Topology of Graph Configuration Spaces

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1 Introduction

Aaron Abrams and Robert Ghrist tell us that configuration spaces have been used by roboticists for years for planning robot motion. What makes motion planning for robots moving about on a graph interesting, is that it needs a more global understanding of the configuration space than in planning problems that allow the robots a greater freedom of direction in their motions. Besides this, problems of this sort are quite close to real world factory situations in which robots may be confined to guide rails etc. The primary motivation of this paper is to investigate some of the interesting topology that occurs in a "reduced" form of the configuration space of a system of robots on a graph. In particular, we investigate the reasons for some of these "reduced spaces" being manifolds. We also carry out some original work in which we extend the results of Abrams and Ghrist to non-simple graphs.

In [1] they give us an example of a less restricted motion planning problem of a group of robots that are moving about on a factory floor. In order to plan their motion successfully, one does not need to know where every robot is at any given time. In fact, all you need to do is make sure that each robot has a small neighbourhood around it that is free of robots, so that it has a direction in which it can move if it needs to avoid a collision. And the path of one robot does not significantly affect the paths of the other robots. However, in a more restricted situation, such as a group of robots restricted to move along a network of rails, there are a limited number of directions that a robot can take in order to avoid a collision. Consider a situation in which a robot needs to modify its path through the rail network. The network of rails presents it with a finite number of choices of directions in which to move. Thus even for the most fundamental movements, there is a good chance that all or some of its possible directions of movement are blocked by the other robots. In such cases, the paths of some or all of the other robots in the group need to be modified in order to move a particular robot. This translates to knowing the positions of all the robots in the system at all times, and speaking in terms of the configuration space of the system, this translates to understanding the configuration space more globally. We model systems of robots on a rail network by a group of labelled point like robots restricted to move along a graph.

Abrams and Ghrist try to take the "buffer neighbourhood" solution for the system which had a group of robots moving about on a factory floor, and adapt it to the more restricted problem of a group of robots restricted to move along a graph. All graphs have a discretized metric that measures the shortest path between two vertices in the graph. Abrams and Ghrist describe the "buffer neighbourhood" in terms of this discretized metric. This gives rise to a Discretized Configuration Space that much of the work in this paper centers around.

The discretization of the configuration space, provided that the graph is large enough, disallows most of the configurations that make the problem of coordinating movement of the robots a global one, i.e. it allows us to move a robot without having to disturb the positions of the other robots, and partially turns the problem of controlling the robots to a local one.

A startling result mentioned in [1] is that the "discretized" spaces of K_5 and $K_{3,3}$ for a system of 2 robots are 2-manifolds homeomorphic to a 6-holed torus and a 4-holed torus respectively. Abrams [2] tells us that there are the only two graphs that give us 2-manifolds for a system of 2 robots.

One of the first questions that we posed to ourselves was, “If the structure of the discretized space is such that the question of an orientation is a meaningful one, then is the discretized space always orientable ?” This question was the driving force behind much of the work done in producing this paper.

Pseudomanifolds are a generalization of manifolds that retain enough of the properties of manifolds to give the concept of an orientation meaning. In fact, they are the most general structures that retain a meaningful concept of orientation. We begin by considering the generalized case of a system of k robots on a graph and set up some basic definitions. We then restrict ourselves to a system of 2 robots on a graph and attempt to study the kinds of graphs that give us 2-pseudomanifolds. Using our own methods, we recover Abrams' result, that the completely connected graph of 5 vertices, K_5 and the bipartite graph $K_{3,3}$ are the only graphs that give us discretized spaces that are 2-manifolds when we have two robots free to move on them. We add to this by extending his result to non-simple graphs where we prove that the completely connected graph of 4 vertices K_4 with each of its edges doubled, and the 4-Cycle graph with each of its edges doubled are the only graphs that give us 2-pseudomanifolds as their discretized spaces when we have two robots free to move about on them.

2 Examples

Let us look at some examples of Entire Spaces, Configuration Spaces and Discretized Spaces. All the definitions and notation that we introduce in this section are precisely defined in the next section.

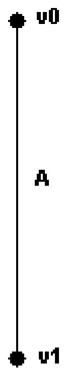

Figure 1: Graph Γ

Consider the graph Γ in 1, if there is one robot R_0 free to move over it, then the entire space of the system can be represented by the closed interval $[0, 1]$. Each value in the closed interval, appropriately scaled, represents a position of R_0 on Γ. Since there is only one robot in this system, there can be no collisions and so the entire space, the configuration space and the discretized space are the same.

Now consider two robots R_0 and R_1, moving on the graph Γ in Fig. 1. The entire space of this system can be represented by $[0, 1] \times [0, 1]$, i.e. the unit square. We let the x co-ordinate represent the position of R_0 on Γ and the y co-ordinate represent the position of R_1 on Γ. Each point in the square represents a possible state of the system, including collisions of the robots. Collisions occur when both robots are at the same position on Γ, points in the entire space whose x and y co-ordinates are equal represent collision points. Thus all diagonal points in the unit square represent collision points.

We name each cell by an ordered pair. The first entry in the ordered pair tells us where robot R_0 is on the graph and the second entry in the ordered pair tells us where robot R_1 is on the graph. For example (A, v_0) is a cell in which all the configurations have robot R_0 on edge A and robot R_1 at vertex v_0.

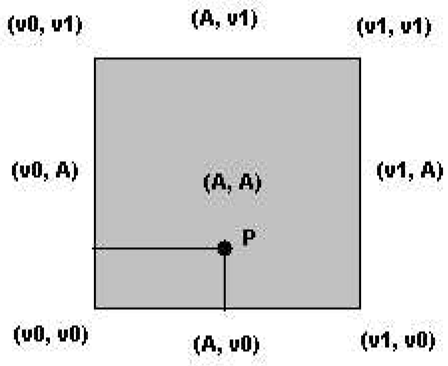

Figure 2: Entire Space

There are four possible configurations in which R_0 and R_1 are constrained to vertices of Γ, one in which R_0 and R_1 are both at v_0, R_0 and R_1 are both at v_1, R_0 is at v_0 and R_1 is at v_1, and one in which R_0 is at v_1 and R_1 is at v_0. These give rise to four 0-cells named (v_0, v_0), (v_1, v_1), (v_0, v_1) and (v_1, v_0) respectively. There are four possible sets of configuration points in which one robot is fixed at a vertex while the other is free to move along edge a, one in which R_0 is fixed at v_0 while R_1 moves along a, R_0 is fixed at v_1 while R_1 moves along a, R_1 is fixed at v_0 while R_0 moves along A and one in which R_1 is fixed at v_1 while R_0 moves along a. These give us four 1-cells, (v_0, A), (v_1, A), (A, v_0) and (A, v_1) respectively. There is only one set of configuration points in which both R_0 and R_1 are free to move along edge a. We represent this by a 2-cell (A, A). Taking all the 0-cells, 1-cells and 2-cells together and identifying points that represent the same configuration we get the "entire space" as in Fig.2. The point P in Fig. 2, represents a configuration in which R_0 is approximately one-half of the way along the edge a, from vertex v_0, and R_1 is approximately one-quarter of the way along edge a, from vertex v_0.

All the points along the diagonal of (A, A) represent configurations in which both the robots are at the same point on the All the points along the diagonal of (A, A) represent configurations in which R_0 and R_1 are at the same position

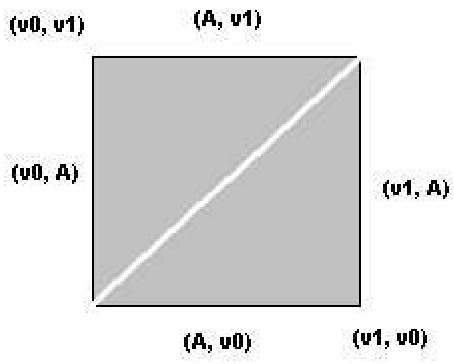

Figure 3: Configuration Space

along the edge A in Γ, i.e. they are collision points. Upon deleting these points, we get the "configuration space" as in Fig.4.

Figure 4: Discretized Space

The only configurations in which R_0 and R_1 are at least an edge apart are the two configuration points represented by the 0-cells (v_0, v_1) and (v_1, v_0). So we have the "discretized space" as in Fig. 3.

3 Definitions and Notation

Basic Assumptions: We consider a system of N independent and distinct robots called $r_i (i = 1, 2, ..., N)$ on a graph Γ. Each of the robots is considered to be a 0-dimensional point on Γ.

When two or more robots occupy the same position on the graph, we term the *entire* configuration of robots on the graph a **collision**.

Consider an edge E in graph Γ. Let u and v be the vertices connected by E. We consider the edge E to include the vertices u and v.

Definition 3.1. We consider a k-cell to be I^k, where $I = [0, 1]$. A k-cell is the configuration space of a system of N robots, in which exactly k of them are free to move along a particular edge of a graph, and the remaining $(N - k)$ of them are restricted to vertices of the graph. Each of the k Cartesian components of the k-cell; i.e. each of the k copies of I; are taken to represent the positions of one of the k robots along the edge of the graph to which it is restricted.

However, we consider a 0-cell to represent any number of robots, each restricted to a particular vertex of the graph.

It is important to note that for a given k-cell, all the robots that are allowed to move are allowed to do so on exactly one edge.

As a form of notation, we represent a k-cell($k \leq N$) by an N-tuple of the form $(y_1, ..., y_N)$. Where y_i is a vertex or an edge of Γ. A vertex in the i'th entry of the N-tuple implies that the robot r_i, is restricted to that vertex of Γ, while edge in the i'th entry of the N-tuple implies that the robot r_i is restricted to move along that edge of Γ. Thus for a 0-cell, each x_i is a vertex of Γ, for a 2-cell, exactly two y_i's are edges in Γ, and in general for a k-cell, exactly k y_i's are edges in Γ. Vertices are generally represented by lower case letters, while edges are generally represented by upper case letters.

We'd like to start with the whole configuration space first, including those configurations that involve collisions between robots. Things get interesting when you consider only those configurations that do not involve collisions.

Definition 3.2. The entire space $E^N(\Gamma)$, is defined to be the set of all possible configuration points, *including* collisions of the robots $r_i (i = 1, 2, ..., N)$ on the graph Γ. This is referred to as $\Gamma \times \Gamma ... \times \Gamma$(N times) in [1]. We consider all the possible k-cells(k = 0, 1, 2, ... , N) for all the possible permutations of robots on the edges and vertices of Γ, and we then identify all the cells at points that represent identical configurations of robots on Γ. The resulting space is $E^N(\Gamma)$.

Ofcourse this is not of much use as we would like to have something that would help us to prevent the robots from colliding. So we come up with what we call the Configuration Space, which is basically the Entire Space, with the collision points deleted. However, even this new definition still does not allow us to look at the problem from a purely topological point of view. This is because notions such as distance do matter in our configuration space, as we have to keep track of where the robots are on our graph on a very fine level, to prevent them from colliding. This prevents us from looking at our configuration space from a purely topological point of view as this may require us to carry out various transformations on it that are not distance preserving and so present us with a difficulty of translating whatever results we obtain back into the language of whatever control scheme we are using to control our robots.

We present the definition of the Configuration Space verbatim, as it appears in the article.

Definition 3.3. [1]The Configuration Space $\mathcal{C}^N(\Gamma)$ is defined to be $E^N(\Gamma) - \Delta$. Where $\Delta = \{x | x \in E^N(\Gamma)$ and x is a collision$\}$.

We now present Abrams' and Ghrist's adaptation of the "buffer neighbourhood" solution to the problem of a system of robots moving about on a graph.

What we would like is for each robot to have an ϵ neighbourhood around it that is kept free of robots. However, if we allow ϵ to be real valued, then the metric that is needed to specify this, forms a very artificial construction on the graph. What Aaron and Abrams do, is to specify this ϵ neighbourhood around each robot in terms of the discretized metric of the graph. Choosing an ϵ that is too large results in a useless situation in which no robots can move, the smallest non-zero value that can be chosen is 1. We don't need an ϵ neighbourhood around each robot that has an ϵ larger than 1. What this amounts to, is keeping all the robots at least 1 edge apart from each other. We call all the points in the configuration space that satisfy these conditions, the Discretized space.

The discretized space is basically built out of chunks of the configuration space that do not contain any collision points. This allows us to stop worrying about the exact positions of the robots on the graph and to start working in a non-distance preserving and purely topological setting. The definition involves restricting the robots to configurations in which they are all kept at least an edge apart from each other. The definition plays off well against the discretized structure of the graph, allowing us to talk about the cell structure of the space in terms of edges and vertices that are at least an edge apart. This allows us to talk about the configuration spaces purely in terms of the graph structure, without any reference whatsoever to the robots and makes it very easy for us to translate our results from working on the space, back into the concrete notions of the positions of the robots on the graph.

First we define formally the discretized metric that arises naturally out of the graph structure.

Definition 3.4. The edge metric $d(v_i, v_j)$, where v_i and v_j are two vertices in Γ, is defined to be the number of edges traversed by the shortest path from v_i to v_j in Γ. Given edges E, F in Γ, with vertices v_{1E}, v_{2E} and v_{1F}, v_{2F}, and for a vertex v, all in Γ, we define $d(E, v)$ to be $\min\{d(v_{1E}, v), d(v_{2E}, v)\}$, and $d(E, F)$ to be $\min\{d(v_{1E}, v_{1F}), d(v_{1E}, v_{2F}), d(v_{2E}, v_{1F}), d(v_{2E}, v_{2F})\}$.

Definition 3.5. [1]Consider all the r_i's on the graph Γ. If r_i is on a vertex v_i of Γ then let $x_i = v_i$, if r_i is on an edge E_i of Γ, then let $x_i = E_i$. We define the Discretized space $D^N(\Gamma)$ to be the subset of all configurations of the r_i's, such that $d(x_i, x_j) \geq 1, \forall i, j$ with $i \neq j$. It can also be defined as $E^N(\Gamma) - \tilde{\Delta}$. Where $\tilde{\Delta} = \{(y_1, ..., y_N) | (y_1, ..., y_N) \cap \Delta \neq \phi\}$.

