Two Approaches to Proving

Goldbach's Conjecture

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A Brief Introduction to Goldbach's Conjecture

In 1741 Goldbach made his most famous contribution in mathematics with the conjecture that all even numbers can be expressed as the sum of two primes (currently, his conjecture is stated as "all even numbers greater than 2 can be expressed as the sum of two primes" since 1 is no longer considered a prime as it was in Goldbach's time [1].) As of yet, no proof of Goldbach's Conjecture has been found. This conjecture has been shown to be correct for a large amount of numbers using numerical calculations. Some examples are 10 = 3 + 7, 18 = 7 + 11, 100 = 3 + 97, and so on.

The study of Goldbach's conjecture has led to very great achievements since 1920. For example, I. M. Vinogradov proved in1937 that every sufficiently large odd number can be expressed as the sum of three primes. Also, the investigation of Goldbach's conjecture has been a catalyst for the creation and development of several number-theoretic methods which are useful in number theory and other fields of mathematics.

In 1900, the mathematician Hilbert gave a famous speech at the 2^{nd} International Congress of Mathematics held in Paris where he proposed 23 problems for mathematicians in the 20^{th} century. Goldbach's conjecture was part of one of those problems. Then in 1921, Hardy said that Goldbach's conjecture is not only one of the most famous and difficult problems in number theory, but also in the whole of mathematics.

The first great achievements toward Goldbach's conjecture were obtained in the 1920s. The first of which was in 1923 when, using the "circle method", British mathematicians Hardy and Littlewood proved that every sufficiently large odd integer is the sum of three odd primes and almost all even integers are sums of two primes, provided that the grand Riemann hypothesis is assumed to be true. In 1919, Norwegian mathematician Brun established, using his sieve method, that every large even number is the sum of two numbers each having at most nine prime factors. Then in 1930, using Brun's method along with his own idea of the "density" of an integer sequence, Russian mathematician Schnirelman proved that every sufficiently large integer is the sum of at most c primes for a fixed number c. Then in 1937, the Russian mathematician Vinogradov was able to remove, using the circle method and his method on the estimation of exponential sum with prime variable, the dependence on the grand Riemann hypothesis and therefore provide unconditional proofs of Hardy and Littlewood's findings. And finally, after improvements on Brun's method and his result, Chinese mathematician Chen Jing Run was able to prove that every large even integer is the sum of a prime and a product of at most two primes in 1966 [2].

Unfortunately, my mathematical background is not sufficient to understand the advanced methods of these mathematicians. However, this did not end my curiosity for the problem. Since learning about it in high school I have fiddled with it much in my spare time. In fact, it was in high school that I realized an equivalent statement to Goldbach's conjecture which seemed to make it more approachable. It was not until the latter part of my college years that I had the mathematics background to start attacking this equivalent statement. The purpose of my research was to approach Goldbach's Conjecture using this equivalent statement in hopes of discovering insight into an alternative way of proving Goldbach's Conjecture. For my research Mathematica was the primary tool to collect data. The different functions used to generate the data can be found in the Appendix.

An Equivalent Statement of Goldbach's Conjecture

An equivalent statement to Goldbach's Conjecture is that for every integer $n \geq 2$ there exists an integer j such that $n+j$ and $n-j$ are prime numbers.

Theorem 1: For $n \geq 2$, $2n = p+q$ where p and q are prime numbers if and only if there exists an integer j such that $n+j$ and $n-j$ are prime numbers.

Proof: Suppose $2n = p+q$ where p and q are prime numbers. Observe that $n-(q-n) = 2n-q = p$ and $n+(q-n) = q$. Thus there exists an integer $j = q-n$ such that $n+j$ and $n-j$ are prime numbers. Now suppose there exists an integer j such that $n+j$ and $n-j$ are prime numbers. Then $(n+j)+(n-j) = 2n$.

Choose an integer $n \geq 2$. Let $k = \pi\left(\sqrt{2n}\right)$ where $\pi(x)$ equals the number of primes less than or equal to x. The Chinese Remainder Theorem guarantees that there exists an integer j such that $n+j$ and $n-j$ are not divisible by the primes 2, 3, …, p_k.

Theorem 2: For all $n \geq 2$, there exists an integer j such that $n+j$ and $n-j$ are not divisible by the primes 2, 3, …, p_k.

Proof: Choose a_i such that $a_i \not\equiv \pm n (\text{mod } p_i)$ for $1 \leq i \leq k$. By the Chinese Remainder Theorem there exists a solution modulo α_k (where $\alpha_k = 2 \cdot 3 \cdot ... \cdot p_k$ is the ith primorial, and $\alpha_0 = 1$ for convenience) to the following system of congruence equations:

$$
\begin{aligned}
&j \equiv a_1 (\bmod 2) \\
&j \equiv a_2 (\bmod 3) \\
&\ldots \\
&j \equiv a_k (\bmod p_k)
\end{aligned}
$$

So then, $j \equiv a_i \not\equiv \pm n (\bmod p_i)$. Thus $n \pm j \not\equiv 0 (\bmod p_i)$ for $1 \le i \le k$. Therefore $n + j$ and $n - j$ are not divisible by the primes 2, 3, …, p_k.

If $n \pm j \le 2n$, then $n \pm j$ is prime since $\sqrt{n \pm j} \le \sqrt{2n}$ and $n \pm j$ is not divisible by all of the prime numbers less than or equal to $\sqrt{2n}$ (this is the sieve of Eratosthenes.) However, j is a solution modulo α_k, so j can be such that $n \pm j$ is significantly greater than $2n$. Its value is dependent on how a_i is chosen for $1 \le i \le k$. However, if j is in the right range, particularly if $|j| \le n - 2$, then $n + j$ and $n - j$ are prime. Moreover, if for every $n \ge 2$ there exists such a j then Goldbach's conjecture is true.

Let's look at an example. Let $n = 52$, then $k = \pi\left(\sqrt{2 \cdot 52}\right) = 4$. Observe that

$$
\begin{aligned}
&\pm n \equiv 0 (\bmod 2) \\
&\pm n \equiv 1,2 (\bmod 3) \\
&\pm n \equiv 2,3 (\bmod 5) \\
&\pm n \equiv 3,4 (\bmod 7)
\end{aligned}
$$

One possible way of choosing a_i is to have $a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 0$. This gives $j = 105$ which is greater than $52 - 2 = 50$. However, if we instead let $a_1 = 1, a_2 = 0, a_3 = 4, a_4 = 2$ we get $j = 9 < 50$ and thus $n + j = 52 + 9 = 61$ and $n + j = 52 - 9 = 43$ are both prime.

