0, y_0)$. *Since the set of nonnegative integers less than or equal to* m_0 *is finite,* $m = \min\{L(x,y) : x, y \in X\}$.

Choose arbitrary $\delta > 0$. *We may assume* $\delta < 1$. *Let n be the smallest integer such that* $a^{\mathrm{n}} \le \delta$. *Note that if we consider a set B composed of all the n-tails of a sequence with its first n elements fixed, then B has diameter* a^n. *There are* 2^n *such sets in X because there at* 2^n *possible combinations of the first n terms in a sequence. Thus we need* 2^n *of sets like B to cover X, label this family of sets* $\{B_{n,i}\}_{i=1}^{2^n}$ *and call these sets fundamental sets. For the general k case, there are* k^n *such sets.*

For any m we can consider a fundamental set B_m *that consists of the possible m-tails for a fixed first m terms. This set is open in the Cantor set since for any* $\varepsilon < a^m$ *the* ε *ball about any point in* B_m *is contained in* B_m. *Since the Cantor set is compact, for any cover by* B_m*'s we have an associated finite subcover.*

Choose an arbitrary countable cover $\{E_j\}$ *for X. Note that any set* E_j *must be contained in some* B_m, *where* $diam(E_j) = a^m$, *since otherwise its diameter would be at least* a^{m-1}. *So then we associate an open cover* $\{B_{m,j}\}_{m,j}$. *Choose a finite subcover* $\{B_{m,l}\}_{finite}$. *Since our* $\{B_{m,j}\}_{m,j}$ *have the same diameters as the* $\{E_j\}$ *we have* $\sum_{m,j}\left(diam(E_j)\right)^s \ge \sum_{m,j}\left(diam(B_{m,j})\right)^s$ *for any s. Further, since* $\{B_{m,l}\}_{finite}$ *considers only certain sest from the family* $\{B_{m,j}\}_{m,j}$ *we see* *that* $\sum_{j}\left(diam(B_{m,j})\right)^{s} \geq \sum_{finite}\left(diam(B_{m,l})\right)^{s}$ *. So then* $\sum_{j}\left(diam(E_{j})\right)^{s} \geq \sum_{finite}\left(diam(B_{m,l})\right)^{s}$ *. This holds for any value of s.*

We now argue that for these covers by fundamental sets that for $s > \log_{a}(\frac{1}{2})$ *we have* $\sum_{finite}\left(diam(B_{m,l})\right)^{s} \geq \sum_{i=1}^{2^{n}}\left(diam(B_{n,i})\right)^{s}$ *. Here n is as chosen in the third paragraph of this proof. Then by the definition of infimum* $\sum_{i=1}^{2^{n}}\left(diam(B_{n,i})\right)^{s} = \inf_{\{E_j\}}\left\{\sum_{j}\left(diam(E_{j})\right)^{s}\right\}$*. Choose an arbitrary such cover* $\left\{B_{m,l}\right\}_{finite}$*. Consider a single set* $B_{n,i}$*, either* $\left\{B_{m,l}\right\}_{finite}$ *breaks* $B_{n,i}$ *into multiple sets of diameter* a^{n}*, or* $\left\{B_{m,l}\right\}_{finite}$ *breaks* $B_{n,i}$ *into sets of smaller diameter. In the first case, we get* $\sum_{l}\left(diam(B_{m,l})\right)^{s} > \left(diam(B_{n,i})\right)^{s}$*, where* $\left\{B_{m,l}\right\}$ *is the portion of* $\left\{B_{m,l}\right\}_{finite}$ *that covers* $B_{n,i}$*. This is since* $2\left(diam(B_{n,i})\right)^{s} \leq \sum_{l}\left(diam(B_{m,l})\right)^{s}$ *.*