We now classify the spaces we get into broad classes called k-complexes, cell complexes that have at least one k-cell. What this tells us is that the graph has k edges that are at least an edge apart from each other. This helps us a lot in some of our proofs.

Definition 3.6. A cell complex C, in which the cell of highest dimension is a k-cell and all cells of dimension lower than k are contained in the closure of some k-cell, is called a k-cell complex.

Definition 3.7. The boundary of a k-cell complex is defined to be the union of all (k-1)-cells that do not lie in the intersection of at least 2 k-cells.

4 Boundaryless Spaces

Giving the spaces $E^N(\Gamma)$ and $D^N(\Gamma)$ a cursory glance, one notices that they are cell complexes. It is possible for any number of k-cells to intersect each other in a (k-1)-cell. We would like to be able to put orientations on these spaces, but this is not always permissible. Thus we need to restrict our questions of orientation to those spaces that take the form of pseudomanifolds, or pseudomanifolds with boundary.

Proposition 4.1. *Given that $E^2(\Gamma)$ is a 2-cell complex, then $E^2(\Gamma)$ is a 2-cell complex without boundary if and only if ever vertex in Γ has valence ≥ 2.*

Proof. First, let us assume that $D^2(\Gamma)$ is a 2-cell complex without boundary. Thus every 1-cell is the intersection of at least two 2-cells.Let if possible that a vertex v in Γ has valence 1. Let E be the edge having v as a vertex. Since $E^2(\Gamma)$ is a 2-cell complex, $\exists$ a 2-cell (F, G) in $E^2(\Gamma)$. Consider the 1-cell (v, F). Since $E^2(\Gamma)$ is a 2-cell complex without boundary, (v, F) is the intersection of at least two 2-cells. One of the 2-cells intersecting (v, F) is (e, F). Let (H, F) be any other 2-cell intersecting (v, F). Thus, (v, F) is common to both (E, F) and (H, F), which implies that edges h and e share the same vertex v in Γ. Thus, v has valence 2. This is a contradiction. So we must conclude that our assumption that $\exists$ a vertex v in Γ with valence 1 is false. Therefore, every vertex of Γ has valence ≥ 2.

Conversely, let us assume that every vertex in Γ has valence ≥ 2. Consider an arbitrary vertex v and an arbitrary edge E in Γ. (v, E) and (E, v) will be 1-cells in $E^2(\Gamma)$. v has valence ≥ 2, and therefore $\exists$ two edges F and G with v as a vertex. The 2-cells (F, E) and (G, E) intersect in the 1-cell (v, E), and the 2-cells (E, F) and (E, G) intersect in the 1-cell (E, v). Since v and E were arbitrary, we can conclude that every 1-cell in $E^2(\Gamma)$ is the intersection of two or more 2-cells. Thus $E^2(\Gamma)$ is a 2-cell complex without boundary. □

Proposition 4.2. *Given that $D^2(\Gamma)$ is a 2-cell complex, it is without boundary if and only if $\forall$ edge E in Γ, $\forall$ vertex v such that $d(v, E) \geq 1$, $\exists$ two edges F and G sharing the vertex v, such that $d(F, E) \geq 1$ and $d(G, E) \geq 1$.*

Proof. Assume that $D^2(\Gamma)$ is a 2-cell complex without boundary. $\therefore$ Every 1-cell is the intersection of at least two 2-cells. Let E be any edge in Γ, and v be any vertex such that $d(E, v) \geq 1$. $\therefore$ (E, v) is a 1-cell in $D^2(\Gamma)$, and $\exists$ two 2-cells (E, F) and (E, G) intersecting in (E, v). This implies that edges F and G share the vertex v in Γ, and the definition of $D^2(\Gamma)$ tells us that $d(E, F) \geq 1$ and $d(E, G) \geq 1$.

Conversely, assume that $\forall$ edge E in Γ, $\forall$ vertex v in Γ such that $d(E, v) \geq 1$, $\exists$ two edges F and G, sharing the vertex v, such that $d(E, F) \geq 1$ and $d(E, G) \geq 1$. Any 1-cell in Γ is of the form (b, M) or (M, b), where M is an edge in Γ and b is a vertex such that $d(M, b) \geq 1$. $\therefore$ $\exists$ two edges P and Q, sharing the vertex b, such that $d(M, P) \geq 1$ and $d(M, Q) \geq 1$. So, by definition the 2-cells (P, M), (Q, M), (M, P) and (M, Q) exist in $D^2(\Gamma)$, and (b, M) is the intersection of (P, M) and (Q, M), and (M, b) is the intersection of (M, P) and (M, Q). $\therefore$ $D^2(\Gamma)$ is a 2-cell complex without boundary. □

Corollary 4.3. *Given that $D^2(\Gamma)$ is a 2-cell complex, if it is without boundary, then every vertex v in Γ has valence ≥ 3.*

Proof. Let v be a vertex in Γ. As $D^2(\Gamma)$ is a 2-cell complex, there must be an edge E in Γ such that $d(E, v) \geq 1$. $\therefore$ by Theorem 4.2, $\exists$ two edges F and G that share vertex v. $\therefore$ the remaining two vertices of F and G are v_F and v_G respectively. Valence $v_F \geq 2$, for if valence of v_F were 1, then the 1-cell (v_F, G) would not lie in the intersection of two or more 2-cells, and $D^2(\Gamma)$ would have a boundary. $\therefore$ let H be an edge in Γ, sharing the vertex v_F with F. We must have $d(H, v) = 0$ or $d(H, v) \geq 1$. If $d(H, v) = 0$, then the edges F, G and H share vertex v and valence $v = 3$. If $d(H, v) \geq 1$, then by Theorem 4.2, $\exists$ two edges M, N that share the vertex v, such that $d(M, H) \geq 1$ and $d(N, H) \geq 1$. If $d(G, H) = 0$, then the edges M and N must be distinct from F and G. $\therefore$ Edges

F, G, M and N share vertex v. $\therefore$ valence $v \geq 4$. If $d(G, H) \geq 1$, then there must exist an edge K, distinct from G, that has v as a vertex, such that $d(K, H) \geq 1$. The edges K and G would be 2 edges sharing v as a vertex with $d(K, H) \geq 1$ and $d(G, H) \geq 1$. $\therefore$ We have 3 edges F, G and K that share vertex v. So we have valence $v \geq 3$. Thus, in either case, valence $v \geq 3$. $\therefore$ Since v is arbitrary, we must conclude that $\forall$ vertices v in Γ, valence $v \geq 3$. □

5 Pseudomanifolds

5.1 Generalized Case of k Robots

We have based the following definitions in this sub-section, on those found in Munkres [3], by re-stating the definition in terms of cells rather than simplices.

Definition 5.1. A k-cell complex K in which every (k-1)-cell lies in the intersection of exactly two k-cells, is called a k-pseudomanifold.

Note that the above definition is looser than the traditional definition of a k-pseudomanifold, in that we are not requiring that there be a sequence of k-cells between any two k-cells in K. As a result of this, the k'th homology group of K, $H_k(K)$ may have more than one generator.

We call these spaces pseudomanifolds because they fail to be manifolds in cases, where they are not locally Euclidean. A k-pseudomanifold K, can fail to be locally Euclidean at an m-cell in K, where $(k-2) \geq m \geq 0$. We call such an m-cell a singularity.

We define the Star of an m-cell and use a special case of this definition with $m = 0$, to define a singularity at a vertex in a 2-pseudomanifold in the next section. However we state the generalized definition of a Star of an m-cell anyway.

Definition 5.2. Let t be an m-cell of a k-cell complex K. We define the Star S of t to be the union of the interiors of all the cells in K that have t as a face.

5.2 Specific Case of 2 Robots

Despite the definitions being given for k dimensions, we will be confining ourselves to 2-dimensional discretized spaces, i.e. $D^2(\Gamma)$ spaces, for the length of this paper. And for all the above definitions, we will henceforth assume the dimension k to be 2. So henceforth we will be talking about 2-pseudomanifolds, which may fail to be locally Euclidean precisely at 0-cells.

We now proceed to state and prove one of the central ideas of this paper. The Pseudomanifold Criterion. What we want to know is, what kind of graphs give us pseudomanifolds. What this reduces to, is asking, for what types of graphs, does every 1-cell lie in the intersection of exactly two 2-cells in $D^2(\Gamma)$. We see from Theorem 4.2, that the conditions for making sure that every 1-cell lies in the intersection of at least two 2-cells boils down to an inequality. Making that condition "exactly two 2-cells" is simply a matter of changing that to an equality.

Theorem 5.1. (Pseudomanifold Criterion) *Given that $D^2(\Gamma)$ is a 2-complex, every 1-cell is the intersection of exactly two 2-cells if and only if $\forall$ edges E in*

Γ, ∀ vertices v such that $d(E, v) \geq 1$*, ∃ exactly two edges F and G such that* $F \bigcap G = v$*, and* $d(E, F) \geq 1$ *and* $d(E, G) \geq 1$*.*

Proof. Assume that $D^2(\Gamma)$ is a 2-complex without boundary. Let E be any edge in Γ. Let v be any vertex such that $d(E, v) \geq 1$. Consider the 1-cell (E, v), it is the intersection of exactly two 2-cells, call them (E, F) and (E, G). The definition of $D^2(\Gamma)$ tells us that $d(E, F) \geq 1$ and $d(E, G) \geq 1$. And the fact that (E, v) is the intersection of (E, F) and (E, G) tells us that v is that $F \bigcap G = v$.

Conversely, assume that ∀ edges E in Γ, ∀ vertices v such that $d(E, v) \geq 1$, ∃ exactly two edges F and G, such that $F \bigcap G = v$ and $d(E, F) \geq 1$ and $d(E, G) \geq 1$. All 1-cells in $D^2(\Gamma)$ are of the form (X, a) or (a, X). The definition of $D^2(\Gamma)$ tells us that $d(X, a) \geq 1$. ∴ ∃ exactly two edges L and M, such that $L \bigcap M = a$ and $d(L, X) \geq 1$ and $d(M, X) \geq 1$. So we have $(X, L), (X, M), (L, X)$ and (M, X) as 2-cells in $D^2(\Gamma)$. The 1-cell (X, a) is the intersection of exactly two 2-cells, namely (X, L) and (X, M), and the 1-cell (a, X) is the intersection of exactly two 2-cells, namely (L, X) and (M, X). ∴ Every 1-cell in $D^2(\Gamma)$ is the intersection of exactly two 2-cells in $D^2(\Gamma)$. □

We call graphs whose $D^2(\Gamma)$ spaces are pseudomanifolds, Pseudomanifold graphs, and those whose $D^2(\Gamma)$ spaces are manifolds, Manifold graphs. Manifold graphs are a subset of Pseudomanifold graphs.

6 Singularities

2-pseudomanifolds, fail to be manifolds due to the presence of 0-cells that do not have any Euclidean neighborhoods. As we mentioned earlier, these vertices are called singularities. Due to the fact that we are now working with 2-pseudomanifolds, 0-cells are the only cells are the only cells which can be singularities. We define more clearly what a singularity in a 2-pseudomanifold is. Note that this definition relies on the definition of a Star of an m-cell($m = 0$ in this case) which can be found in the second sub-section of the previous section.

Definition 6.1. In a 2-pseudomanifold M, consider the star S of a 0-cell t. We call t a singularity, if $S - t$ is disconnected. We call the closures of each of the connected components of $S - t$ the Star Components of S. Note that in a 2-pseudomanifold, singularities can only occur in dimension 0, and hence the only type of singularities in 2-pseudomanifolds occur at 0-cells.

In the previous section, we got certain concrete conditions that pseudomanifold graphs need to satisfy. However, we would naturally like to know what further conditions we need to put on this subset of graphs so that their $D^2(\Gamma)$ spaces are manifolds. What this amounts to, is finding conditions to impose on the pseudomanifold, so that we filter out all those graphs whose $D^2(\Gamma)$ spaces contain singularities. So naturally, we begin to study those properties of pseudomanifold graphs that give rise to singularities.

As a run up to the Singularity Theorem, we attempt to provide the reader with an intuitive understanding its basic ideas. Consider a graph Γ such that $D^2(\Gamma)$ is a 2-pseudomanifold has a singularity at a 0-cell (v_a, v_b). We look at a local section of the graph, i.e. the vertices v_a, v_b and all the edges connected to them. The singularity in $D^2(\Gamma)$ will look something like Figure 5. Each pyramid would be a star component of the singularity. We have drawn the cell structure of each of the star components F and G in Figure 6. Note that one has to identify the 0-cell (v_a, v_b) in both the star components to view the singularity as it actually occurs in $D^2(\Gamma)$. Figure 7 shows the positions of the edges in Γ that form the 2-cells of F and G.The edges that form the 2-cells of F that have v_a or v_b as a vertex are of the form ${E_a}^F$ and ${E_b}^F$ respectively. And the edges that form the 2-cells of G that have v_a or v_b as a vertex are of the form ${E_a}^G$ and ${E_b}^G$ respectively.

It is apparent from Figure 7 that for all edges of the form ${E_a}^F$ and ${E_b}^G$, we have $d({E_a}^F, {E_b}^G) = 0$. For if this were not true, a glance at the diagram tells us that the graph would violate the Pseudomanifold Criterion Theorem. Similarly for all edges of the form ${E_a}^G$ and ${E_b}^F$, we must have $d({E_a}^G, {E_b}^F) = 0$. And the only way that these distance conditions can be achieved is by having vertices v_c and v_d such that all edges of the form ${E_b}^F$ has vertices v_b and v_d, and all edges of the form ${E_b}^G$ has vertices v_b and v_c. This is shown in Figure 8.