The Sequence Approach

A special sequence was created to generate a j, given an integer seed s, such that $n+j$ and $n-j$ are not divisible by the primes 2, 3, …, p_k. It is defined as follows:

(1) Given a positive integer n, let $k=\pi\left(\sqrt{2n}\right)$ and let s be some integer. Then j_i^s is defined as follows:

$$j_0^s = s\,,\ \ j_i^s = \begin{cases} j_{i-1}^s, \text{ If } j_{i-1}^s \not\equiv \pm n (\text{mod } p_i) \\ j_{i-1}^s + \alpha_{i-1}, \text{ If } j_{i-1}^s \equiv \pm n (\text{mod } p_i) \text{ and } j_{i-1}^s + \alpha_{i-1} \not\equiv \pm n (\text{mod } p_i) \\ j_{i-1}^s + 2\alpha_{i-1}, \text{ If } j_{i-1}^s \equiv \pm n (\text{mod } p_i) \text{ and } j_{i-1}^s + \alpha_{i-1} \equiv \pm n (\text{mod } p_i) \end{cases}$$

<u>Theorem 3</u>: Given $n \geq 2$, let $k=\pi\left(\sqrt{2n}\right)$, and let s be some integer. Then $n \pm j_k^s$ is not divisible by the primes 2, 3, …, p_k.

<u>Proof</u>: If $j_{i-1}^s \not\equiv \pm n (\text{mod } p_i)$, then $j_i^s = j_{i-1}^s \not\equiv \pm n (\text{mod } p_i)$. If $j_{i-1}^s \equiv \pm n (\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \not\equiv \pm n (\text{mod } p_i)$, then $j_i^s = j_{i-1}^s + \alpha_{i-1} \not\equiv \pm n (\text{mod } p_i)$. Observe that p_i is relatively prime to $\alpha_{i-1} = 2 \cdot 3 \cdot \ldots \cdot p_{i-1}$. Suppose $j_{i-1}^s \equiv \pm n (\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \equiv \pm n (\text{mod } p_i)$. This is only possible if $i > 1$. Thus $2 < p_i$. So then $j_{i-1}^s \not\equiv j_{i-1}^s + \alpha_{i-1} (\text{mod } p_i)$ and $j_{i-1}^s \not\equiv j_{i-1}^s + 2\alpha_{i-1} (\text{mod } p_i)$, but since $j_{i-1}^s \equiv \pm n (\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \equiv \pm n (\text{mod } p_i)$, $j_i^s = j_{i-1}^s + 2\alpha_{i-1} \not\equiv \pm n (\text{mod } p_i)$. Therefore, in all cases $j_i^s \not\equiv \pm n (\text{mod } p_i)$. Furthermore, for $1 \leq i \leq r$, $j_r^s = j_i^s + c_i\alpha_i + c_{i+1}\alpha_{i+1} + \ldots + c_{k-1}\alpha_{k-1}$, where $c_i, c_{i+1}, \ldots, c_{k-1}$ are constants (which are 0, 1, or 2.) But, p_i divides $\alpha_i, \alpha_{i+1}, \ldots, \alpha_k$. Thus

$j_r^s \equiv j_i^s + 0 \equiv j_i^s \not\equiv \pm n (\bmod\ p_u)$. Then $j_k^s \not\equiv \pm n (\bmod\ p_i)$ for $1 \le i \le k$ and therefore $n \pm j_k^s$ is not divisible by the primes 2, 3, …, p_k.

For an example, again let's look at $n = 52$. Then $k = \pi\left(\sqrt{2 \cdot 52}\right) = 4$. Observe that

$$\pm n \equiv 0 (\bmod 2)$$
$$\pm n \equiv 1,2 (\bmod 3)$$
$$\pm n \equiv 2,3 (\bmod 5)$$
$$\pm n \equiv 3,4 (\bmod 7)$$

Let's use the seed $s = 1$. Then $j_0^s = j_0^1 = 1$. Observe that $1 \not\equiv \pm n (\bmod 2)$. So then $j_1^1 = j_0^1 = 1$. Observe that $1 \equiv n (\bmod 3)$ and $1 + \alpha_1 = 1 + 2 \equiv 0 \not\equiv \pm n (\bmod 3)$. So then $j_2^1 = j_1^1 + \alpha_1 = 1 + 2 = 3$. Observe that $3 \equiv -n (\bmod 5)$ and $3 + \alpha_2 = 3 + 6 \equiv 2 \equiv n (\bmod 5)$. So then $j_3^1 = j_2^1 + 2\alpha_2 = 3 + 2 \cdot 6 = 15$. Observe that $15 \equiv 1 \not\equiv \pm n (\bmod 7)$. So then $j_4^1 = j_3^1 = 15$. $52 + 15 = 67$ and $52 - 15 = 37$. 67 and 37 are not divisible by 2, 3, 5, or 7. Also note that 37 and 67 are prime numbers.

For a given n, j_k^s (where $k = \pi\left(\sqrt{2n}\right)$) behaves somewhat erratically for varying values of s. The following are values for j_k^s, given $n = 100$ and s:

Table 1 $n = 100$, $\{ s \text{ , } j_k^s \}$

{-100,-99}	{-69,2241}	**{-38,-3}**	**{-7,-3}**	**{24,27}**	**{55,57}**	{86,297}
{-99,-99}	**{-68,-63}**	**{-37,-3}**	**{-6,-3}**	**{25,27}**	**{56,57}**	{87,297}
{-98,-63}	**{-67,-63}**	**{-36,-3}**	**{-5,-3}**	**{26,27}**	**{57,57}**	{88,123}
{-97,-63}	**{-66,-63}**	**{-35,-3}**	**{-4,-3}**	**{27,27}**	**{58,63}**	{89,123}
{-96,-63}	**{-65,-63}**	**{-34,-3}**	**{-3,-3}**	**{28,63}**	**{59,63}**	{90,123}
{-95,-63}	**{-64,-63}**	**{-33,-3}**	**{-2,3}**	**{29,63}**	**{60,63}**	{91,123}
{-94,-63}	**{-63,-63}**	**{-32,-27}**	**{-1,3}**	**{30,63}**	**{61,63}**	{92,123}
{-93,-63}	**{-62,-57}**	**{-31,-27}**	**{0,3}**	**{31,63}**	**{62,63}**	{93,123}
{-92,123}	**{-61,-57}**	**{-30,-27}**	**{1,3}**	**{32,63}**	**{63,63}**	{94,99}
{-91,123}	**{-60,-57}**	**{-29,-27}**	**{2,3}**	**{33,63}**	{64,2379}	{95,99}
{-90,123}	**{-59,-57}**	**{-28,-27}**	**{3,3}**	**{34,39}**	{65,2379}	{96,99}
{-89,123}	**{-58,-57}**	**{-27,-27}**	**{4,39}**	**{35,39}**	{66,2379}	{97,99}
{-88,123}	**{-57,-57}**	{-26,189}	**{5,39}**	**{36,39}**	{67,2379}	{98,99}
{-87,123}	{-56,189}	{-25,189}	**{6,39}**	**{37,39}**	{68,2379}	
{-86,-81}	{-55,189}	{-24,189}	**{7,39}**	**{38,39}**	{69,2379}	
{-85,-81}	{-54,189}	{-23,189}	**{8,39}**	**{39,39}**	**{70,81}**	
{-84,-81}	{-53,189}	{-22,189}	**{9,39}**	**{40,81}**	**{71,81}**	
{-83,-81}	{-52,189}	{-21,189}	{10,231}	**{41,81}**	**{72,81}**	
{-82,-81}	{-51,189}	{-20,231}	{11,231}	**{42,81}**	**{73,81}**	
{-81,-81}	**{-50,-39}**	{-19,231}	{12,231}	**{43,81}**	**{74,81}**	
{-80,2241}	**{-49,-39}**	{-18,231}	{13,231}	**{44,81}**	**{75,81}**	
{-79,2241}	**{-48,-39}**	{-17,231}	{14,231}	**{45,81}**	**{76,81}**	
{-78,2241}	**{-47,-39}**	{-16,231}	{15,231}	**{46,81}**	**{77,81}**	
{-77,2241}	**{-46,-39}**	{-15,231}	{16,231}	**{47,81}**	**{78,81}**	
{-76,2241}	**{-45,-39}**	{-14,231}	{17,231}	**{48,81}**	**{79,81}**	
{-75,2241}	**{-44,-39}**	{-13,231}	{18,231}	**{49,81}**	**{80,81}**	
{-74,2241}	**{-43,-39}**	{-12,231}	{19,231}	**{50,81}**	**{81,81}**	
{-73,2241}	**{-42,-39}**	{-11,231}	{20,231}	**{51,81}**	{82,297}	
{-72,2241}	**{-41,-39}**	{-10,231}	{21,231}	**{52,57}**	{83,297}	
{-71,2241}	**{-40,-39}**	{-9,231}	**{22,27}**	**{53,57}**	{84,297}	
{-70,2241}	**{-39,-39}**	**{-8,-3}**	**{23,27}**	**{54,57}**	{85,297}	

The values that are bolded are ones where j_k^s is such that $n \pm j_k^s$ are prime numbers. A good portion of the values in the table fit such a condition. This observation combined with other observations of data leads to the following conjecture:

<u>Conjecture 1</u>: For all integers $n \geq 2$ there exists an integer s such that $\left| j_k^s \right| \leq n - 2$ where j_k^s is as defined in *(1)*. Thus implying that $n \pm j_k^s$ are prime numbers. It follows from this that Goldbach's conjecture is true since $\left(n + j_k^s\right) + \left(n - j_k^s\right) = 2n$.

Also, note from the data in *Table 1* that there is a great amount of repetition in the values for j_k^s for $n = 100$ (but as we will see soon, there is repetition in values of j_k^s for all n .) If we look at values of n modulo 6 we get some explanation for the repetition. We are looking at n modulo 6 because $\alpha_2 = 6$ (we also could look at n modulo $\alpha_3 = 30, \alpha_4 = 210$, etc.) However, before we do that, we must show that for $n \geq 2$ and two integers s_1 and s_2, $1 \leq s_1 \leq s_2 \leq k$, if $j_u^{s_1} = j_v^{s_2}$ where $1 \leq u \leq v \leq k$, then $j_k^{s_1} = j_k^{s_2}$. Suppose $j_u^{s_1} = j_v^{s_2}$. Observe that, as was shown in the proof for Theorem 3, $j_u^{s_1} = j_v^{s_2} \not\equiv \pm n (\bmod p_i)$ for $1 \leq i \leq v$. Thus by the definition of j_i^s, $j_u^{s_1} = j_{u+1}^{s_1} = j_{u+2}^{s_1} = ... = j_v^{s_1}$. So then $j_v^{s_1} = j_v^{s_2}$. It follows that $j_k^{s_1} = j_k^{s_2}$.

Now let $n \equiv 0 (\bmod 6)$. Then $n \equiv 0 (\bmod 2)$ and $n \equiv 0 (\bmod 3)$. We will consider the seeds $s, s+1, s+2, s+3, s+4,$ and $s+5$ where $s \equiv 0 (\bmod 6)$ in order to discover the behavior of all seeds modulo 6. Then $j_0^s = s \equiv 0 \equiv n (\bmod 2)$ and $s + \alpha_1 = s + 1 \equiv 1 \not\equiv \pm n (\bmod 2)$. Thus $j_1^s = s+1$. Observe that $s+1 \equiv 1 \not\equiv \pm n (\bmod 3)$. Then $j_2^s = s+1$. Now consider the seed $s+1$. $j_0^{s+1} = s+1$, but $j_1^s = s+1$. Then $j_k^s = j_k^{s+1}$. Consider the seed $s+2$. Then $j_0^{s+2} = s+2 \equiv 0 (\bmod 2)$ and $s + 2 + \alpha_1 = s + 2 + 1 \equiv 1 \not\equiv \pm n (\bmod 2)$. Then $j_1^{s+2} = s+3$. Observe that $s+3 \equiv 0 \equiv n (\bmod 3)$ and $s + 3 + \alpha_2 = s + 3 + 2 \equiv 2 \not\equiv \pm n (\bmod 3)$. Thus $j_2^{s+2} = s+5$. Now consider the seed $s+3$. $j_0^{s+3} = s+3 = j_1^{s+1}$. So then $j_k^{s+2} = j_k^{s+3}$. Consider the seed $s+4$. $j_0^{s+4} = s+4 \equiv 0 (\bmod 2)$ and $s + 4 + \alpha_1 = s + 4 + 1 \equiv 1 \not\equiv \pm n (\bmod 2)$. Thus $j_1^{s+4} = s+5 = j_2^{s+2}$. So then $j_k^{s+2} = j_k^{s+4}$. Finally, consider the seed $s+5$. $j_0^{s+5} = s+5 = j_2^{s+2}$. Thus $j_k^{s+2} = j_k^{s+5}$. Therefore for $n \equiv 0 (\bmod 6)$ and $s \equiv 0 (\bmod 6)$, $j_k^s = j_k^{s+1}$ and $j_k^{s+2} = j_k^{s+3} = j_k^{s+4} = j_k^{s+5}$. This approach can be used on the other values of n modulo 6 to get similar results. The results can be summarized as follows:

(2) For $s \equiv 0 (\mathrm{mod}\, 6)$,

If $n \equiv 0 (\mathrm{mod}\, 6)$	$j_k^s = j_k^{s+1}$ and $j_k^{s+2} = j_k^{s+3} = j_k^{s+4} = j_k^{s+5}$
If $n \equiv 1,5 (\mathrm{mod}\, 6)$	$j_k^{s+1} = j_k^{s+2} = j_k^{s+3} = j_k^{s+4} = j_k^{s+5} = j_k^{s+6}$
If $n \equiv 2,4 (\mathrm{mod}\, 6)$	$j_k^{s-2} = j_k^{s-1} = j_k^s = j_k^{s+1} = j_k^{s+2} = j_k^{s+3}$
If $n \equiv 3 (\mathrm{mod}\, 6)$	$j_k^{s-1} = j_k^s = j_k^{s+1} = j_k^{s+2}$ and $j_k^{s+3} = j_k^{s+4}$

This result can be tested on an example. Let $n = 126$. So then $n \equiv 0 (\mathrm{mod}\, 6)$.

Table 2 $n = 126$, $\{s, j_k^s\}$.

{0,217}
{1,217}
{2,2315}
{3,2315}
{4,2315}
{5,2315}
{6,217}
{7,217}
{8,17}
{9,17}
{10,17}
{11,17}
{12,283}

So the pattern in *(2)* seems to hold. Note that the relations in *(2)* do not indicate when new values for j_k^s will occur, but rather which ones are definitely the same. A more specific set of relations can most likely be found by looking at n modulo higher primorials.

Another interesting aspect of the j_i^s series is that for $n \geq 2$, it does not converge.

<u>Theorem 4</u>: For $n \geq 2$ and an integer s, j_i^s does not converge.

<u>Proof:</u> Suppose j_i^s converged. Then $j_l^s = j_{l+1}^s = j_{l+2}^s = \ldots$ for some $l \geq 1$. It was shown in the proof for Theorem 3 $j_l^s \not\equiv \pm n (\mathrm{mod}\, p_i)$ for $1 \leq i \leq l$.

Then $n + j_l^s = p_{m_1}^{r_1} p_{m_2}^{r_2} \cdot \ldots \cdot p_{m_t}^{r_t}$ for some $t \geq 1$ where $p_{m_1}, p_{m_2}, \cdot \ldots \cdot p_{m_t}$ are distinct primes with, $r_1, r_2, \ldots, r_t$ being their respective non-negative exponents and with

$m_i > l$. But then $n + j_l^s = p_{m_1}^{r_1} p_{m_2}^{r_2} \cdot \ldots \cdot p_{m_t}^{r_t} \equiv 0 (\mathrm{mod}\, p_{m_1})$. Thus $j_l^s \equiv -n (\mathrm{mod}\, p_{m_1})$. Then $j_{m_1}^s \geq j_l^s + \alpha_{m_1 - 1} > j_l^s$. This is a contradiction. Therefore j_i^s does not converge.

Now we will investigate, for a given n , which values of s give j_k^s such that $\left| j_k^s \right| \leq n - 2$. The following table is a table of values of j_k^s such that $\left| j_k^s \right| \leq n - 2$ for $n = 250$ and for $-3000 \leq s \leq 98$.