In the latter case, we will say without loss of generality that we break a single set $B_{n,i}$ *into 2 fundamental sets* $\left\{B_{n+1,l}\right\}$ *of diameter* a^{n+1}*. Assume that our cover* $\left\{B_{m,l}\right\}_{finite}$ *consists of these two sets along with the other* $\left\{B_{n,i}\right\}$*. We can make this assumption because if we decomposed* $\left\{B_{n,i}\right\}$ *any further it would reduce to breaking up similar sets that would increase the sum in question by a similar argument. Then we get:*

$$\sum_{l}\left(diam(B_{m,l})\right)^{s} = (2^{n}-1)(a^{n})^{s} + 2(a^{n+1})^{s} = 2^{n}(a^{ns}) - a^{ns} + 2(a^{s})(a^{ns})$$

$$>_{(1)}\ 2^{n}(a^{ns}) - a^{ns} + 2(\frac{1}{2})(a^{ns}) = 2^{n}(a^{ns}) - a^{ns} + (a^{ns}) = 2^{n}(a^{ns}).$$

The inequality (1) uses the fact that $s < \log_{a}(\frac{1}{2})$*, which gives us* $a^{s} > \frac{1}{2}$*.*

So then $\sum_{j}\left(diam(E_{j})\right)^{s} > \sum_{i=1}^{2^{m}}\left(diam(B_{i})\right)^{s}$ *for any countable cover* $\left\{E_{j}\right\}$ *and* $s > \log_{a}(\frac{1}{2})$*. Then by the definition of infimum we know* $\sum_{i=1}^{2^{n}}\left(diam(B_{n,i})\right)^{s} = \inf_{\{E_j\}}\left\{\sum_{j}\left(diam(E_{j})\right)^{s}\right\}$*. We now consider the Hausdorff dimension of X.*

Fix $\delta > 0$*. Construct* $\left\{B_i\right\}$ *as above. Note* $\left\{\sum_{i=1}^{2^m}\left(diam(B_{n,i})\right)^s\right\} \geq \inf_{\{\mathrm{E_j}\}}\left\{\sum_j\left(diam(E_j)\right)^s\right\}$ *by the definition of infimum and that* $\inf_{\{\mathrm{E_j}\}}\left\{\sum_j\left(diam(E_j)\right)^s\right\} \geq 0$. *As* δ *approaches 0, the n response to* δ *approaches* ∞.

So then $\lim_{\delta\to 0}\sum_{i=1}^{2^n}\left(diam(B_{n,i})\right)^s = \lim_{n\to\infty} 2^n a^{ns} = \lim_{n\to\infty}(2a^s)^n$. *For* $\mathrm{s} > \log_a(\frac{1}{2})$*,* $\lim_{\delta\to 0}\sum_{i=1}^{2^n}\left(diam(B_i)\right)^s = \lim_{n\to\infty}(2a^s)^n \leq \lim_{n\to\infty} r^n = 0, \text{where } \mathrm{r} = 2a^s < 1$. *For* $s < \log_a(\frac{1}{2})$*, by our work above,* $\sum_{i=1}^{2^m}\left(diam(B_i)\right)^s = \inf_{\{\mathrm{E_j}\}}\left\{\sum_j\left(diam(E_j)\right)^s\right\}$*. Then* $\lim_{\delta\to 0} S_\delta(s) = \lim_{n\to\infty}(2a^s)^n \geq \lim_{n\to\infty} r^n \to \infty, \text{since } r = 2a^s > 1$. *Thus for* $\mathrm{s} > \log_a(\frac{1}{2})$*,* $\lim_{\delta\to 0} S_\delta(x) = 0$ *and for* $\mathrm{s} < \log_a(\frac{1}{2})$*,* $\lim_{\delta\to 0} S_\delta(x) = \infty$*. Therefore the Hausdorff dimension of X is* $\log_a(\frac{1}{2})$*. Similarly, if we consider the general k case, we require* $\lim_{n\to\infty}(\mathrm{k}a^s)^n$ *to be either 0 or* ∞*. This limit depends on the value of* $\mathrm{r} = \mathrm{k}a^s$*, and so we get the Hausdorff dimension of the k-Cantor set to be* $\log_a(\frac{1}{\mathrm{k}})$.

Then two k-Cantor sets with different metrics will have different Hausdorff dimension and thus are not bilipschitz equivalent.

We wish to consider the relationship between the metric we place on the Cantor sets and the spaces that they define. We will prove that this relationship is a continuous one. Since the spaces are compact, we may apply the Gromov-Hausdorff distance to the spaces.