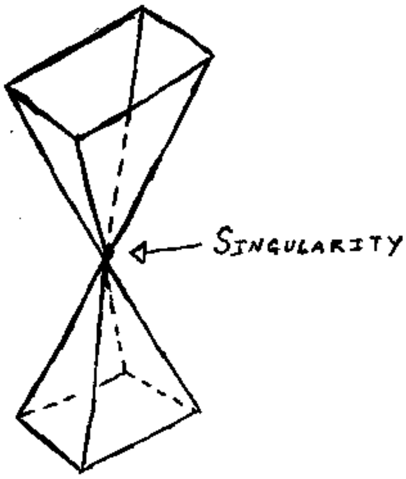

Figure 5: Singularity

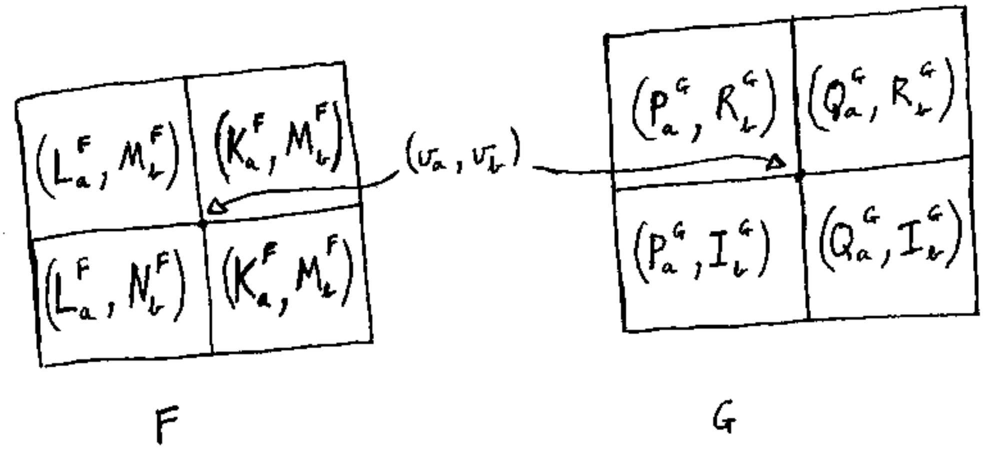

Figure 6: Cell Structure

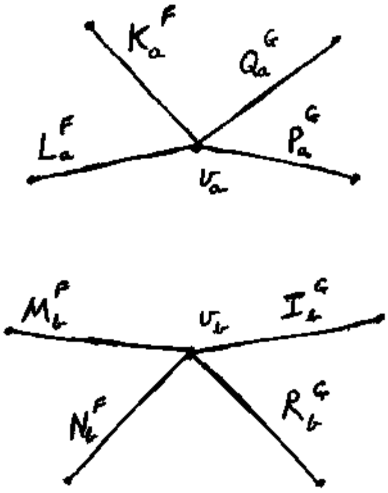

Figure 7: Star Component Edges

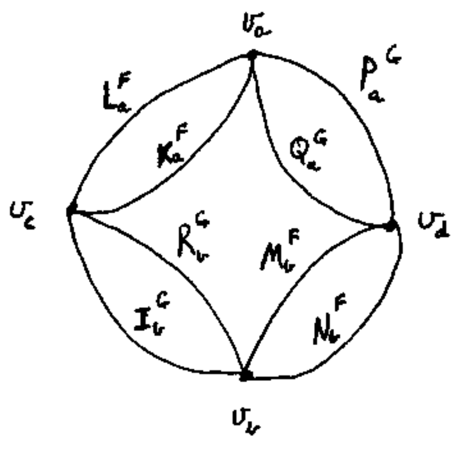

Figure 8: Graph Structure

The the presence of a singularity in $D^2(\Gamma)$ dictates the form our graph will take locally about the singularity.

The Singularity Theorem is now formally presented, it tells us what a pseudomanifold type graph Γ must look like if $D^2(\Gamma)$ has a singularity.

Theorem 6.1. (Singularity Theorem) *Given that $D^2(\Gamma)$ is a pseudomanifold, with a singularity at (v_a, v_b). Let S be the star of (v_a, v_v). Consider two distinct star components of S, call them F and G. Then there exist two vertices v_c and v_d in Γ that have the following properties,*

1. *Every 2-cell in F is of the form $({E_a}^F, {E_b}^F)$, where every edge ${E_a}^F$ has vertices v_a and v_c, and every edge of the form ${E_b}^F$ has vertices v_b and v_d.*

2. *Every 2-cell in G is of the form $({E_a}^G, {E_b}^G)$, where every edge of the form ${E_a}^G$ has vertices v_a and v_d, and every edge of the form ${E_b}^G$ has vertices v_b and v_c.*

Proof. Given that (v_a, v_b) is a singularity in $D^2(\Gamma)$. Let S be the star of (v_a, v_b). Let F and G be two distinct star components of S. All the 2-cells in F and G, intersect the 0-cell (v_a, v_b). Thus, all the edges in Γ, whose ordered pairs form the 2-cells in F and G, have either v_a or v_b as a vertex. Thus, all the 2-cells in F are of the form $({E_a}^F, {E_b}^F)$, where every edge ${E_a}^F$ has v_a as a vertex and every edge ${E_b}^F$ has v_b as a vertex, and all the 2-cells in G are of the form $({E_a}^G, {E_b}^G)$, where every edge ${E_a}^G$ has v_a as a vertex and every edge ${E_b}^G$ has v_b as a vertex.

Thus we can separate all the edges that form 2-cells in F into two sets A_F and B_F as follows, let A_F and B_F be the sets of edges with vertices v_a and v_b respectively, whose ordered pairs form the 2-cells in F. Thus, for all edges ${L_a}^F$ in A_F we can find two edges ${M_b}^F$ and ${N_b}^F$ in B_F such that $d({L_a}^F, {M_b}^F) \geq 1$ and $d({L_a}^F, {N_b}^F) \geq 1$. And for every edge ${R_b}^F$ in B_F we can find two edges ${S_a}^F$ and ${T_a}^F$ in A_F such that $d({R_b}^F, {S_a}^F) \geq 1$ and $d({R_b}^F, {T_a}^F) \geq 1$. This is all because the edges in A_F and B_F are all the edges that form the 2-cells in F.

Thus we can separate all the edges that form 2-cells in G into two sets A_G and B_G as follows, let A_G and B_G be the sets of edges with vertices v_a and v_b respectively, whose ordered pairs form the 2-cells in G. Thus, for all edges ${I_a}^G$ in A_G we can find two edges ${J_b}^G$ and ${K_b}^G$ in B_G such that $d({I_a}^G, {J_b}^G) \geq 1$ and $d({I_a}^G, {K_b}^G) \geq 1$. And for every edge ${X_b}^G$ in B_G we can find two edges ${Y_a}^G$ and ${Z_a}^G$ in A_G such that $d({X_b}^G, {Y_a}^G) \geq 1$ and $d({X_b}^G, {Z_a}^G) \geq 1$. This is all because the edges in A_G and B_G are all the edges that form the 2-cells in G.

We assert that, $\forall$ edges of the form ${E_a}^F \in A_F$, $\forall$ edge ${E_b}^G \in B_G$, we have $d({E_a}^F, {E_b}^G) = 0$.Let if possible that our assertion is false. $\therefore$ $\exists$ an edge ${P_a}^F \in A_F$ and an edge ${Q_b}^G \in B_G$, such that $d({P_a}^F, {Q_b}^G) \geq 1$. $\therefore$ $\exists$ a 2-cell $({P_a}^F, {Q_b}^G)$, which has the 0-cell (v_a, v_b) as a vertex. Now ${Q_b}^G \in B_G$, $\therefore$ $\exists$ two edges ${R_a}^G, {S_a}^G \in A_G$ such that $d({Q_b}^G, {R_a}^G) \geq 1$ and $d({Q_b}^G, {S_a}^G) \geq 1$. So we have $d({Q_b}^G, v_a) \geq 1$ and three edges ${P_a}^F, {R_a}^G$ and ${S_a}^G$, which share vertex v_a, and $d({Q_b}^G, {P_a}^F) \geq 1, d({Q_b}^G, {R_a}^G) \geq 1$ and $d({Q_b}^G, {S_a}^G) \geq 1$. $\therefore$ By the Pseudomanifold Criterion 5.1, $D^2(\Gamma)$ is not a pseudomanifold. This is a contradiction. $\therefore$ our assumption that $\exists$ an edge ${P_a}^F \in A_F$ and an edge ${Q_b}^G \in B_G$, such that $d(P, Q) \geq 1$, is false. In fact, no such edges exist.

$\therefore$ $\forall$ edges ${E_a}^F \in A_F$, $\forall$ edges ${E_b}^G \in B_G$, we have $d({E_a}^F, {E_b}^G) = 0$. This implies that $\exists$ a vertex v_c in Γ such that $\forall$ edges ${E_a}^F \in A_F$, $\forall$ edges ${E_b}^G \in B_G$, ${E_a}^F$ and ${E_b}^G$ share the vertex v_c.

Therefore all edges of the form ${E_a}^F$ have vertices v_b and v_c and all edges of the form ${E_b}^G$ have vertices v_c and v_b.

A similar argument tells us that there is a vertex v_d in Γ such that all the edges of the form ${E_a}^G$ have vertices v_a and v_d, and all the edges of the form ${E_b}^F$ have vertices v_d and v_b. □

Corollary 6.2. *Given a graph Γ, such that $D^2(\Gamma)$ is a 2-pseudomanifold. If Γ is simple, then $D^2(\Gamma)$ is a 2-manifold.*

Proof. If $D^2(\Gamma)$ has a singularity, then by the Singularity Theorem 6.1 there are at least two edges that share the same vertices, and so Γ must be non simple. Therefore, the contrapositive of the previous statement is true, i.e. if Γ is simple, then $D^2(\Gamma)$ has no singularities. If $D^2(\Gamma)$ has no singularities, then $D^2(\Gamma)$ is a 2-manifold. Therefore, if $D^2(\Gamma)$ is a 2-pseudomanifold and Γ is a simple graph, then $D^2(\Gamma)$ is a 2-manifold. □

What this tells us, is that all pseudomanifold graphs that are simple graphs are actually manifold graphs.

7 Structure Results

Our understanding of pseudomanifold graphs is very local. All we know about them is encapsulated in the Pseudomanifold Criterion; given an edge and a separate vertex; the number of edges that may meet in that vertex, and how they are positioned in the graph with respect to the edge. This does not tell us what the graph looks like globally.

However, we can attempt to construct a pseudomanifold graph. Since our graph cannot consist of one edge alone, we can put down an edge E and a vertex v that is separate from it. Then we add two edges F, G so that they both share vertex v, and $d(E,F) \geq 1, d(E,G) \geq 1$. Note that these two edges are required by the Pseudomanifold Criterion. We then re assess each of the vertices in relation to the newly added edges of the graph. The process soon gets out of hand, and you will be very lucky indeed to successfully construct a pseudomanifold graph in this manner. However, you will intuitively understand why there must be a limit to the size of these graphs. You will notice that as you add more edges to the graph, the further the newly added edge from the original seed edge and vertex, the greater the probability that your newly added edge violates the rules laid down by the Pseudomanifold Criterion, in relation to some vertex of the graph. And you will notice that the vertices in relation to which the newly added edges violate the Pseudomanifold Criterion, are generally far away from the newly added edges.

We now make more concrete the notions that have been intuited out above.

The first step is the Halo Lemma, which basically tells us that given an edge in a pseudomanifold graph, all the vertices in the graph form a halo around this edge, so that none of them are greater than an edge's distance from it. It tells us that pseudomanifold graphs are quite compact and cannot be very spread out creatures.

Lemma 7.1. (Halo Lemma) *If Γ is a graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then $\forall$ edge E, $\forall$ vertex v, we must have $0 \leq d(E,v) \leq 1$.*

Proof. Let E be an arbitrary edge of Γ let v be an arbitrary vertex. $D^2(\Gamma)$ is a 2-pseudomanifold. $\therefore$ $D^2(\Gamma)$ is a 2-complex without boundary. $\therefore$ By Corollary 4.3 valence of $v \geq 3$. If v is a vertex of E, then by definition $d(E, v) = 0$. Now, if v is not a vertex of E, let if possible, that $d(E, v) \geq 2$. Let F, G and H be three edges such that $F \bigcap G \bigcap H = v$, i.e. F, G and H shaver vertex v. We must have $d(E, F) \geq 1$, $d(E, G) \geq 1$ and $d(E, H) \geq 1$, for if this were not true then by definition we would have $d(E, v) = 1$ in contradiction to our assumption. So (E, F), (E, G) and (E, H) are 2-cells in $D^2(\Gamma)$, and they all intersect in a 0-cell (E, v). This contradicts the fact that $D^2(\Gamma)$ is a pseudomanifold. $\therefore$ Our assumption that $d(E, v) \geq 2$ is false. So we must have $0 \leq d(E, v) \leq 1$. □

It is easy to see that the Halo Lemma translates to there being a maximum diameter for a pseudomanifold graph. As it turns out, the shortest path between any two vertices cannot exceed a distance of two edges.

Proposition 7.2. (Maximum Diameter Law)*If Γ is a graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then $\forall$ pair of vertices v, $f \in \Gamma$, we must have $0 \leq d(v, f) \leq 2$.*

Proof. Let v, f be arbitrary vertices in Γ. If $v = f$ then $d(v, f) = 0$. If $v \neq f$ then let if possible that $d(v, f) \geq 3$. $\therefore$ By definition, the shortest path between v and f in Γ spans a minimum of 3 edges. Let E be an edge on the shortest path from v to f that has v as a vertex. Let r be the second vertex of E. Let F be the edge along the shortest path from v to f, that has r as a vertex. Let s be the second vertex of F. We have $d(v, s) = 2$, $d(s, f) \geq 1$ and $d(r, s) = 1$. We have $d(E, f) = \min d(v, f), d(r, f) = 2$. This is a contradiction to the Halo Lemma 7.1. $\therefore$ Our assumption that $d(v, f) \geq 3$ is false. $\therefore$ We must have $0 \leq d(v, f) \leq 2$. □

The next Corollary gives us more of an idea of how the pseudomanifold graphs look. None of the edges can be separated by a shortest path of more than two edges distance.