Table 3 $n = 250\ ,\{s, j_k^s\},\ -30000 \le s \le 248\ .$

{-29858,207}	{-2503,-189}	{-408,-189}	{-197,-189}	{-88,-87}	{15,21}	{166,171}
{-29857,207}	{-2502,-189}	{-407,-189}	{-196,-189}	{-87,-87}	{16,21}	{167,171}
{-29856,207}	{-2501,-189}	{-406,-189}	{-195,-189}	{-68,147}	{17,21}	{168,171}
{-29855,207}	{-2500,-189}	{-405,-189}	{-194,-189}	{-67,147}	{18,21}	{169,171}
{-29854,207}	{-2499,-189}	{-404,-189}	{-193,-189}	{-66,147}	{19,21}	{170,171}
{-29853,207}	{-2498,-183}	{-403,-189}	{-192,-189}	{-65,147}	{20,21}	{171,171}
{-29852,183}	{-2497,-183}	{-402,-189}	{-191,-189}	{-64,147}	{21,21}	{172,207}
{-29851,183}	{-2496,-183}	{-401,-189}	{-190,-189}	{-63,147}	{22,27}	{173,207}
{-29850,183}	{-2495,-183}	{-400,-189}	{-189,-189}	{-62,-57}	{23,27}	{174,207}
{-29849,183}	{-2494,-183}	{-399,-189}	{-188,-183}	{-61,-57}	{24,27}	{175,207}
{-29848,183}	{-2493,-183}	{-398,-189}	{-187,-183}	{-60,-57}	{25,27}	{176,207}
{-29847,183}	{-2348,207}	{-397,-183}	{-186,-183}	{-59,-57}	{26,27}	{177,207}
{-29828,207}	{-2347,207}	{-396,-183}	{-185,-183}	{-58,-57}	{27,27}	{178,183}
{-29827,207}	{-2346,207}	{-395,-183}	{-184,-183}	{-57,-57}	{52,57}	{179,183}
{-29826,207}	{-2345,207}	{-394,-183}	{-183,-183}	{-56,-21}	{53,57}	{180,183}
{-29825,207}	{-2344,207}	{-393,-183}	{-182,-147}	{-55,-21}	{54,57}	{181,183}
{-29824,207}	{-2343,207}	{-338,-123}	{-181,-147}	{-54,-21}	{55,57}	{182,183}
{-29823,207}	{-2342,-27}	{-337,-123}	{-180,-147}	{-53,-21}	{56,57}	{183,183}
{-2750,-189}	{-2341,-27}	{-336,-123}	{-179,-147}	{-52,-21}	{57,57}	{184,189}
{-2749,-189}	{-2340,-27}	{-335,-123}	{-178,-147}	{-51,-21}	{82,87}	{185,189}
{-2748,-189}	{-2339,-27}	{-334,-123}	{-177,-147}	{-38,207}	{83,87}	{186,189}
{-2747,-189}	{-2338,-27}	{-333,-123}	{-176,-171}	{-37,207}	{84,87}	{187,189}
{-2746,-189}	{-2337,-27}	{-272,-57}	{-175,-171}	{-36,207}	{85,87}	{188,189}
{-2745,-189}	{-2318,207}	{-271,-57}	{-174,-171}	{-35,207}	{86,87}	{189,189}
{-2744,-189}	{-2317,207}	{-270,-57}	{-173,-171}	{-34,207}	{87,87}	{202,207}
{-2743,-189}	{-2316,207}	{-269,-57}	{-172,-171}	{-33,207}	{88,123}	{203,207}
{-2742,-189}	{-2315,207}	{-268,-57}	{-171,-171}	{-32,-27}	{89,123}	{204,207}
{-2741,-189}	{-2314,207}	{-267,-57}	{-152,-147}	{-31,-27}	{90,123}	{205,207}
{-2740,-189}	{-2313,207}	{-248,-213}	{-151,-147}	{-30,-27}	{91,123}	{206,207}
{-2739,-189}	{-2222,123}	{-247,-213}	{-150,-147}	{-29,-27}	{92,123}	{207,207}
{-2720,-189}	{-2221,123}	{-246,-213}	{-149,-147}	{-28,-27}	{93,123}	{208,213}
{-2719,-189}	{-2220,123}	{-245,-213}	{-148,-147}	{-27,-27}	{94,99}	{209,213}
{-2718,-189}	{-2219,123}	{-244,-213}	{-147,-147}	{-26,-21}	{95,99}	{210,213}
{-2717,-189}	{-2218,123}	{-243,-213}	{-128,-123}	{-25,-21}	{96,99}	{211,213}
{-2716,-189}	{-2217,123}	{-230,-189}	{-127,-123}	{-24,-21}	{97,99}	{212,213}
{-2715,-189}	{-2192,123}	{-229,-189}	{-126,-123}	{-23,-21}	{98,99}	{213,213}
{-2714,-189}	{-2191,123}	{-228,-189}	{-125,-123}	{-22,-21}	{99,99}	
{-2713,-189}	{-2190,123}	{-227,-189}	{-124,-123}	{-21,-21}	{112,147}	
{-2712,-189}	{-2189,123}	{-226,-189}	{-123,-123}	{-20,21}	{113,147}	
{-2711,-189}	{-2188,123}	{-225,-189}	{-122,-87}	{-19,21}	{114,147}	
{-2710,-189}	{-2187,123}	{-224,-189}	{-121,-87}	{-18,21}	{115,147}	
{-2709,-189}	{-2138,207}	{-223,-189}	{-120,-87}	{-17,21}	{116,147}	
{-2708,-183}	{-2137,207}	{-222,-189}	{-119,-87}	{-16,21}	{117,147}	
{-2707,-183}	{-2136,207}	{-221,-189}	{-118,-87}	{-15,21}	{118,123}	
{-2706,-183}	{-2135,207}	{-220,-189}	{-117,-87}	{-14,21}	{119,123}	
{-2705,-183}	{-2134,207}	{-219,-189}	{-110,-99}	{-13,21}	{120,123}	
{-2704,-183}	{-2133,207}	{-218,-213}	{-109,-99}	{-12,21}	{121,123}	
{-2703,-183}	{-2108,207}	{-217,-213}	{-108,-99}	{-11,21}	{122,123}	
{-2540,-189}	{-2107,207}	{-216,-213}	{-107,-99}	{-10,21}	{123,123}	
{-2539,-189}	{-2106,207}	{-215,-213}	{-106,-99}	{-9,21}	{142,147}	
{-2538,-189}	{-2105,207}	{-214,-213}	{-105,-99}	{-8,207}	{143,147}	
{-2537,-189}	{-2104,207}	{-213,-213}	{-104,-99}	{-7,207}	{144,147}	
{-2536,-189}	{-2103,207}	{-212,-207}	{-103,-99}	{-6,207}	{145,147}	
{-2535,-189}	{-440,-189}	{-211,-207}	{-102,-99}	{-5,207}	{146,147}	
{-2534,-189}	{-439,-189}	{-210,-207}	{-101,-99}	{-4,207}	{147,147}	
{-2533,-189}	{-438,-189}	{-209,-207}	{-100,-99}	{-3,207}	{154,189}	
{-2532,-189}	{-437,-189}	{-208,-207}	{-99,-99}	{-2,213}	{155,189}	
{-2531,-189}	{-436,-189}	{-207,-207}	{-98,147}	{-1,213}	{156,189}	
{-2530,-189}	{-435,-189}	{-206,-171}	{-97,147}	{0,213}	{157,189}	
{-2529,-189}	{-434,-189}	{-205,-171}	{-96,147}	{1,213}	{158,189}	
{-2510,-189}	{-433,-189}	{-204,-171}	{-95,147}	{2,213}	{159,189}	
{-2509,-189}	{-432,-189}	{-203,-171}	{-94,147}	{3,213}	{160,171}	
{-2508,-189}	{-431,-189}	{-202,-171}	{-93,147}	{10,21}	{161,171}	
{-2507,-189}	{-430,-189}	{-201,-171}	{-92,-87}	{11,21}	{162,171}	
{-2506,-189}	{-429,-189}	{-200,-189}	{-91,-87}	{12,21}	{163,171}	
{-2505,-189}	{-410,-189}	{-199,-189}	{-90,-87}	{13,21}	{164,171}	
{-2504,-189}	{-409,-189}	{-198,-189}	{-89,-87}	{14,21}	{165,171}	

