ON THE ENUMERATION OF CYCLE-EQUIVALENT CLASSES OF SEQUENTIAL DYNAMICAL SYSTEMS

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ABSTRACT. Sequential dynamical systems (SDSs) is a class of dynamical systems that can be used to model many complex systems. Two SDSs are cycle equivalent if their periodic orbits are isomorphic as directed graphs. This means that the two systems have similar long-term dynamics. This paper approaches the problem of applying the vertex functions sequentially to the states of the vertices in a given graph Y, using an update order instead of applying all the vertex functions to Y in parallel. Three forms of equivalence for SDSs are functional equivalence, dynamical equivalence, and cycle equivalence. This paper describes upper bounds for the number of equivalence classes for each of these forms of equivalence, focusing mainly on cycle equivalence and its bound, $\bar{\kappa}$. This paper gives new results for the evaluation of $\bar{\kappa}$ for certain families of graphs.

1. BACKGROUND AND DEFINITIONS

1.1. **Basic Definitions.**

The following are basic definitions as given in [2]. *Sequential dynamical systems* (SDSs) are dynamical systems constructed from a finite undirected graph Y where each vertex has a state, a squence of vertex functions, and a word w over the vertex set of Y. The SDS *map* is constructed as the composition of the functions in the order specified by w. The *phase space* of the map $\phi\colon K^n \longrightarrow K^n$ is the directed graph $\Gamma(\phi)$ with vertex set K^n and edge set $\big\{(\mathbf{y}, \phi(\mathbf{y})) \mid \mathbf{y} \in K^n\big\}$. Two SDSs are *cycle equivalent* if their periodic orbits are isomorphic as directed graphs.

1.2. **Functional Equivalence.**

Two SDSs are *functionally equivalent* if their SDS maps are identical as functions. Let $\sim_Y$ be the equivalence relation on S_Y defined by $\pi \sim_Y \pi'$ iff π can be constructed from π' by using a series of transpositions of entries k and $k+1$ under the condition that $\{w_k, w_{k+1}\}$ is not an edge of Y. The set of equivalence classes is denoted as $S_Y/{\sim_Y}$.

Let $\mathsf{Acyc}(Y)$ denote the set of acyclic orientations of Y. In [4] it is shown that there is a bijection

$$f_Y\colon S_Y/{\sim_Y} \longrightarrow \mathsf{Acyc}(Y)\ . \tag{1.1}$$

A permutation $\pi \in S_Y$ defines a linear order on v[Y]r by $\pi_k = i <_\pi j = \pi_\ell$ iff $k < \ell$. This order defines an acyclic orientation O_Y^π where $O_Y^\pi(\{v, v'\})$ equals (v, v') if $v <_\pi v'$ and

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(v', v) otherwise. The map f_Y in (1.1) sends $[\pi]_Y \in S_Y/{\sim_Y}$ to O_Y^π. It follows that

$$\alpha(Y) = |\mathsf{Acyc}(Y)| \tag{1.2}$$

is a sharp upper bound for the number of functionally non-equivalent permutation SDSs that can be obtained by varying the update order, meaning that for any graph Y, there is a function that can be applied to the vertices of Y such that $alpha(Y)$ is exactly the number of functionally non-equivalent permutation SDSs.

1.3. Dynamical Equivalence.

Two finite dynamical systems ϕ, ψ are *dynamically equivalent* if there exists a bijection h such that

$$\phi \circ h = h \circ \psi\ . \tag{1.3}$$

The number of orbits $\bar{\alpha}(Y)$ under the action $\mathsf{Aut}(Y)$ on $S_Y/{\sim_Y}$ is an upper bound for the number of $\mathsf{Aut}(Y)$-invarient *SDS* maps up to dynamically equivalence. The number of orbits is given by

$$\bar{\alpha}(Y) = \frac{1}{|\mathsf{Aut}(Y)|} \sum_{\gamma \in \mathsf{Aut}(Y)} \alpha(\langle\gamma\rangle \backslash Y)\ , \tag{1.4}$$

where the orbit graph $\langle\gamma\rangle \backslash Y$ has vertices the vertex orbits of v$[Y]$ under $\langle\gamma\rangle$, and edges the edge orbits of e$[Y]$ under $\langle\gamma\rangle$. This bound is sharp for some graphs, but not known in general.