Corollary 7.3. *If Γ is graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then $\forall$ pairs of edges E, F in Γ, we must have* $0 \leq d(E, F) \leq 1$.

Proof. This is a consequence of the Maximum Diameter Law 7.2. □

Now while the above results have told us about how compact our graphs need to be, how the various edges and vertices fit together is still opaque. We are interested in producing examples of pseudomanifold graphs, and the fact that they are compact does not help us much, as there are an enormous number of ways that we can connect any number of vertices to each other, even if they cannot stray too far from each other. However, the Valence Restriction Theorem tells us that each vertex in a pseudomanifold graph that is simple, must have a valence that is either 3 or 4. This puts a tight limit on the number of ways that we can connect our vertices to each other.

Theorem 7.4. (Valence Restriction Theorem for Simple Graphs)
If Γ is a simple graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then $\forall$ vertex v in Γ, $3 \leq$ valence $v \leq 4$.

Proof. $D^2(\Gamma)$ is without boundary, $\therefore$ by Corollary 4.3, valence $v \geq 3$. Let v be an arbitrary vertex of Γ. Let if possible that valence $v \geq 5$. Let E be any edge that has v as a vertex. Let r be the other vertex of E. As $D^2(\Gamma)$ is a 2-pseudomanifold, $\exists$ an edge F, such that $d(E, F) \geq 1$. By the Pseudomanifold Theorem 5.1, $\exists$ an edge G, with v as a vertex, such that $d(F, G) \geq 1$. Let H, I and J be the three other edges that have v as a vertex. Let v_{F1} and v_{F2} be the vertices of F. If H, I or J have $r \neq v_{F1}, v_{F2}$ as a vertex, then $\exists$ an edge $K = H, I$ or J, such that $d(F, K) \geq 1$. So we would have three 2-cells (E, F). (G, F) and (K, F) that intersect in the 1-cell (v, F), in contradiction to the fact that $D^2(\Gamma)$ is a pseudomanifold. Thus we must conclude that the edges H, I and F must have v as one vertex, and either v_{F1} or v_{F2} as their other vertex. By the pigeonhole principle, two edges from the three(H, I and J), must share the same two vertices, namely, v and v_{F1} or v and v_{F2}. This is in contradiction to the fact that Γ is a simple graph. $\therefore$ Our assumption that valence $v \geq 5$ is false. $\therefore$ We must conclude that $3 \leq$ valence $v \leq 4$. □

Things are a little more complicated when we are dealing with non-simple pseudomanifold graphs, because it is possible to connect more edges between vertices without violating the Pseudomanifold Criterion.

However, the Muscle Lemma tells us that we can only have a maximum of two edges that share the same vertices. So it turns out, that non-simple pseudomanifold graphs cannot have many more edges than their simple counterparts.

Lemma 7.5. (Muscle Lemma) *If Γ is a graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then there can be no more than 2 edges that share the same pair of vertices.*

Proof. Let if possible that u and v be vertices in Γ with at least three edges sharing them as vertices. Let E, F and G be three of these edges. As $D^2(\Gamma)$ is a 2-pseudomanifold, $\exists$ an edge H, such that $d(E, H) \geq 1$. $\therefore$ We must also have $d(F, H) \geq 1$ and $d(G, H) \geq 1$. $\therefore \exists$ three 2-cells (E, H), (F, H) and (G, H) that intersect in the same 1-cell (v, H). This contradicts the fact that $D^2(\Gamma)$ is a pseudomanifold. $\therefore$ Our assumption that $\exists$ two vertices u and v with at least 3 edges sharing them as vertices, is false. So we must conclude that no more than 2 edges may share the same two vertices in Γ. □

The Valence Restriction Theorem for non-simple pseudomanifold graphs puts a limit on the number of ways that we can connect the various vertices to each other. At first, it is a bit alarming to see that we can have as many as 6 edges at a vertex in one of these graphs. But, as we shall see in the next section, the number of ways in which these 6 edges can connect up with other vertices is very limited, and it in no way implies that a vertex may have 6 neighbors.

Theorem 7.6. (Valence Restriction Theorem for Non-Simple Graphs) *If Γ is a non-simple graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then $\forall$ vertex v in Γ, we must have $3 \leq$ valence $v \leq 6$.*

Proof. Let if possible that $\exists$ a vertex v of Γ with valence ≥ 7. Let E be an edge in Γ such that $d(E, v) \geq 1$. Let v_{E1} and v_{E2} be the vertices of E. As $D^2(\Gamma)$ is a pseudomanifold, $\exists$ two edges F and G that share v as a vertex such that $d(E, F) \geq 1$ and $d(E, G) \geq 1$. Valence $v = 7$, $\therefore \exists$ at least five more edges H, I, J, K and L that share v as a vertex. Let X = H, I, J, K or L, be an edge that has v and r $\neq v_{E1}, v_{E2}$ as its vertices. $\therefore d(E, X) \geq 1$, $d(E, F) \geq 1$ and $d(E, G) \geq 1$. This implies that there are three 2-cells (E, X), (E, F) and (E, G) that intersect in the 1-cell (E, v). This contradicts the fact that $D^2(\Gamma)$ is a pseudomanifold. $\therefore$ Edges H, I, J, K and L must have v_{E1} or v_{E2} as their second vertex. There are 5 edges with 2 choices as their second vertex. $\therefore$ By the pigeonhole principle, three of these edges have either v_{E1} or v_{E2} as their second vertex. $\therefore$ Three edges share the same pair of vertices, v and v_{E1}, or v and v_{E2}. This contradicts Lemma 7.5. $\therefore$ Our assumption that valence $v \geq 7$ is false. $\therefore$ We must conclude that $3 \leq$ valence $v \leq 6$. □

8 Pseudomanifold Type Graphs

The subtle difference between the valence of a vertex and the number of neighbors of a vertex comes into play only in non-simple graphs. In simple graphs, the terms are synonymous. However, in non-simple graphs, a vertex can have a valence that far exceeds the number of its neighbors. Of course, in non-simple pseudomanifold graphs, in light of the Muscle Lemma, the valence of a vertex can be at most twice the number of its neighbors.

As it turns out, the maximum number of neighbors that a vertex may have, in a pseudomanifold graph, be it simple or non-simple, is 4. Since valence and number of neighbors are synonymous for simple graphs, an appeal to the Valence Restriction Theorem for Simple Graphs takes care of that case, and we only need to work toward proving it true for non-simple graphs.

Proposition 8.1. (Maximum Neighbour Law) *If Γ is a graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then $\forall$ vertex v in Γ, v can have at most 4 vertices adjacent to it.*

Proof. Let if possible that $\exists$ a vertex v in Γ, such that there are at least 5 distinct vertices v_1, v_2, v_3, v_4 and v_5 adjacent to it.

$D^2(\Gamma)$ is a 2-pseudomanifold, $\therefore \exists$ an edge E such that $d(E, v) \geq 1$. Let the vertices of E be v_{E1} and v_{E2}. There are two cases, one in which both v_{E1} and v_{E2} are adjacent to v, and one in which only one of v_{E1} or v_{E2} is adjacent to v.

Case 1, in which v_{E1} and v_{E2} are adjacent to v. $\therefore$ Two of the vertices v_1, v_2, v_3, v_4 or v_5 are the vertices v_{E1} and v_{E2}. Without loss of generality, let $v_1 = v_{E1}$ and $v_2 = v_{E2}$. $\therefore$ Vertices $d(E, v_3) \geq 1, d(E, v_4) \geq 1$ and $d(E, v_5) \geq 1$, this contradicts the Pseudomanifold Criterion 5.1.

Case 2, in which only one of v_{E1} or v_{E2} is adjacent to v. Without loss of generality, let v_{E1} be adjacent to v. $\therefore$ Exactly one of the vertices v_1, v_2, v_3, v_4 or v_5, is the vertex v_{E1}. Without loss of generality, let $v_1 = v_{E1}$. The vertex v_{E2} is not adjacent to v, $\therefore v_{E2} \neq v_2, v_3, v_4$ or v_5. $\therefore d(v_{E2}, v_2) \geq 1, d(v_{E2}, v_3) \geq 1, d(v_{E2}, v_4) \geq 1$ and $d(v_{E2}, v_5) \geq 1$. Also $d(v_1, v_2) \geq 1, d(v_1, v_3) \geq 1, d(v_1, v_4) \geq 1$ and $d(v_1, v_5) \geq 1$, and $v_1 = v_{E1}$. $\therefore d(E, v_2) \geq 1, d(E, v_3) \geq 1, d(E, v_4) \geq 1$ and $d(E, v_5) \geq 1$. This contradicts the Pseudomanifold Theorem 5.1.

$\therefore$ From Case 1 and Case 2, we see that our assumption that $\exists$ a vertex v in Γ, such that v has at least 5 distinct vertices adjacent to it, is false.

$\therefore$ We must conclude that for a vertex v in Γ, v can have at most 4 distinct vertices adjacent to it. □

If every vertex in a pseudomanifold graph can have at most 4 neighbours, and if its maximum diameter is a shortest path of 2 edges, then there is an upper limit to the number of vertices that it can have. Too many vertices, and you end up violating either the Pseudomanifold Criterion, the limit on the diameter or the maximum number of neighbours.

We can do a very rough "linear" calculation of the maximum number of vertices in a pseudomanifold graph, and get an upper limit of 17. What we do is make a tree, in which the height of the tree obeys the Maximum Diameter Law, and each node of the tree obeys the Maximum Neighbours law. Many thanks to Dr. Parry for showing me this way of estimating the upper limit on the number of vertices in pseudomanifold graphs.

I state the result in two parts. In the first part, I calculate the upper limit on the maximum number of vertices that we can have in a graph in which each vertex can have a maximum of M neighbours, and every pair of vertices have a shortest path between them consisting of a maximum of k edges.

Proposition 8.2. (Graph Limit) *Given a graph Γ, such that $\forall$ vertex v in Γ, the maximum number of vertices adjacent to v is $M(M \geq 1)$, and $\forall$ vertices u, v in Γ, $d(u,v) \leq k(k \geq 2)$. If N is the number of vertices in Γ, then*
$N \leq (1+M) + \sum_{i=1}^{k-1} M(M-1)^i$.

Proof. Let v be a vertex in Γ, it has at most M adjacent vertices. Call these vertices the 0'th generation. Consider the 0'th generation. Each of them have at most (M-1) adjacent vertices distinct from v. Call this the 1'st generation. $\forall$ Vertex u, in the i'th generation$(i \geq 1)$, u can have at most (M-1) adjacent vertices that are distinct from a vertex in the (i-1)'th generation. The highest generation that we can have is the (k-1)'th generation, for if we had a k'th generation, then $d(v,u) > k$ for a vertex u from the k'th generation. $\therefore (1+M) + \sum_{i=1}^{k-1} M(M-1)^i$ is an upper bound on the number of vertices in Γ. $\therefore$ If N is the number of vertices in Γ, we must have $N \leq (1+M) + \sum_{i=1}^{k-1} M(M-1)^i$. □

In pseudomanifold graphs, each vertex can have a maximum of 4 neighbours, and every pair of vertices can have a shortest path between them of a maximum of 2 edges. So we have $M = 4, k = 2$.

Corollary 8.3. *If Γ is a graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then the number of vertices in Γ cannot exceed 17.*

Proof. Apply the Graph Limit 8.2 with $M = 4$ and $k = 2$. □

Throughout this section, we have been working toward reducing the possible range of pseudomanifold graphs. The following result, the Four Lemma, is the clincher of this section. It tells us that if we have a vertex with 4 neighbours in our pseudomanifold graph, then the graph must consist of exactly 5 vertices.

It is important to note that for a vertex v, the statement "v has n neighbours in Γ" translates to "v has valence n in Γ" if Γ is simple. This is not always true if Γ is non-simple.

The reason that we wrote the following lemma in terms of "number of neighbours of a vertex", is because it allows us to apply this lemma in the setting of non-simple graphs.

Lemma 8.4. (Four Lemma) *Given a graph Γ such that $D^2(\Gamma)$ is a 2-pseudomanifold, then if $\exists$ a vertex $v \in \Gamma$, such that v is adjacent to 4 vertices, then $\forall$ vertex $u \in \Gamma$, $0 \leq d(v,u) \leq 1$ and Γ has exactly 5 vertices, namely v, v_1, v_2, v_3 and v_4.*

Proof. Let v be a vertex in Γ, that is adjacent to 4 vertices, namely v_1, v_2, v_3 and v_4. $\therefore \exists$ edges E_1, E_2, E_3 and E_4 that meet in vertex v and have v_1, v_2, v_3 and v_4 as their vertices. Let if possible that $\exists$ vertex w $\in \Gamma$, such that $d(v, w) = 2$. Clearly, $w \neq v_1, v_2, v_3, v_4$. Let F be an edge in Γ that has w as a vertex. By Lemma 7.1, $d(F, v) \leq 1$. Now $d(F, v) \neq 0$, for this would imply that $d(v, w) = 1$. $\therefore d(F, v) = 1$. $\therefore$ the other vertex of F has to be v_1, v_2, v_3 or v_4, as we have $d(v, w) = 2$. Without loss of generality, let F have w and v_1 as vertices. So we must have $d(F, E_2) = 1, d(F, E_3) = 1$ and $d(F, E_4) = 1$, and E_2, E_3 and E_4 meet in v, with $d(F, v) = 1$. However $D^2(\Gamma)$ is a 2-pseudomanifold, so there is a contradiction with the Pseudomanifold Criterion 5.1. So no such vertex w can exist. So $\forall$ vertex $u \in \Gamma$, $0 \leq d(v, u) \leq 1$.

Let if possible that $\exists$ a vertex y $\in \Gamma$, that is distinct from the vertices v, v_1, v_2, v_3 and v_4. $\therefore d(v, y) = 1$, and so v is adjacent to 5 vertices. This contradicts the Maximum Neighbour Law 8.1, as $D^2(\Gamma)$ is a 2-pseudomanifold. So no such vertex y can exist, and the vertices v, v_1, v_2, v_3 and v_4 are exactly the vertices of Γ. □

The Four Lemma is very helpful in allowing us to reduce the number of possibilities that we need to consider when searching for pseudomanifold graphs.