$n-2$ is an obvious upper bound for s since we want $\left|j_k^s\right| \le n-2$. However, the lower bound is not so obvious. Moreover there are curious gaps of values for s which give $\left|j_k^s\right| \le n-2$ which we can see in *Table 2* where the gap is between $s=-440$ and $s=-2103$. It will be shown that these gaps can be predicted and that a lower bound for s, such that $\left|j_k^s\right| \le n-2$, can be found. However, first we must show that if $j_{i-1}^s \equiv \pm n(\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \equiv \pm n(\text{mod } p_i)$ (or in other words $j_i^s = j_{i-1}^s + 2\alpha_{i-1}$) then $2n+\alpha_{i-1} \equiv 0(\text{mod } p_i)$ or $2n-\alpha_{i-1} \equiv 0(\text{mod } p_i)$. Suppose $j_{i-1}^s \equiv \pm n(\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \equiv \pm n(\text{mod } p_i)$. Then either $j_{i-1}^s \equiv n(\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \equiv -n(\text{mod } p_i)$ or $j_{i-1}^s \equiv -n(\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \equiv n(\text{mod } p_i)$. If $j_{i-1}^s \equiv n(\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \equiv -n(\text{mod } p_i)$, then $n \equiv -n-\alpha_{i-1}(\text{mod } p_i)$ and $2n+\alpha_{i-1} \equiv 0(\text{mod } p_i)$. If $j_{i-1}^s \equiv -n(\text{mod } p_i)$ and $j_{i-1}^s + \alpha_{i-1} \equiv n(\text{mod } p_i)$, then $-n \equiv -\alpha_{i-1} + n(\text{mod } p_i)$ and $2n-\alpha_{i-1} \equiv 0(\text{mod } p_i)$. Therefore $2n+\alpha_{i-1} \equiv 0(\text{mod } p_i)$ or $2n-\alpha_{i-1} \equiv 0(\text{mod } p_i)$.

Again, let us look at $n=250$. Observe the following table:

Table 4 $n=250$, $k=\pi\left(\sqrt{2\cdot 250}\right)=8$.

i	$2n+\alpha_{i-1}(\text{mod } p_i)$	$2n-\alpha_{i-1}(\text{mod } p_i)$
1	1	1
2	1	0
3	1	4
4	5	1
5	6	4
6	2	10
7	15	16
8	5	7

So then for $n = 250$, $j_i^s = j_{i-1}^s + 2\alpha_{i-1}$ is only possible for $i = 2$. Then j_k^s equals a sum comprising of the terms s, α_0, α_1 or $2\alpha_1$, α_2, α_3, α_4, α_5, α_6, and α_7. The maximum value for j_k^s involving only the terms s, α_0, α_1 or $2\alpha_1$, α_2, α_3, and α_4, is $s + 1 + 2(2) + 6 + 30 + 210 = s + 251$. Thus in order for $\left|j_k^s\right| \leq n - 2 = 248$, $|s + 251| \leq 248$ or $s \geq -499$. The next largest possible value for j_k^s is $s + \alpha_5 = s + 2310$. If $j_k^s = s + 2310$ then in order for $\left|j_k^s\right| \leq 248$, $|s + 2310| \leq 248$ or $s \leq -2062$. Thus there is a gap $[-2061, 450]$ where for all $s \in [-2061, 450]$, $\left|j_k^s\right| > 248$. If we look at *Table 3* we see that this true. Similarly, when considering maximum values for j_k^s involving only the terms s, α_0, α_1 or $2\alpha_1$, α_2, α_3, α_4, and α_5, we find that for $s \in [-29781, -2810]$, $\left|j_k^s\right| > 248$ which again is verified by *Table 3*. The lower bound can be found by looking at the maximum possible value for j_k^s which is $j_k^s = s + \alpha_0 + 2\alpha_1 + 1\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = s + 543101$. So then if $s < -543349$, then $\left|j_k^s\right| > 248$. This method can be used for other values of n.

A Counting Approach

Another way to try and show that there always exists a j such that $|j| \leq n - 2$ (and therefore $n + j$ and $n - j$ are prime) is by using counting methods. For a given $n \geq 2$, let $k = \pi\left(\sqrt{2n}\right)$. We want j such that $j \not\equiv \pm n \pmod{p_i}$ for $1 \leq i \leq k$. Then for each i there is either one or two "bad" values for j modulo p_i. More specifically, if $n \equiv 0 \pmod{p_i}$ then there is one bad value and if $n \not\equiv 0 \pmod{p_i}$ there are two bad values.

The maximum number of solutions for j that give $0 \leq j \leq n - 2$ is $(n - 2) + 1 = n - 1$. The idea is to cut out the bad values modulo p_i starting with $p_1 = 2$ for $1 \leq i \leq k$ and hopefully being left with a positive amount of good values. Let's look at $n = 100$. Then $k = \pi\left(\sqrt{2n}\right) = \pi\left(\sqrt{2 \cdot 100}\right) = 6$ and the maximum number of solutions for j is $100 - 1 = 99$. Observe that $100 \equiv 0 \pmod 2$. So we want to cut all j where

$j \equiv 0 (\mathrm{mod}\, 2)$. This is at most $\left\lceil 1 \cdot \frac{99}{2} \right\rceil = 50$. This leaves j where $j \equiv 1 (\mathrm{mod}\, 2)$. From these we cut the bad values modulo 3. Observe that $\pm 100 \equiv 1{,}2 (\mathrm{mod}\, 3)$. So then we cut from the remaining values, values of j such that $j \equiv 1 (\mathrm{mod}\, 2)$ and $j \equiv 1{,}2 (\mathrm{mod}\, 3)$ which is at most $\left\lceil 1 \cdot 2 \cdot \frac{99}{2 \cdot 3} \right\rceil = 33$. This leaves j where $j \equiv 1 (\mathrm{mod}\, 2)$ and $j \equiv 0 (\mathrm{mod}\, 3)$. From these we cut the bad values modulo 5. Observe that $\pm 100 \equiv 0 (\mathrm{mod}\, 5)$. So then we cut from the remaining values, values of j such that $j \equiv 1 (\mathrm{mod}\, 2)$, $j \equiv 0 (\mathrm{mod}\, 3)$, and $j \equiv 0 (\mathrm{mod}\, 5)$ which is at most $\left\lceil 1 \cdot 1 \cdot 1 \cdot \frac{99}{2 \cdot 3 \cdot 5} \right\rceil = 4$. This method is continued all the way up to $p_k = p_6 = 13$ where find that there are at least 4 good values for j. Therefore there are at least four ways to write $2 \cdot 100 = 200$ as the sum of two primes (there are in fact 8 ways.) In general, we find the following:

For $n \geq 2$,

(3) Let $G(n)$ be defined as the number of distinct ways $2n$ can be written as the sum of two primes (for example, $G(5) = 2$.)