Consider a map *F* from (0, 1) to the set of metric spaces on the 2-Cantor set. For given $a \in (0,1)$, define $F(a) = (X, d_a)$, where *X* is the 2-Cantor set.

Proposition: The map F **as defined above is continuous.**

Proof:

Fix a_1 *in* $(0,1)$*. Choose arbitrary* $\varepsilon > 0$*. Let* $\delta = \frac{\varepsilon(1-a_1)}{2}$.

Choose $a_2 \in B_\delta(a_1)$.

Pick x, x_1*, and* x_2 *in* X*. Note that X=Y. Since X and Y are the same space and we are working with the identity map it suffices to consider only one set of the inequalities. Let f and g be the identity map for use in the definition of the Gromov-Hausdorff distance between* (X, d_{a_1}) *and* (Y, d_{a_2}).

Note that $d_{a_1}\left(x, g(f(x))\right) = d_{a_1}\left(x, x\right) = 0$ *and* $d_{a_2}\left(x, x\right) = 0 < \varepsilon$ *. So then we consider:*

$$d_{GH}(X,Y) \le \left|d_{a_1}(x_1,x_2) - d_{a_2}(f(x_1), f(x_2))\right| = \left|d_{a_1}(x_1,x_2) - d_{a_2}(x_1,x_2)\right|$$

$$= \left|a_1^{L(x_1,x_2)} - a_2^{L(x_1,x_2)}\right| \le \left|a_1 - a_2\right| * \left|a_1^{L(x_1,x_2)-1} a_2^0 + a_1^{L(x_1,x_2)-2} a_2^1 + \ldots + a_1^1 a_2^{L(x_1,x_2)-2} + a_1^0 a_2^{L(x_1,x_2)-1}\right|$$

$$\le \left|a_1 - a_2\right| * \left|L(x_1,x_2) \max\left\{a_1, a_2\right\}^{L(x_1,x_2)}\right| \le \left|a_1 - a_2\right| * \left|\sum_{i=1}^{L(x_1,x_2)} \max\left\{a_1, a_2\right\}^i\right|$$

$$\le \left|a_1 - a_2\right| * \left|\sum_{i=1}^{\infty} \max\left\{a_1, a_2\right\}^i\right| \le \left|a_1 - a_2\right| * \frac{1}{1 - \max\left\{a_1, a_2\right\}} \le \delta * \frac{1}{1 - (a_1 + \delta)}$$

. Since $\delta = \frac{\varepsilon(1 - a_1)}{2}$*, we have*

$$d_{GH}(X,Y) \le \frac{\varepsilon(1 - a_1)}{2} * \frac{1}{1 - a_1 - \frac{\varepsilon(1 - a_1)}{2}} = \varepsilon * \frac{(1 - a_1)}{2 - 2a_1 - \varepsilon(1 - a_1)} < \varepsilon \text{ for } \varepsilon < 1$$

. So we have $d_{GH}(X,Y) < \varepsilon$ *. So the map F is continuous.*

Then we have a continuous relationship between our choice of ultrametrics d_a and the spaces they define on the 2-Cantor set and analogously for the k-Cantor sets. This concludes what we set out to accomplish in this paper. The question remains of whether a stronger relationship between these Cantor sets and general self-similar sets exists. Due to the complexity of many self-similar sets, it would be incredibly useful to be able to relate them to simpler sets such as the Cantor sets.

References

Self-Similar Geometry, Cantor sets and definitions:

[1] Semmes, Stephen and David, Guy. Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure. Clarendon Press: Oxford. (1997).

Self-Similar Examples, pictures:

[2] http://en.wikipedia.org/wiki/Sierpinski_carpet
[3] http://mathworld.wolfram.com/

Gromov-Hausdorff Distance Definition:

[4] Gromov, M. Metric Structures for Riemannian and Non-Riemannian Spaces. (Translation by S.M. Bates of 1981 French edition) Birkhauser: Boston (1999).

[5] Hitchin, N. "Global Differential Geometry", Mathematics Unlimited-2001 and Beyond. Eds. B. Engquist and W. Schmid. Springer-Verlag: Berlin. (2001) pg. 577-591.

Product Topology, Tychonoff Theorem:

[6] Munkres, James R. Topology: A First Course. Prentice Hall: New Jersey. (1975).