1.4. Cycle Equivalence.

Two finite dynamical systems are *cycle equivalent* if the restrictions of the maps to their respective sets of periodic points are dynamically equivalent. This is the same as if the sizes of periodic orbits and their frequency match.

In [2], it is shown that two SDS maps that differ only in a shift in the update order are cycle equivalent. The bijection in (1.1) identifies $O_Y^\pi \in \mathsf{Acyc}(Y)$ with an update order. By moving the first element to the last element of the update order, a new orientation of a source vertex O_Y^σ is constructed from O_Y^π by converting π_1 from a source to a sink. Such a conversion is called a *click*. This gives rise to an equivalence relation $\sim_\kappa$ on $\mathsf{Acyc}(Y)$. Two orientaions O_Y, $O'_Y \in \mathsf{Acyc}(Y)$ where O_Y can be transformed into O'_Y by a sequence of clicks are κ-equivalent. The number of κ-equivalence classes is $\kappa(Y)$ and is an upper bound for the number of cycle equivavalent permutation SDS maps. It is conjected that this bound is sharp, but this an open problem.

In [1], for any v $\in v[Y]$, every κ-class contains exactly one orientation such that v is the unique source.

The group $\mathsf{Aut}(Y)$ acts on $\mathsf{Acyc}(Y)/{\sim_\kappa}$. The number of orbits in $\mathsf{Acyc}(Y)/{\sim_\kappa}$ under $\mathsf{Aut}(Y)$ is denoted $\bar{\kappa}(Y)$, and by Burnside's Lemma

$$\bar{\kappa}(Y) = \frac{1}{|\mathsf{Aut}(Y)|} \sum_{\gamma \in \mathsf{Aut}(Y)} |\mathsf{Fix}(\gamma)|\ , \tag{1.5}$$

where $\mathsf{Fix}(\gamma) = \{[O_Y] : \gamma[O_Y] = [O_Y]\}$.

If Y has a unique degree vertex v then every automorphism γ fixes v. Also, each $[O_Y]$ of Y has a unique element where v is the unique source. Let O_Y^v be the unique element in $[O_Y]$ where v is the only source.

From [3], if $\gamma \in \mathsf{Aut}(Y)$ fixes a vertex v of Y then $[O_Y] \in \mathsf{Fix}(\gamma)$ if and only if $\gamma O_Y^v = O_Y^v$.

Example 1.1. In this example, we will look at the equivalence classes of $Y = \mathsf{Circ}_4$ with vertices labeled 1, 2, 3, and 4, as well as $Y' = Y$ plus an edge between vertices 2 and 4. The following are various values for Y: $\alpha(Y) = 14$, $\bar{\alpha}(Y) = 3$, $\kappa(Y) = 3$, and $\bar{\kappa}(Y) = 2$. The following are various values for Y': $\alpha(Y') = 18$, $\bar{\alpha}(Y') = 6$, $\kappa(Y') = 4$, and $\bar{\kappa}(Y') = 2$. A traversal of the $\bar{\kappa}$-classes for both Y and Y' is $\{(1234),(1243)\}$.

Example 1.2. In this example, we will focus on the graph $Y = Q_2^3$, which is the binary 3-cube. The following are various values of Y: $\alpha(Y) = 1862$, $\bar{\alpha}(Y) = 54$, $\kappa(Y) = 133$, and $\bar{\kappa}(Y) = 8$. A traversal of the $\bar{\kappa}$-classes for Y is $\{$(01243567), (10243567), (20314567), (23104567), (40236157), (31204567), (34012567), and (32015467)$\}$. This means that for any funcion on the vertices of Y, every update order will lead to the same cycle structure as one of these eight update orders. The following is the cycle structures for each of the eight update orders:

Using NOR functions, number of cycles of a given size

Update Order	2	3	5	7	11	13	15	17
(01243567)	6	1	4	0	0	0	0	0
(10243567)	6	0	2	0	0	1	0	0
(20314567)	8	0	1	2	0	0	0	0
(23104567)	4	0	0	0	2	0	0	0
(40236157)	4	0	2	0	0	0	0	1
(31204567)	4	2	2	0	1	0	0	0
(34012567)	4	0	4	1	0	0	0	0
(32015467)	0	0	4	0	0	0	1	0

The implication of this list is that each $\bar{\kappa}$-equivalence class gives a unique periodic orbit configuration, which in turn implies that the bound $\kappa(Y)$ is sharp for Q_2^3.