(4) Let c_i^n be defined as

$$c_i^n = \begin{cases} 1, \text{ If } p_i \mid n \text{ or } i = 1 \\ 2, \text{ If } p_i \nmid n \end{cases}$$

(5) Let $$H(n) = (n-1) - \left\lceil \frac{n-1}{2} \right\rceil - \sum_{i=2}^{k} \left\lceil \frac{c_i^n \cdot (n-1) \cdot \prod_{j=1}^{i-1} (p_j - c_j^n)}{\alpha_i} \right\rceil$$

Then $G(n) \geq H(n)$.

Table 4 $\{n, H(n), G(n)\}$, $2 \le n \le 300$.

{2,0,1}	{42,5,8}	{82,2,5}	{122,2,9}	{162,11,20}	{202,5,11}	{242,6,14}	{282,15,24}
{3,1,1}	{43,2,5}	{83,2,6}	{123,9,16}	{163,5,7}	{203,5,13}	{243,15,23}	{283,7,13}
{4,1,1}	{44,1,4}	{84,9,13}	{124,3,6}	{164,4,10}	{204,13,20}	{244,5,9}	{284,6,13}
{5,0,2}	{45,7,9}	{85,3,9}	{125,5,9}	{165,19,24}	{205,8,13}	{245,11,19}	{285,26,31}
{6,1,1}	{46,2,4}	{86,1,6}	{126,13,16}	{166,5,6}	{206,5,11}	{246,16,22}	{286,8,11}
{7,1,2}	{47,1,5}	{87,6,11}	{127,4,9}	{167,4,11}	{207,13,21}	{247,7,13}	{287,7,16}
{8,0,2}	{48,5,7}	{88,3,7}	{128,3,8}	{168,14,19}	{208,6,10}	{248,4,13}	{288,16,26}
{9,2,2}	{49,3,3}	{89,2,7}	{129,9,14}	{169,5,9}	{209,6,11}	{249,15,23}	{289,7,12}
{10,1,2}	{50,3,6}	{90,9,14}	{130,7,10}	{170,7,13}	{210,23,30}	{250,10,13}	{290,8,19}
{11,1,3}	{51,6,8}	{91,4,6}	{131,4,9}	{171,12,17}	{211,7,11}	{251,5,15}	{291,17,25}
{12,3,3}	{52,2,5}	{92,1,8}	{132,12,16}	{172,5,10}	{212,4,12}	{252,20,27}	{292,7,12}
{13,1,3}	{53,2,6}	{93,6,13}	{133,6,8}	{173,4,9}	{213,13,21}	{253,8,15}	{293,7,13}
{14,0,2}	{54,5,8}	{94,2,5}	{134,4,9}	{174,11,16}	{214,5,9}	{254,5,14}	{294,19,29}
{15,3,3}	{55,4,6}	{95,3,8}	{135,15,19}	{175,10,13}	{215,7,14}	{255,25,32}	{295,10,16}
{16,1,2}	{56,3,7}	{96,7,11}	{136,5,7}	{176,5,10}	{216,14,19}	{256,6,11}	{296,6,15}
{17,0,4}	{57,6,10}	{97,3,7}	{137,4,11}	{177,11,20}	{217,6,13}	{257,5,14}	{297,20,27}
{18,2,4}	{58,3,6}	{98,2,9}	{138,11,16}	{178,4,9}	{218,5,11}	{258,17,23}	{298,7,12}
{19,1,2}	{59,3,6}	{99,8,13}	{139,5,7}	{179,4,10}	{219,13,21}	{259,9,11}	{299,9,15}
{20,1,3}	{60,10,12}	{100,4,8}	{140,9,14}	{180,18,22}	{220,10,14}	{260,11,17}	{300,25,32}
{21,3,4}	{61,3,4}	{101,3,9}	{141,12,16}	{181,4,8}	{221,7,13}	{261,17,24}	
{22,1,3}	{62,1,5}	{102,8,14}	{142,4,8}	{182,5,14}	{222,14,21}	{262,6,11}	
{23,1,4}	{63,7,10}	{103,4,7}	{143,6,12}	{183,10,18}	{223,6,12}	{263,6,15}	
{24,3,5}	{64,2,3}	{104,3,7}	{144,11,17}	{184,3,8}	{224,6,13}	{264,18,25}	
{25,2,4}	{65,2,7}	{105,15,19}	{145,6,10}	{185,4,14}	{225,20,27}	{265,8,14}	
{26,0,3}	{66,7,9}	{106,4,6}	{146,3,8}	{186,11,18}	{226,5,12}	{266,8,17}	
{27,2,5}	{67,3,6}	{107,2,8}	{147,11,19}	{187,5,10}	{227,4,12}	{267,15,22}	
{28,1,3}	{68,2,5}	{108,8,13}	{148,4,8}	{188,3,11}	{228,15,24}	{268,6,13}	
{29,1,4}	{69,6,8}	{109,3,7}	{149,4,11}	{189,13,22}	{229,5,9}	{269,6,14}	
{30,4,6}	{70,4,7}	{110,5,9}	{150,14,21}	{190,6,13}	{230,9,16}	{270,21,30}	
{31,2,3}	{71,3,8}	{111,9,11}	{151,5,9}	{191,4,10}	{231,21,28}	{271,7,10}	
{32,0,5}	{72,6,11}	{112,4,7}	{152,3,10}	{192,12,19}	{232,5,12}	{272,6,13}	
{33,3,6}	{73,3,6}	{113,2,7}	{153,12,15}	{193,5,12}	{233,4,13}	{273,21,30}	
{34,1,2}	{74,2,5}	{114,8,12}	{154,5,8}	{194,4,9}	{234,15,24}	{274,6,11}	
{35,2,5}	{75,10,12}	{115,6,9}	{155,6,12}	{195,19,27}	{235,10,15}	{275,10,19}	
{36,4,6}	{76,3,4}	{116,2,7}	{156,12,17}	{196,5,11}	{236,4,13}	{276,17,23}	
{37,1,5}	{77,3,8}	{117,9,15}	{157,4,9}	{197,4,11}	{237,14,23}	{277,7,11}	
{38,0,5}	{78,7,11}	{118,3,9}	{158,3,10}	{198,13,21}	{238,9,14}	{278,6,11}	
{39,3,7}	{79,2,5}	{119,5,9}	{159,10,15}	{199,5,7}	{239,5,11}	{279,16,23}	
{40,2,4}	{80,3,8}	{120,13,18}	{160,6,11}	{200,6,14}	{240,22,29}	{280,11,18}	
{41,1,5}	{81,7,10}	{121,5,8}	{161,5,11}	{201,12,17}	{241,6,11}	{281,7,14}	

Table 5 $\{n, H(n), G(n)\}$, $10000 \le n \le 11000$ in increments of 10.