9 The Special Graphs

As you may have guessed, there are not many graphs; simple or non-simple; that are pseudomanifold graphs. We work out exactly which graphs they are in this section. However, due to the subtle difficulties that non-simple pseudomanifold graphs present us, we have treated simple and non-simple pseudomanifold graphs separately, in different sub-sections.

9.1 Simple Graphs

We now recover two results proved by Abrams in [2], that the only simple graphs for which the discretized spaces generated by two robots are 2-manifolds, are K_5 and $K_{3,3}$. By the Singularity Theorem 6.1 we see that $D^2(K_5)$ and $D^2(K_{3,3})$ have to be 2-pseudomanifolds without singularities, and hence 2-manifolds. In [1] Abrams and Ghrist conclude that the 2-manifolds thus obtained are orientable, and by calculating their Euler Characteristic from the number of 0, 1 and 2-cells they conclude that the genus of $D^2(K_5)$ is 6 and that of $D^2(K_{3,3})$ is 4. Thus $D^2(K_5)$ and $D^2(K_{3,3})$ are homeomorphic to the 6-holed torus and 4-holed torus respectively.

We use a good idea suggested by Dr. Peter Haskell, in which we divide the set of simple pseudomanifold graphs into two subsets, one set consisting of all graphs that contain at least one vertex of valence 4, and the other set containing no vertices of valence 4. As valence can only take values 3 or 4 in simple pseudomanifold graphs, this amounts to our two subsets consisting of the set of simple pseudomanifold graphs that have at least one vertex of valence 4, and the set of simple pseudomanifold graphs that has all vertices with valence 3.

This idea makes short work of narrowing down the search for which graphs exactly are pseudomanifold graphs.

Theorem 9.1. **(K_5 Theorem)** *Given a simple graph Γ, such that $D^2(\Gamma)$ is a 2-pseudomanifold, then if $\exists$ a vertex $v \in \Gamma$, such that valence v is 4, then Γ is the graph K_5.*

Proof. Let v_1 have valence 4 in Γ. Since Γ is a simple graph, this tells us that v_1 is adjacent to 4 vertices, let us call them v_2, v_3, v_4 and v_5. By Lemma 8.4, Γ has exactly 5 vertices, namely v_1, v_2, v_3, v_4 and v_5.

We claim that each of the vertices has valence 4. Suppose this were not true, then $\exists$ a vertex that has valence 3, as the vertices can have valence either 3 or 4. Without loss of generality, let v_2 have valence 3, so this tells us that v_2 is adjacent to exactly 3 vertices in Γ, as this is a simple graph. As v_1 has valence 4, v_2 must be adjacent to v_1, by Lemma 8.4. Let F_1 be the edge that has v_1 and v_2 as vertices. Without loss of generality, let v_3 be another vertex that is adjacent to v_2. Let F_2 be the edge that has v_2 and v_3 as vertices. Let E be the edge that has v_1 and v_3 as vertices. Now v_2 has valence 3, so let us call the third edge that has it as a vertex F_3. Note that F_3 cannot have v_1 or v_3 as its other vertex as Γ is a simple graph. So we have $d(E, v_2) = 1$, and only one edge F_1 with v_2 as a vertex and $d(E, F_1) = 1$. As $D^2(\Gamma)$ is a 2-pseudomanifold with boundary, this contradicts the Pseudomanifold Criterion 5.1. So no such vertex of valence 3 exists in Γ. $\therefore$ every vertex in Γ has valence 4. As Γ is simple and has exactly 5 vertices, this implies, that every vertex is adjacent to every other vertex. Again, since this is a simple graph, this tells us that Γ is the graph K_5. □

Theorem 9.2. **($K_{3,3}$ Theorem)** *Given a simple graph Γ, such that $D^2(\Gamma)$ is a 2-pseudomanifold, if every vertex in Γ has valence 3, then Γ is the graph $K_{3,3}$.*

Proof. Let v_1 be a vertex in Γ. Since it has valence 3, and Γ is a simple graph, it must be adjacent to exactly 3 vertices, call them v_2, v_3 and v_4. Let A_1, A_2 and A_3 be the edges that connect vertices v_1, v_2 and v_1, v_3 and v_1, v_4 respectively.

If Γ has exactly 4 vertices, v_1, v_2, v_3 and v_4, then each vertex is adjacent to every other vertex as Γ is simple, and they all have valence 3. Let E be the edge that has v_1 and v_2 as vertices. Consider vertex v_3. It has valence 3, and is adjacent to vertices v_1 and v_2. Let F_1 and F_2 be the edges that connect vertices v_3, v_1 and v_3, v_2 respectively. Let F_3 be the third edge that has v_3 as a vertex. So we have $d(E, v_3) = 1$ and $\exists$ exactly 1 edge F_1 with v_3 as a vertex, such that $d(E, F_1) = 1$. As $D^2(\Gamma)$ is a 2-pseudomanifold, this contradicts the Pseudomanifold Criterion 5.1. So Γ cannot have exactly 4 vertices.

Γ cannot have exactly 5 vertices as the sum of the valences of all the vertices would be 15, which is an odd number. This contradicts a basic result from graph theory that tells us that this number needs to be even.

Let Γ have ≥ 6 vertices. Let v_5 and v_6 be any two vertices that are both distinct from v_1, v_2, v_3, v_4. So we must have $d(v_1, v_5) > 1$ and $d(v_1, v_6) > 1$, as v_1 has valence 3 and is adjacent to vertices v_2, v_3 and v_4. $\therefore$ there can be no edge F, that has v_5 and v_6 as vertices. As if this were true, it would imply that $d(F, v_1) > 1$, which contradicts Lemma 7.1 as $D^2(\Gamma)$ is a 2-pseudomanifold.

If Γ has ≥ 6 vertices, then for any vertex $u \in \Gamma$, such that u is distinct from v_1, v_2, v_3 and v_4, u cannot be adjacent to v_1 as its valence is 3 and it is adjacent to v_2, v_3 and v_4, and u cannot be adjacent to any vertex other than v_1, v_2 and v_3, as we have just proved above. So u must be adjacent to the vertices v_1, v_2 and v_3, as u has valence 3 and Γ is a simple graph.

Γ must have exactly 6 vertices. For if this were not true, then let x, y and z be three vertices that are distinct from v_1, v_2, v_3 and v_4. So as proved above, x, y and z must each be adjacent to all three of the vertices v_2, v_3 and v_4. So the vertex v_2 would be adjacent to the vertices v_1, x, y, z, which contradicts the fact that v_2 has valence 3.

Let us give the names v_5 and v_6 to the two vertices that are distinct from v_1, v_2, v_3 and v_4. The vertex v_1 is adjacent to v_2, v_3, v_4, as proved above, the vertex v_5 is adjacent to v_2, v_3, v_4 and the vertex v_6 is adjacent to v_2, v_3, v_4. This tells us that Γ is the graph $K_{3,3}$. □

9.2 Non-Simple Graphs

The work in this section is original and extends the work done by Abrams and Ghrist to non-simple graphs. In particular, we show that certain non-simple graphs yield discretized spaces that are 2-pseudomanifolds and that there are only two such non-simple graphs, K_4 with each edge doubled and the 4-Cycle Graph with each edge doubled.

The non-simple pseudomanifold type graphs are a bit trickier to handle than the simple ones. The first problem that one encounters, is the relatively large range within which the valence of a vertex in these graphs can fall. Valences can potentially range from 3 to 6. One expects that due to this, the potential number of vertices in our graph will skyrocket. However, this is not the case, as The Maximum Neighbour Law puts an upper bound on the number of neighbours that a vertex in the graph can have. Thus, rather than increasing the number of pseudomanifold type graphs through an increase in the number of vertices, the increase in range of the valence translates to an increase in the potential number of connections between vertices via edges, and the various permutations of these connections. Even here, the Muscle Lemma comes to our rescue and severely limits the number of edges that we can have between vertices.

All the tools that we have developed thus far, probe the internal edge structure of the graph by dealing directly or indirectly with the valence. Clearly this approach will not work with non-simple graphs. However, we can think of all the possible non-simple pseudomanifold type graphs as variations of various basic “seed graphs”. By this I mean that the various non-simple graphs are obtained by “fattening up” certain basic simple graphs.

The “basic simple graphs” that I mentioned above, are obtained by taking a non-simple pseudomanifold type graph and re-drawing it with a single edge connecting two vertices if and only if they were adjacent. This is obviously a simple graph, and we call it the Collapse of the original non-simple graph and denote it by Γ_C. We define it formally below.

Definition 9.1. Given a graph Γ, we define the collapse of Γ to be the graph Γ_C, with the same vertices as Γ, and exactly one edge connecting two vertices if and only if they are adjacent in Γ.

The key to understanding non-simple pseudomanifold type graphs, is understanding their Collapse's. Toward this goal, we define below, relative valence, something which we have only been intuitively aware of until this point. The relative valence of a vertex v with respect to an edge E, is the number of edges with v as a vertex, that do not come in contact with E. The formal definition follows.

Definition 9.2. The relative valence if a vertex v; with respect to an edge E such that $d(E, v) \geq 1$; denotes the number k, of edges F_i $(i = 1, 2, ..., k)$ that have v as a vertex such that $d(E, F_i) \geq 1$. We denote the relative valence of vertex v with respect to edge E by $R(v, E)$. And we call the edges F_i such that $d(E, F_i) \geq 1$, the free edges at v with respect to E.

Note that in a graph Γ such that $D^2(\Gamma)$ is a 2-pseudomanifold, the relative valence of every vertex v, with respect to every edge E that does not have v as a vertex, is exactly equal to 2. Furthermore, given a vertex v and an edge E that does not have v as a vertex, a relative valence of 1 for v with respect to E, implies that there is a 2-cell in $D^2(\Gamma)$, at least one of whose faces lies in the closure of only one 2-cell, and so forms a boundary.

What the Collapse Structure Theorem tells us, is that the discretized space of the collapse of a non-simple pseudomanifold type graph, is basically a 2-pseudomanifold with boundary, i.e. the discretized space contains some points whose neighbourhoods are homeomorphic to the half plane.

Theorem 9.3. (Collapse Structure Theorem) *Given a non-simple graph* Γ *such that* $D^2(\Gamma)$ *is a 2-pseudomanifold, then* $D^2(\Gamma_C)$ *is not a 2-pseudomanifold and* ∀ *vertices* v*,* ∀ *edges* E *in* Γ_C *with* $d(E, v) \geq 1$ *we must have* $1 \leq R(v, E) \leq 2$.

Proof. ∃ two vertices a, b and edges A and b in Γ, that share vertices a and b, as Γ is a non-simple graph. ∃ an edge E, with vertices e, f, in Γ such that $d(E, A) \geq 1$, as $D^2(\Gamma)$ is a 2-pseudomanifold. $\therefore d(E, B) \geq 1$ and $d(E, a) \geq 1$. We must have $R(a, E) = 2$ in Γ. ∴ edges A and b are the only two edges in Γ such that $d(E, A) \geq 1$ and $d(E, B) \geq 1$. $\therefore d(e, a) \geq 1, d(e, b) \geq 1, d(f, a) \geq 1, d(f, b) \geq 1$.

Let if possible that $D^2(\Gamma_C)$ be a 2-pseudomanifold. Let I be the unique edge with vertices e, f in Γ_C and let J be the unique edge with vertices a, b in Γ_C. $\therefore d(I, a) \geq 1$ and $d(I, J) \geq 1$. Now $D^2(\Gamma_C)$ is a 2-pseudomanifold, ∴ ∃ an edge K in Γ_C such that K has a as a vertex and $d(I, K) \geq 1$ by the Pseudomanifold Criterion 5.1. As Γ_C must be a simple graph, the other vertex v, of K must be distinct from a and b. It is obvious that v must be distinct from e and f as we would have $d(E, K) = 0$ if this were not true. ∴ vertices a and v are adjacent in Γ. ∴ ∃ an edge F in Γ with vertices a and v. Now v is distinct from e and f, so $d(e, v) \geq 1, d(f, v) \geq 1, d(e, a) \geq 1, d(f, a) \geq 1$, so we must have $d(E, F) \geq 1$. ∴ we have three edges, a, b and F such that $d(E, A) \geq 1, d(E, B) \geq 1, d(E, F) \geq 1$. However $D^2(\Gamma)$ is a 2-pseudomanifold and this contradicts the Pseudomanifold Criterion 5.1. So our assumption that $D^2(\Gamma_C)$ is a 2-pseudomanifold is false.

Let if possible that ∃ a vertex v and an edge E in Γ_C, with $d(E, v) \geq 1$, such that $R(v, E) = 0$. If e and f are the vertices of E, then this implies that for any edge X in Γ_C with vertex v, the other vertex of X must be identical to either e or f. ∴ v is only adjacent to vertices e or f. ∴ ∀ edges Y in Γ, with vertex v, the other vertex of Y must be e or f. ∃ an edge G with vertices e and f in Γ, as Γ_C tells us that e and f are adjacent in Γ. $\therefore R(v, G) = 0$. This contradicts the fact that $D^2(\Gamma)$ is a 2-pseudomanifold by the Pseudomanifold Criterion 5.1. So our assumption that ∃ a vertex v and an edge E in Γ_C, with $d(E, v) \geq 1$, such that $R(v, E) = 0$ is false. So ∀ vertices v and edges E in Γ_C, with $d(E, v) \geq 1$, we must have $1 \leq R(v, E)$.