2. THE BOUND $\bar{\kappa}$ FOR SPECIAL FAMILIES OF GRAPHS

Where Y and Y' are graphs such that $Y \supset Y'$, let $Y \ominus Y'$ be the graph such that it has the vertex set v$[Y]\backslash$v$[Y']$ and the edges $\{$v,v$'\}$ where both v$\in$v$[Y]$ and v$'$ $\in$v$[Y]$.

Theorem 2.1. *For graphs Y and Y' such that $Y \supset Y'$, if every $\gamma \in \mathsf{Aut}(Y)$ fixes Y' and $|\mathsf{Aut}(Y)|$ equals $|\mathsf{Aut}(Y \ominus Y')|$, then $\bar{\kappa}(Y) = \bar{\alpha}(Y \ominus Y')$.*

Proof. We have

$$\bar{\kappa}(Y) = \frac{1}{|\mathsf{Aut}(Y)|} \sum_{\gamma \in \mathsf{Aut}(Y)} |\mathsf{Fix}(\gamma)| \, ,$$

where $\mathsf{Fix}(\gamma) = \{[O_Y] : \gamma[O_Y] = [O_Y]\}$. Let v be a vertex in Y'. Because all $\gamma \in \mathsf{Aut}(Y)$ fix v, we conclude $\mathsf{Fix}(\gamma) = \{O_Y^v : \gamma O_Y^v = O_Y^v\}$.

Because every $\gamma \in \mathsf{Aut}(Y)$ fixes Y', there is a mapping $\theta : \gamma \in \mathsf{Aut}(Y) \longrightarrow \gamma' \in \mathsf{Aut}(Y \ominus Y')$ such that γ' maps any vertex in $Y \ominus Y'$ to the same vertex as γ. Therefore, θ is injective, and since $|\mathsf{Aut}(Y)| = |\mathsf{Aut}(Y \ominus Y')|$, θ is surjective meaning there are no automorphisms of $Y \ominus Y'$ that cannot also be applied as an automorphism to Y. Simply, a $\gamma \in \mathsf{Aut}(Y)$ fixes Y if and only if $\theta(\gamma)$ fixes $Y \ominus Y'$, and a $\gamma \in \mathsf{Aut}(Y \ominus Y')$ fixes $Y \ominus Y'$ if and only if $\theta^{-1}(\gamma)$ fixes Y.

Therefore, for a $\gamma \in \mathsf{Aut}(Y)$, $|\mathsf{Fix}(\gamma)| = \alpha(\langle\theta(\gamma)\rangle \backslash (Y \ominus Y'))$.

Now we can say,

$$\bar{\kappa}(Y) = \frac{1}{|\mathsf{Aut}(Y)|} \sum_{\gamma \in \mathsf{Aut}(Y)} |\mathsf{Fix}(\gamma)| = \frac{1}{|\mathsf{Aut}(Y \ominus Y')|} \sum_{\gamma \in \mathsf{Aut}(Y \ominus Y')} \alpha(\langle\gamma\rangle \backslash (Y \ominus Y')) = \bar{\alpha}(Y \ominus Y'),$$

□

where Y is a graph and v is a vertex in Y, $Y \ominus v$ creates a new graph with a vertex set v[Y]$\backslash$ v and an edge set that contains only {v,v′}∈e[Y] such that $v \neq$v and $v \neq$v′.

Corollary 2.2. *If Y has a unique vertex v adjacent to all other vertices in Y, then $\bar{\kappa}(Y) = \bar{\alpha}(Y \ominus v)$.*

Proof. In Theorem 2.1 let Y' be v. Since v is a unique degree vertex in Y, every $\gamma \in \mathsf{Aut}(Y)$ fixes v. Because $Y \ominus v$ has no vertex with an edge to all other vertices in $Y \ominus v$, $|\mathsf{Aut}(Y)| = |\mathsf{Aut}(Y \ominus v)|$ so the conditions in Theorem 2.1 hold true. □

Where Y is a graph and v is a vertex not in Y, the operation $Y \oplus v$ creates a new graph with a vertex set $v \cup$ v[Y] and an edge set that contains e[Y] as well as {v,v′} for every v′ ∈v[Y].

Example 2.3. Using the graphs from Example 1.1, we can find that $\bar{\kappa}(Y \oplus v) = \bar{\alpha}(Y) = 3$ and $\bar{\kappa}(Y' \oplus v) = \bar{\alpha}(Y') = 6$.

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