{10000,200,231}	{10150,258,283}	{10300,210,228}	{10450,250,272}	{10600,221,229}	{10750,223,239}	{10900,225,239}
{10010,301,329}	{10160,205,232}	{10310,208,232}	{10460,211,233}	{10610,216,223}	{10760,221,226}	{10910,224,248}
{10020,422,443}	{10170,432,467}	{10320,447,464}	{10470,443,456}	{10620,456,491}	{10770,455,478}	{10920,610,635}
{10030,223,239}	{10180,203,220}	{10330,209,219}	{10480,214,231}	{10630,216,237}	{10780,295,318}	{10930,223,232}
{10040,201,223}	{10190,202,239}	{10340,241,249}	{10490,213,235}	{10640,277,297}	{10790,242,266}	{10940,223,237}
{10050,430,460}	{10200,458,477}	{10350,459,480}	{10500,533,547}	{10650,457,475}	{10800,458,470}	{10950,469,473}
{10060,201,225}	{10210,205,225}	{10360,260,273}	{10510,214,234}	{10660,242,248}	{10810,237,254}	{10960,224,232}
{10070,222,237}	{10220,252,274}	{10370,228,247}	{10520,213,249}	{10670,245,266}	{10820,222,244}	{10970,222,237}
{10080,511,527}	{10230,498,524}	{10380,437,460}	{10530,485,490}	{10680,457,450}	{10830,483,502}	{10980,470,487}
{10090,203,219}	{10240,206,234}	{10390,211,231}	{10540,238,254}	{10690,217,232}	{10840,222,228}	{10990,271,300}
{10100,206,240}	{10250,215,233}	{10400,230,260}	{10550,216,233}	{10700,220,240}	{10850,279,289}	{11000,248,272}
{10110,426,455}	{10260,459,483}	{10410,438,453}	{10560,497,521}	{10710,581,605}	{10860,458,467}	
{10120,241,259}	{10270,233,255}	{10420,212,232}	{10570,257,279}	{10720,223,241}	{10870,221,249}	
{10130,204,221}	{10280,209,227}	{10430,257,276}	{10580,225,255}	{10730,235,255}	{10880,235,256}	
{10140,465,497}	{10290,524,554}	{10440,458,457}	{10590,449,481}	{10740,454,476}	{10890,510,527}	

The values in bold in *Table 4* are the values of n where $H(n)=0$. Notice the last of these values is $n=38$. Also, $H(n)$ is a fairly close lower bound of $G(n)$ that grows as $G(n)$ grows. From this the following conjecture is made:

<u>Conjecture 2</u>: For $n>38$, $H(n)>0$. And thus since $G(n)\geq H(n)$, this implies Goldbach's conjecture is true for $n>38$.

Conclusions

The purpose of this research project was to approach Goldbach's Conjecture by using the equivalent statement: For every integer $n\geq 2$, there exists an integer j such that $n+j$ and $n-j$ are prime numbers. Two approaches were by using this statement which resulted in two conjectures.

The first approach was creating the sequence j_i^s to generate a j, namely j_k^s for a given integer $n\geq 2$. This sequence was interesting in that it had a few irregularities, particularly that values for j_k^s were often repeated for succeeding values of s, the distinct values of j_k^s could vary greatly in size, and also that there were curiously large gaps between groups of values of s which gave $\left|j_k^s\right|\leq n-2$ (the bound which determined whether $n\pm j_k^s$ are prime numbers. The repeated values and gaps between groups of values were explained. And finally, it was conjectured that for all integers $n\geq 2$ there exists an integer s such that $\left|j_k^s\right|\leq n-2$ where j_i^s is as defined in *(1)*. Thus implying that $n\pm j_k^s$ are prime numbers. It follows from this that Goldbach's conjecture is true since $\left(n+j_k^s\right)+\left(n-j_k^s\right)=2n$.

The second approach was counting or "cutting out" the "bad" integer values from 0 to $n-2$ and hopefully being left with a positive number of "good" values thereby proving Goldbach's conjecture for that particular n. This resulted in a lower bound for $G(n)$, namely $H(n)$. This lower bound turned out to actually be generally very close to the value of $G(n)$, but more importantly $H(n)$ seemed to be greater than 0 for $n > 38$. It was then conjectured that for $n > 38$, $H(n) > 0$. And thus since $G(n) \geq H(n)$, this implies Goldbach's conjecture is true for $n > 38$.

The conjectures made in this paper are essentially two equivalent statements to Goldbach's conjecture. Hopefully these conjectures reveal new approaches to solving the problem and to make solving this old problem a much easier task.

Bibliography

[1]. Mahoney, Michael S. "Goldbach, Christian." *Dictionary of Scientific Biography*. Ed. Charles Coulston Gillipsie. New York: Scribner 1970-1980.

[2]. Yuan, Wang. *Goldbach Conjecture.* World Scientific Publishing, 1984.

Appendix

Function for calculating Primorials:

```
PrimeFact[0]=1;
PrimeFact[n_]:=Prime[n]*PrimeFact[n-1]
```

Function for calculating j_k^s:

```
js[n_,s_]:=(k=PrimePi[Sqrt[2n]];si=s;i=0;
    While[i<k,a=Mod[n,Prime[i+1]];b=Mod[-
n,Prime[i+1]];If[Mod[si,Prime[i+1]]≠a &&Mod[si,Prime[i+1]]≠
b,u=0,If[Mod[si+PrimeFact[i],Prime[i+1]]≠a
&&Mod[si+PrimeFact[i],Prime[i+1]]≠
b,u=1,u=2]];si=si+(u*PrimeFact[i]);i++];Return[si])
```

Function for calculating $G(n)$:

```
Gn[n_]:=(s=0;i=1;m=PrimePi[n];While[i≤m,If[PrimeQ[2n-
Prime[i]],s++];i++];Return[s])
```

Functions for calculating $H(n)$:

```
cni[n_,i_]:=(c=0;If[Mod[n,Prime[i]] ≠0||i=1,c=1,c=2];Return[c])

GnEst2[n_]:=(k=PrimePi[Sqrt[2n]];sum=Ceiling[(n-
1)/2];Do[s=1;t=1;Do[t=t*(Prime[j]-cni[n,j]),{j,1,i-
1}];s=Ceiling[(t*cni[n,i]*(n-
1))/PrimeFact[i]];sum=sum+s,{i,2,k}];est=n-1-sum;Return[est])
```