Let if possible that ∃ a vertex v and an edge E in Γ_C, with $d(E, v) \geq 1$, such that $R(v, E) = 3$. ∴ ∃ edges a, b and C sharing vertex v in Γ_C. Let

a, b and c be the vertices of a, b and C respectively, distinct from v. Let e and f be the vertices of E. $\therefore$ vertices v, a and v, b and v, c are adjacent in Γ. $\therefore$ $\exists$ edges L, M and N in Γ such that they share vertex v, and have vertices a, b and c respectively as their other vertex. So we have $d(e,a) \geq 1, d(e,b) \geq 1, d(e,c) \geq 1, d(f,a) \geq 1, d(f,b) \geq 1, d(f,c) \geq 1$. $\therefore$ $\exists$ three edges L, M and N such that $d(E,L) \geq 1, d(E,M) \geq 1, d(E,N) \geq 1$ in Γ. However $D^2(\Gamma)$ is a 2-pseudomanifold and so we have a contradiction by the Pseudomanifold Criterion 5.1. So our assumption that $\exists$ a vertex v and an edge E in Γ_C, with $d(E,v) \geq 1$, such that $R(v,E) = 3$ is false.

$\therefore$ $\forall$ vertices v and edges E in Γ_C, with $d(E,v) \geq 1$, we must have $1 \leq R(v,E) \leq 2$. □

The Blowup Theorem gives us a way to obtain the non-simple pseudomanifold type graph back from its Collapse. It tells us that we just need to fatten up those edges that give us the 1-cells that form the boundary.

Theorem 9.4. (Blowup Theorem) *If Γ is a non-simple graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then for vertex v and edge I in Γ_C, such that $d(v,I) \geq 1$, we have $R(v,I) = 1$, with J being the free edge at v with respect to I, such that J connects vertices u and v, if and only if $\exists$ edges a, b in Γ such that A and b share vertices u and v.*

Proof. Γ is a non-simple graph, $\therefore$ $\exists$ vertices u, v and edges a, b that share u and v as vertices. $D^2(\Gamma)$ is a 2-pseudomanifold, $\therefore$ $\exists$ an edge E such that $d(E,A) \geq 1$. This implies that $d(E,B) \geq 1$. $\therefore R(v,E) \geq 2$ in Γ. Since $D^2(\Gamma)$ is a 2-pseudomanifold, it is exactly equal to 2. $\therefore$ $\forall$ edges F in Γ such that F has vertex v and $F \neq$a, $F \neq B$, we must have $d(E,F) = 0$. Let e, f be the vertices of E. In Γ_C, let I be the edge that has vertices e and f and let J be the edge connecting vertices u and v, there is exactly one such edge. Also, $d(I,v) \geq 1$. So $R(v,I) = 1$, with J as the free edge at v with respect to I, in Γ_C.

Conversely. $D^2(\Gamma_C)$ is not a 2-pseudomanifold by the Collapse Structure Theorem 9.3, $\therefore$ $\exists$ a vertex v and an edge I in Γ_C with $d(I,v) \geq 1$ such that $R(v,I) = 1$. Let J be the free edge at v with respect to I. Let u, v be the vertices connected by J. Let I have vertices e and f. The vertices u, v are distinct from the vertices e, f. And since $R(v,I) = 1$, u is the only vertex adjacent to v such that $d(u,E) \geq 1$. Let E be an edge with vertices e and f in Γ, and let A be an edge with vertices u, v in Γ. So A is a free edge at v with respect to E, in Γ. As $D^2(\Gamma)$ is a 2-pseudomanifold, $\exists$ another free edge b in Γ, at v, with respect to E. b has v as a vertex, let b be the other vertex of b. So v is adjacent to vertex b, and $d(b,E) \geq 1$. However v is only adjacent to one vertex, u, such that $d(u,E) \geq 1$. So we must have b identical to u. $\therefore$ the edges A and b share the vertices u, v. □

The above result, the Blowup Theorem, gives us and if and only if relation between edges in the Collapse and the original of a non-simple pseudomanifold type graph, and so gives us a one to one correspondence between pseudomanifold type graphs and their collapse's. We state this as a corollary below.

Corollary 9.5. (Collapse Duality) *Given a non-simple graph Γ, such that $D^2(\Gamma)$ is a 2-pseudomanifold, If Γ_C is its Collapse, then Γ is the only pseudomanifold type graph with Γ_C as its collapse.*

Proof. If Γ is simple, then Γ and Γ_C are graph isomorphic. Also, $R(v, E) = 2$ for every vertex v and every edge E such that $d(v, E) \geq 1$. So by the Blowup Theorem 9.4 no pseudomanifold type graph can be obtained by attaching additional edges to Γ_C.

If Γ is non-simple, then given two vertices u, v in Γ. The Muscle Lemma tells us that there can be a maximum of two edges that share u and v as vertices. Let if possible that there exists a different pseudomanifold type graph Γ_F such that Γ_C is also the Collapse of Γ_F. Since Γ and Γ_F share the same Collapse, it implies that they differ only in the number of edges that connect adjacent vertices. Since any two adjacent vertices are connected by either 1 or 2 edges, this implies that there exists a pair of vertices a, b in Γ_C such that a and b are connected by 2 edges in Γ or Γ_F while being connected by 1 edge in Γ_F or Γ. Without loss of generality, let a and b be connected by 2 edges in Γ and 1 edge in Γ_F. The Blowup Theorem tells us that since a and b are connected by 2 edges in Γ, they must have $R(a, E) = 1$ for some edge E in Γ_C with $d(a, E) \geq 1$. If s and t are the vertices of E, then let J be the edge that connects s and t in Γ_F. And so we must have $R(a, J) = 1$. This contradicts the Pseudomanifold Criterion Theorem. So we cannot have such distinct Γ and Γ_F. □

The Four Lemma tells us that the largest pseudomanifold type graph that we can have, simple or non-simple, must contain a maximum of 5 vertices. The following result rules out the possibility of their being any non-simple pseudomanifold type graphs with 5 vertices.

Proposition 9.6. *If Γ is a non-simple graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, then it cannot have a vertex that is adjacent to 4 vertices.*

Proof. Let if possible that $\exists$ vertex $v \in \Gamma$ such that v is adjacent to 4 vertices. Lemma 8.4 tells us that Γ has exactly 5 vertices, namely v, v_1, v_2, v_3, v_4, and furthermore, that $d(v, v_1) = 1, d(v, v_2) = 1, d(v, v_3) = 1, d(v, v_4) = 1$.

Now if all the vertices are adjacent to 3 vertices, this tells us that Γ_C is the graph K_5, and we know that $D^2(K_5)$ is a 2-pseudomanifold. However, Γ is a non-simple graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, and so we have a contradiction to the Collapse Structure Theorem 9.3.

So we must have at least one vertex v which is adjacent to exactly 2 vertices. Without loss of generality, let v_1 be adjacent to v and v_2. Let A be the edge in Γ_C that connects vertices v, v_2. As v_1 is adjacent to just v and v_2, we must have $R(v_1, A) = 0$. This contradicts the Collapse Structure Theorem 9.3.

As $D^2(\Gamma)$ is a 2-pseudomanifold, every vertex must be adjacent to 2, 3 or 4 vertices. And since Γ has 5 vertices, every vertex can be adjacent to 2 or 3 vertices. We just showed that both of these cases are impossible.

So there can be no non-simple graph Γ, such that $D^2(\Gamma)$ is a 2-pseudomanifold, if Γ has a vertex v, that is adjacent to 4 vertices. □

We now look at the various possibilities for a non-simple pseudomanifold type graphs with exactly 4 vertices.

The K_4 Theorem establishes that the graph K_4, is the collapse of a non-simple pseudomanifold type graph.

Theorem 9.7. **(K_4 Theorem)** *Given that Γ is a non-simple graph such that $D^2(\Gamma)$ is a 2-pseudomanifold, if it has a vertex v, that is adjacent to exactly 3 vertices then Γ_C is the graph K_4.*

Proof. Let v be a vertex in Γ such that v is adjacent to exactly 3 vertices, namely v_1, v_2, v_3. Let if possible that one of these vertices is adjacent to exactly 2 vertices in Γ. Without loss of generality let this vertex be v_1. This can occur in two ways; if v_1 is adjacent to an additional vertex v_4, or if v_1 is adjacent to either one of v_2 or v_3.

If v_1 is adjacent to an additional vertex v_4. Let A be the edge in Γ_C that connects vertices v_1 and v_4. Consider the edge b in Γ_C that connects vertices v and v_2. We have $R(v_1, A) = 1$. A is the free edge at v_1 with respect to b. So we must have edges E and F in Γ that share vertices v_1, v_4. Now the vertex v_2 must be adjacent to another vertex besides v. If it is adjacent to v_3, then let G be an edge having vertices v_2, v_3 in Γ. We have $R(v, G) = 3$ in Γ. This contradicts the Pseudomanifold Theorem 5.1. If on the other hand v_2 is adjacent to v_4, then let H be an edge in Γ that connects v, v_2 We have $R(v_4, H) = 3$ in Γ. This contradicts the Pseudomanifold Criterion 5.1. If on the other hand v_2 is adjacent to an additional vertex v_5. Then let I be an edge in Γ that connects v_2, v_5. We have $R(v, I) = 3$. This contradicts the Pseudomanifold Criterion 5.1.

If v_1 is adjacent to exactly one of v_2 or v_3. Without loss of generality let it be v_2. Let J be the edge in Γ_C that connects vertices v, v_2. We must have $R(v, J) = 0$ in Γ_C. This contradicts the Collapse Structure Theorem 9.3.

So we cannot have a vertex that is adjacent to exactly 2 vertices in Γ.

So every vertex in Γ must be adjacent to exactly 3 vertices. Consider v_1, it is adjacent to v. It is adjacent to 2 other vertices in addition to v. It can do this in two ways. Vertex v_1 can be adjacent to an additional vertex v_4 and exactly one of the vertices v_2, v_3; or v_1 can be adjacent to two additional vertices v_4, v_5; or v_1 can be adjacent to vertices $v1, v_2$.

If v_1 is adjacent to an additional vertex v_4 and (without loss of generality)vertex v_2. Now v_4 must be adjacent to vertices v_2, v_3. For if this were not true we would have $d(v_4, v_2) \geq 1$ or $d(v_4, v_3) \geq 1$, in contradiction to the Maximum Diameter Theorem 7.2. Note that v_1 and v_2 are now adjacent to exactly 3 vertices, and so cannot be adjacent to any additional vertices. And v_3 is adjacent to v, v_4. So v_3 must be adjacent to an additional vertex v_5. Let a, b and C be edges in Γ that connect vertices v_3, v and v_3, v_4 and v_3, v_5 respectively. Let E be an edge connecting v_1, v_2 in Γ. We have $R(v_3, E) \geq 3$ in Γ. This contradicts the Pseudomanifold Theorem 5.1. So this construction is impossible.

If on the other hand v_1 is adjacent to two additional vertices v_4, v_5. Now v_4 must be adjacent to vertices v_2, v_3. For if this were not true we would have $d(v_4, v_2) \geq 1$ or $d(v_4, v_3) \geq 1$, in contradiction to the Maximum Diameter Theorem 7.2. And v_4 cannot be adjacent to any other vertices as it is already adjacent to 3, namely v_1, v_2, v_3. So v_5 cannot be adjacent to v_4. So v_5 must be adjacent to vertices v_2, v_3. For if this were not true we would have $d(v_4, v_2) \geq 1$ or $d(v_4, v_3) \geq 1$, in contradiction to the Maximum Diameter Law 7.2. So we have three vertices v, v_4, v_5, each of which are adjacent to v_1, v_2, v_3, in Γ. This tells us that Γ_C must be the graph $K_{3,3}$. But $D^2(K_{3,3})$ is a 2-pseudomanifold, i.e. $D^2(\Gamma_C)$ is a 2-pseudomanifold. This contradicts the Collapse Structure Theorem 9.3. So this construction is impossible.

If on the other hand we have v_1 adjacent to v_2, v_3. If there is an additional vertex v_4, then v_4 must be adjacent to vertices v_2, v_3. For if this were not true we would have $d(v_4, v_2) \geq 1$ or $d(v_4, v_3) \geq 1$, in contradiction to Maximum Diameter Law 7.2. Let A be the edge in Γ_C that connects vertices v, v_2 and let b be the edge in Γ_C that connects v_1, v_3. We have $R(v_1, A) = 1$, with b as the free edge at v_1 with respect to a. This implies that there are edges E, F in Γ that share vertices v_1, v_3. Call the edge in Γ that connects v_3, v_4, edge G. Call H the edge in Γ that connects v, v_2. So we have three free edges E, F and G at v_3 with respect to H. So we have $R(v_3, H) \geq 3$. This contradicts the Pseudomanifold Theorem 5.1. So we cannot have an additional vertex v_4. Since v_2 and v_3 are adjacent to 3 vertices each, we have v_2 adjacent to v, v_1, v_3; and v_3 adjacent to v, v_1, v_2; and v_1 adjacent to v, v_2, v_3. This tells us that Γ_C is the graph K_4. □

The following result reveals the non-simple graph of which K_4 is a collapse.

Corollary 9.8. *Given a non-simple pseudomanifold type graph* Γ *that contains a vertex* v *that is adjacent to exactly 3 other vertices, then the graph is* K_4*, with each edge doubled.*

Proof. The K_4 Theorem tells us that K_4 is the collapse of Γ. The Blowup Theorem tells us that Γ is K_4 with each edge doubled. The Collapse Duality Corollary 9.5 tells us that is the only non-simple pseudomanifold type graph that has K_4 as a collapse. Thus the only non-simple pseudomanifold type graph that contains a vertex that is adjacent to 3 other vertices is K_4 with each edge doubled. □

The 4-Cycle Theorem establishes that the cycle graph of 4 vertices is the collapse of a non-simple pseudomanifold type graph.

Theorem 9.9. (4-cycle Theorem) *Given that* Γ *is a non-simple graph such that* $D^2(\Gamma)$ *is a 2-pseudomanifold, if it has a vertex* v *that is adjacent to 2 vertices, then* Γ_C *is the cycle graph of 4 vertices.*

Proof. No vertices in Γ can be adjacent to 3 vertices as that would imply that we cannot have v adjacent to exactly 2 vertices, by the K_4 Theorem 9.7. Nor can we have any vertices adjacent to 4 other vertices, by Theorem 9.6. So all the vertices in Γ must be adjacent to exactly 2 vertices each. Let v_1, v_2 be the vertices that are adjacent to v. v_1 must be adjacent to another vertex. This can happen in two ways; either v_1 is adjacent to v_2; or v_1 is adjacent to an additional vertex v_3.

If v_1 is adjacent to v_2, let E be an edge in Γ that connects v_1, v_2. As v is adjacent only to v_1, v_2, we have $R(v, E) = 0$, this contradicts the Pseudomanifold Criterion 5.1. So this construction is impossible.

So the other case must be true, i.e. v_1 must be adjacent to an additional vertex v_3. Now v_3 must be adjacent to v_2 or we would have $d(v_3, v_2) \geq 3$ in contradiction to the Maximum Diameter Law 7.2. So we have v adjacent v_1, v_2; and v_1 adjacent to v, v_3; and v_2 adjacent to v, v_3; and v_3 adjacent to v_1, v_2. So each vertex in Γ is adjacent to exactly 2 other vertices. So none of these vertices can be adjacent to any additional vertices. So Γ can have no additional vertices, as they would need to be adjacent to at least one of v, v_1, v_2, v_3. So we see that Γ_C is the cycle graph of 4 vertices. □

The following result reveals the non-simple graph of which the 4-cycle is a collapse.

Corollary 9.10. *Given a non-simple pseudomanifold type graph* Γ *that contains a vertex* v *that is adjacent to exactly 2 other vertices, then* Γ *is the 4-Cycle graph, with each edge doubled.*

Proof. The 4-cycle Theorem tells us that the 4-cycle graph is the collapse of Γ. The Blowup Theorem tells us that Γ is the 4-Cycle graph with each edge doubled. The Collapse Duality Corollary 9.5 tells us that is the only non-simple pseudomanifold type graph that has the 4-Cycle graph as a collapse. Thus the only non-simple pseudomanifold type graph that contains a vertex that is adjacent to 2 other vertices is the 4-Cycle graph with each edge doubled. □

The next corollary brings together what we already know, i.e. that the only two non-simple pseudomanifold type graphs are the doubled K_4 and the doubled 4-cycle.

Corollary 9.11. *The doubled K_4 and doubled 4-cycle are the only non-simple pseudomanifold type graphs.*

Proof. Consider a non-simple pseudomanifold type graph Γ. Every vertex can be adjacent to a maximum of 4 vertices. If it contains a vertex v that is adjacent to 4 vertices, then by the Four Lemma, Γ must have exactly 5 vertices. Theorem 9.6 tells us that no non-simple pseudomanifold graph with 5 vertices exists. Thus Γ must contain vertices that are adjacent to 2 or 3 vertices. If it contains a vertex v that is adjacent to exactly 3 vertices, then the K_4 Theorem together with Corollary 9.8 tells us that Γ must be the graph K_4 with its edges doubled. If Γ contains a vertex that is adjacent to exactly 2 vertices, then the 4-Cycle Theorem together with Corollary 9.10 tell us that Γ must be the 4-Cycle graph with each of its edges doubled. □

10 The Spaces

We now calculate and state exactly what the 2-pseudomanifolds are for the pseudomanifold graphs that we have obtained in the previous sections.

Let Γ be the non-simple pseudomanifold type graph obtained by doubling each edge of the 4-Cycle graph. Name each of the edges and vertices of the graph as in Figure 9.

Each edge in Γ has exactly two edges disjoint from it. For example, edge E only has edges G and H disjoint from it. Thus we can conclude that there are 16 2-cells in $D^2(\Gamma)$.

Consider the two 2-cells (G, F) and (G, E). They have exactly two 1-cells in common, namely (G, v_3) and (G, v_2), and so they get glued together to form a cylinder. Call this Cylinder 1. The boundary of this cylinder is made up of four 1-cells, namely $(v_1, E), (v_1, F), (v_4, E)$ and (v_4, F). Refer to Figure 10.

Consider the two 2-cells (H, E) and (H, F). They have exactly two 1-cells in common, namely (H, v_3) and (H, v_2), and so they get glued together to form a cylinder. Call this Cylinder 2. The boundary of this cylinder is made up of four 1-cells, namely $(v_1, F), (v_1, E), (v_4, F)$ and (v_4, E). Refer to Figure 11.

Cylinders 1 and 2 have the four 1-cells that make up their boundary, in common with each other, in such a way that they allow us to glue Cylinders 1 and 2 together to form a torus, call it Torus 1.

Torus 1 is made up of four 2-cells, eight 1-cells and four 0-cells. The 0-cells are $(v_1, v_3), (v_1, v_2), (v_4, v_3)$ and (v_4, v_2). See Figure 12.

In a similar way we get three more tori, namely Torus 2, Torus 3 and Torus 4, each of which contain four 0-cells as shown in Figure 13, Figure 14 and Figure 14 respectively.

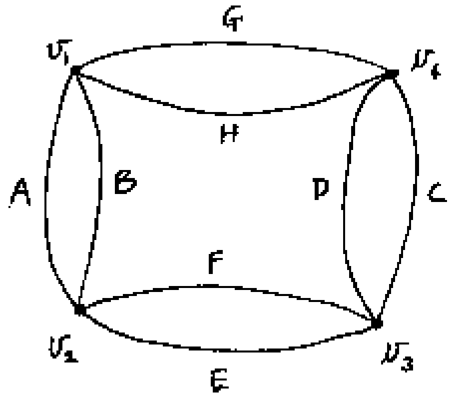

Figure 9: Doubled $K_{2,2}$

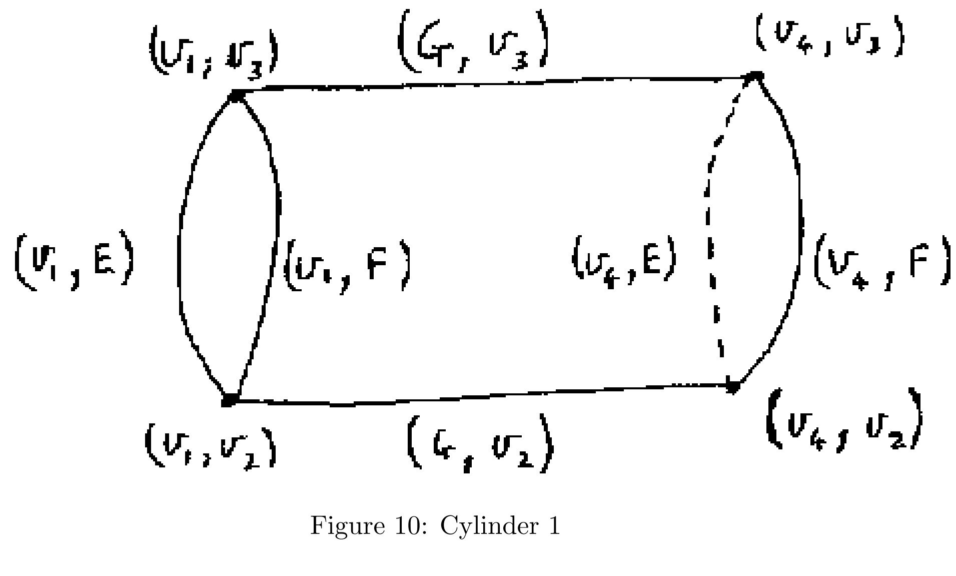

Figure 10: Cylinder 1

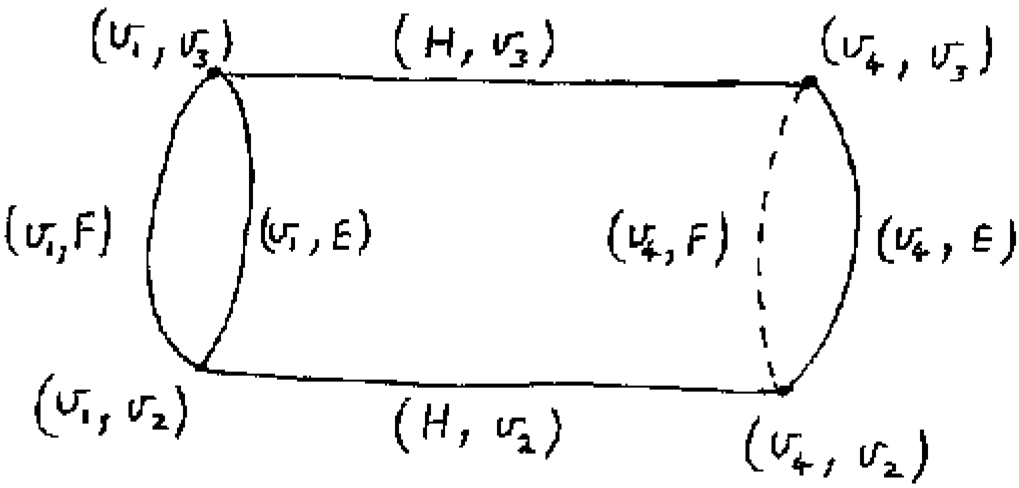

Figure 11: Cylinder 2

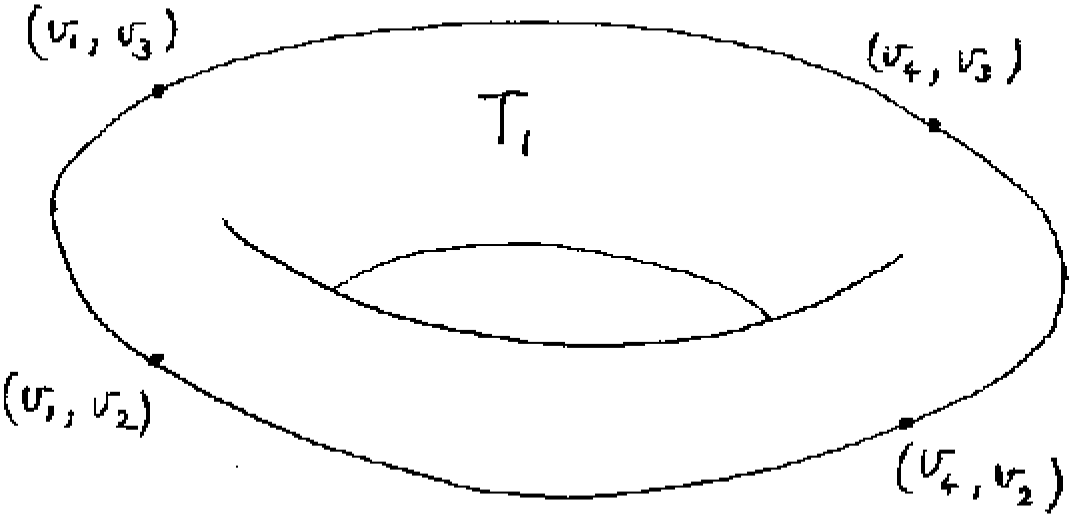

Figure 12: Torus 1

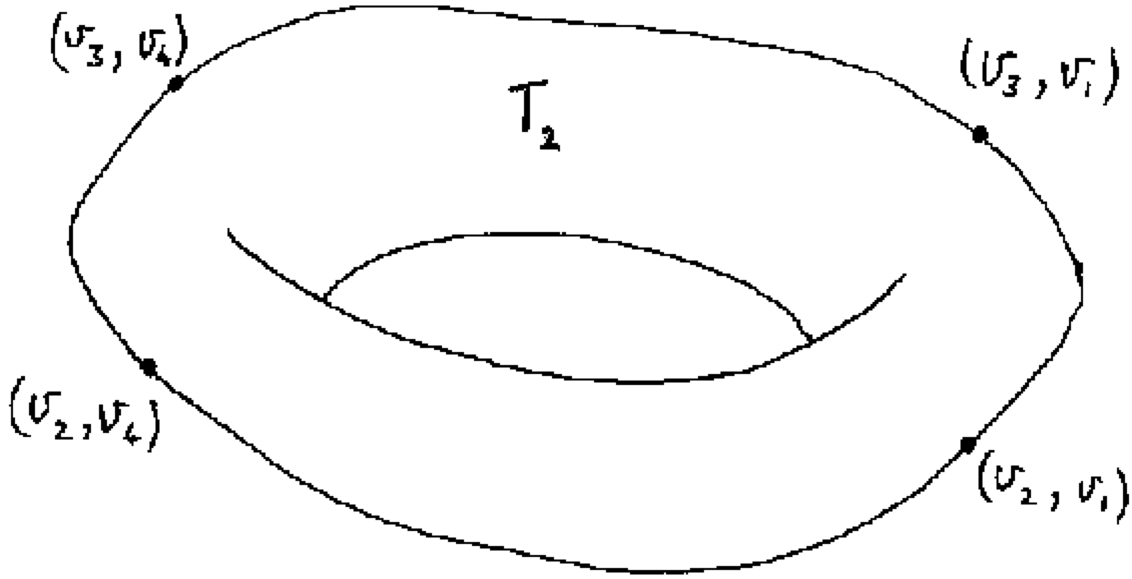

Figure 13: Torus 2

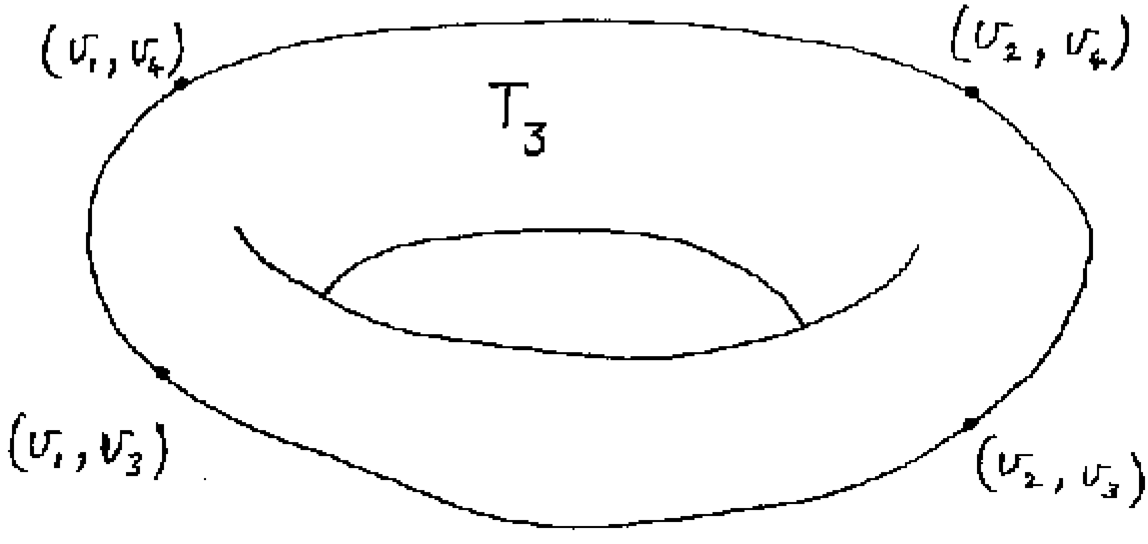

Figure 14: Torus 3

Each torus, has exactly two 0-cells in common with two other tori, sharing each 0-cell with each of them. Torus 1 shares (v_1, v_3) and (v_4, v_2) with Torus 3 and 4 respectively. Torus 2 shares (v_2, v_4) and (v_3, v_1) with Torus 3 and 4 respectively. Thus they are glued together to give us $D^2(\Gamma)$ as shown in Figure 16.

The only other non-simple pseudomanifold type graph is an edge doubled K_4. It can be obtained from our earlier graph by simply adding four new edges I, J, K and L as shown in Figure 17.

The four new edges give rise to two additional tori in the discretized space. Call them Torus 5 and Torus 6. Refer to Figure 18 and Figure 19. They each contain four 0-cells, $(v_1, v_4), (v_1, v_2), (v_3, v_4), (v_3, v_2)$ and $(v_4, v_3), (v_2, v_3), (v_4, v_1)$, (v_2, v_1) respectively. Torus 5 and Torus 6 are disjoint from each other but share each of their four 0-cells with Torus 1, 2, 3 and 4 respectively. The discretized space for this graph is hard to draw and so is not displayed, however it is best described as a cube built out of six tori, in which each torus touches four distinct tori.

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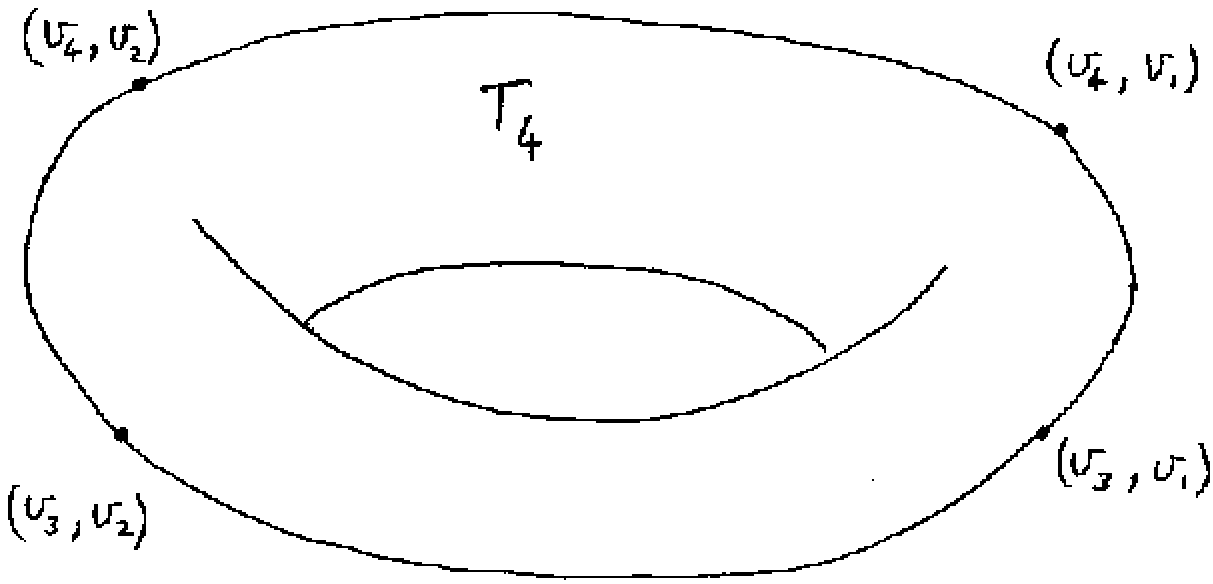

Figure 15: Torus 4

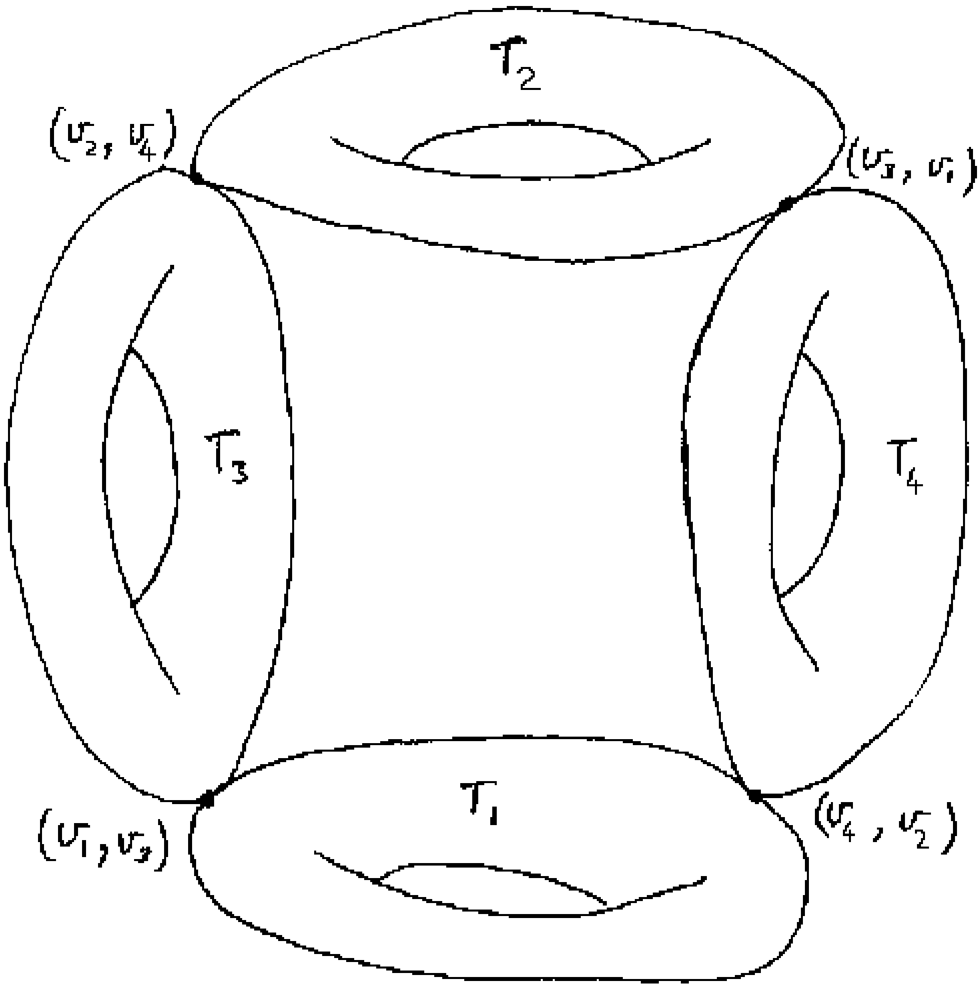

Figure 16: $D^2(\Gamma)$ of doubled 4-Cycle graph

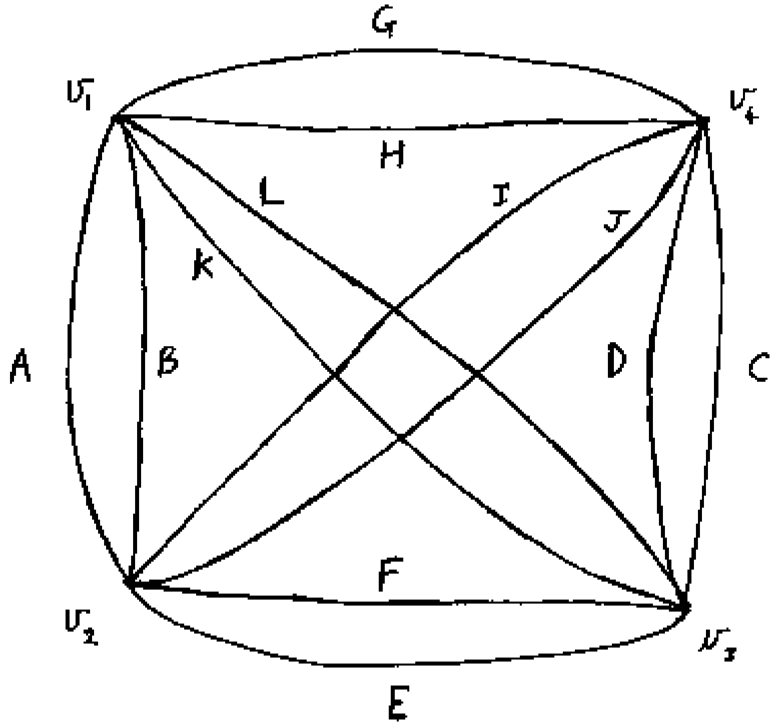

Figure 17: Doubled K_4

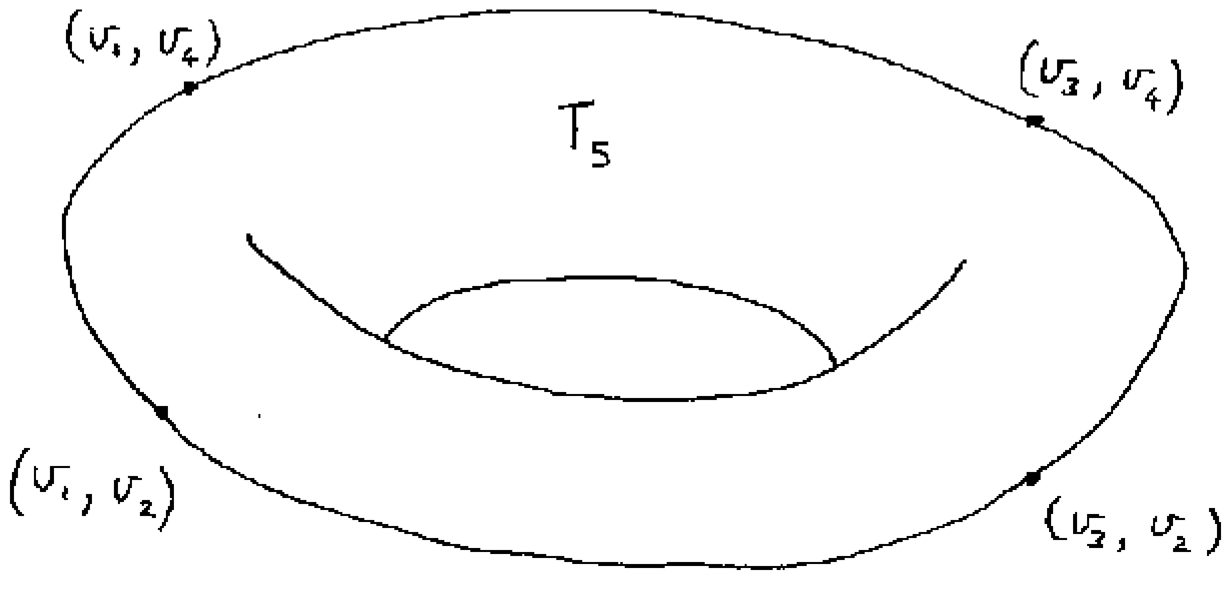

Figure 18: Torus 5

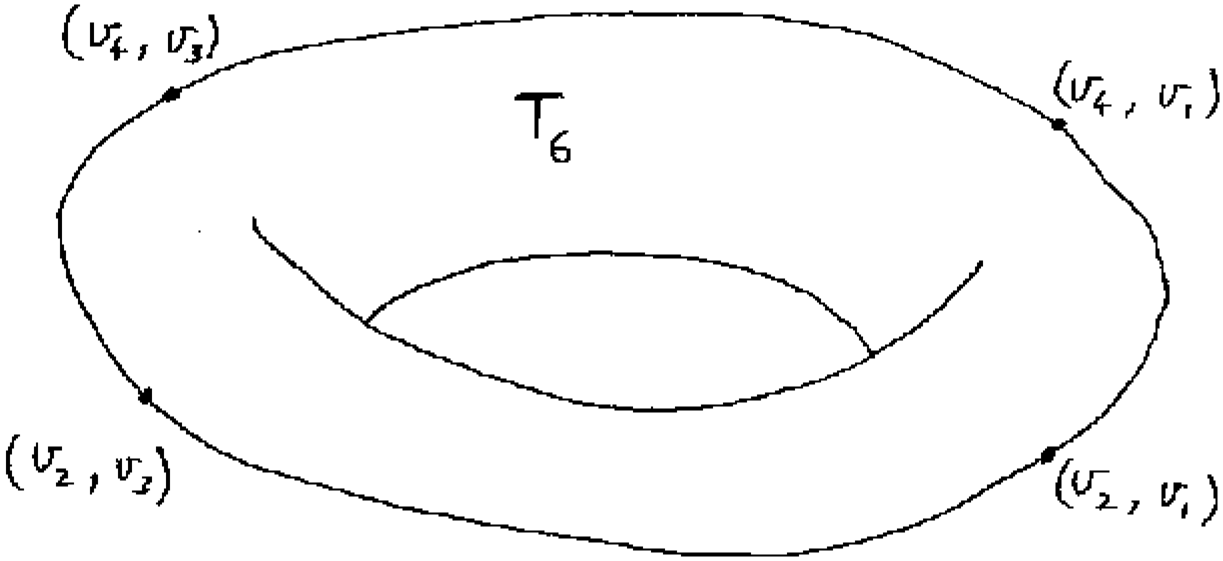

Figure 19: Torus 6