

## Algebra Preliminary Exam Syllabus

In general, the syllabus is determined by the material covered in the course MATH 5125/6, Abstract Algebra, in the year which the student takes the course. This means that the syllabus will depend on the instructor teaching the course that year; however, the following topics are nearly always examined.

### Groups

- Fundamental isomorphism theorem
- Groups acting on sets
- Alternating and symmetric groups
- Sylow Theorems
- Finite simple groups
- Finitely generated abelian groups

### Rings and Fields

- Polynomial rings
- PID's and UFD's
- Prime and maximal ideals
- Field extensions, normal and Galois extensions, separability
- Computing Galois groups

### Modules

- Isomorphism theorems
- Bases, direct sums, free modules, simple modules
- Modules over a PID

**Additional topics** may include:

- Nilpotent and solvable groups
- Free groups, generators and relations
- Noetherian rings and modules
- Hilbert Basis Theorem
- Hilbert Nullstellensatz

- Power series rings
- Projective and injective modules
- Nakayama's lemma
- Jacobson radical
- Artinian rings and modules
- Semisimple rings and Wedderburn structure theorem
- Jordan and rational canonical forms
- Representation theory of finite groups and character theory
- Tensor products

**Books** The following have been used for MATH 5125/6, and each covers most of the above material at the right level.

- "Algebra" by Michael Artin, Prentice Hall, 1991, ISBN 0-13-004763-5, MR: 92g:00001
- "Algebra" by Thomas W. Hungerford, Springer-Verlag, 1980, ISBN 0-387-90518-9, MR: 50 #6693
- "Algebra" (third edition) by Serge Lang, Addison Wesley, 1993, ISBN 0-201-55540-9
- "Basic Algebra" volumes I and II (second editions) by Nathan Jacobson, Freeman, 1985 and 1989, ISBN 0-7167-1480-9 and 0-7167-1079-X, MR: 86d:00001 and MR: 90m:00007
- "Algebra, a graduate course" by I. Martin Isaacs, Brooks/Cole, 1994, ISBN 0-534-19002-2, MR: 95k:00003

**Other Books** The following are good for various sections of MATH 5125/6, but do not cover the whole syllabus.

- "Algebra, a Module Theoretic Approach" by William A. Adkins and Steven H. Weintraub, Graduate Texts in Mathematics no. 136, Springer-Verlag, 1992, ISBN 0-387-97839-9, MR 94a:00001
- "Introduction to Commutative Algebra" by M. F. Atiyah and I. G. Macdonald, Addison-Wesley, 1969, MR: 39 #4129
- "Undergraduate Commutative Algebra" by Miles Reid, London Math. Soc. Student Texts no. 29, Cambridge University Press, 1995, ISBN 0-521-45889-7

- “An Introduction to the Theory of Groups” by Joseph J. Rotman, Graduate Texts in Mathematics no. 148, Springer-Verlag, 1995, ISBN 0-387-94285-8, MR: 95m:20001
- “Rings, Modules and Linear Algebra” by Brian Hartley and Trevor O. Hawkes, Chapman & Hall, 1980, ISBN 0-412-09810-5, MR: 42 #2897
- “Field theory and its classical problems” by Charles Robert Hadlock, Carus Mathematical Monographs no. 19, Mathematical Association of America, 1978, ISBN 0-88385-020-6, MR: 82c:12001
- “Abstract Algebra” by David S. Dummit and Richard M. Foote, Prentice Hall Inc., 1991, ISBN 0-13-004771-6, MR: 92k:00007

## Algebra Preliminary Exam, Spring 1980

1. Let  $A, B$  and  $C$  be finite abelian groups. If  $A \oplus C \cong B \oplus C$ , prove that  $A \cong B$ .
2. Show that there exists no simple group of order 56.
3. Let  $T$  denote the set of all  $5 \times 5$  matrices with eigenvalues 4, 4, 17, 17, 17. Define a relation  $\sim$  on  $T$  by  $M_1 \sim M_2$  if  $M_1$  and  $M_2$  are similar matrices. How many equivalence classes does  $T$  have? Justify your answer. (Assume that the matrices are over  $\mathbb{C}$ .)
4. Give an example of a unique factorization domain (UFD) which is not a principal ideal domain (PID).
5. What is the Galois group of  $x^3 - 10$  over  $\mathbb{Q}$ ? Find all normal subfields of the splitting field.
6. Recall: if  $E$  is the splitting field of a polynomial  $f$  over  $F$ , then  $\text{Gal}(E/F)$  is called the Galois group of  $f$  over  $F$ . The Galois group of  $f$  over  $F$  is said to be transitive if given any two roots  $r_1$  and  $r_2$  of  $f$  in  $E$ , there exists  $\sigma$  in  $\text{Gal}(E/F)$  with  $\sigma(r_1) = r_2$ .
  - (a) Prove that if  $f$  is a separable irreducible polynomial, then the Galois group of  $f$  is transitive.
  - (b) Show that even though the Galois group of  $f$  is transitive, not every permutation of the roots need occur. (Hint: consider  $x^4 - 2$  over  $\mathbb{Q}$ .)
7. Let  $A$  be a local ring with maximal ideal  $\mathfrak{M}$ , let  $k$  be  $A/\mathfrak{M}$ , and let  $M$  be a finitely generated  $A$ -module. Show that if  $\text{Hom}_A(M, k) = 0$ , then  $M = 0$ . (Hint: use Nakayama's lemma.)

## Algebra Preliminary Exam, Fall 1980

1. Suppose that for each prime integer  $p$  dividing the order of a finite group  $G$ , there is a subgroup of index  $p$ . Prove that  $G$  cannot be a nonabelian simple group.
2. A subgroup of a finite group is “ $p$ -local” if it is the normalizer of some Sylow  $p$ -subgroup. Show that the number of  $p$ -local subgroups of a group is congruent to 1 modulo  $p$ .
3. Characterize (with proof) all finitely generated  $\mathbb{Q}x$ -modules with the property that each submodule is a direct summand. ( $\mathbb{Q}$  denotes the field of rational numbers.)
4. Let  $R$  and  $S$  be local Noetherian integral domains with maximal ideals  $M$  and  $N$  respectively. Assume that  $R \subseteq S$  and that  $S$  is a finitely generated  $R$ -module. If there exists a proper ideal  $I$  of  $R$  such that  $I = IS$  in  $R$  and the canonical image of  $R/I$  in  $S/IS$  equals  $S/IS$ , then prove that  $R = S$ .
5. Let  $R$  be an integral domain. For  $x, y \in R$ , define  $xR : yR = \{r \in R \mid yr \in xR\}$ . Let  $\{P_\lambda\}_{\lambda \in \Lambda}$  be a set of prime ideals of  $R$  with the property that if  $x, y \in R$  and  $y \notin xR$ , then  $xR : yR = P_\lambda$  for some  $\lambda \in \Lambda$ . Prove that  $R = \bigcap_{\lambda \in \Lambda} R_{P_\lambda}$ . ( $R_{P_\lambda}$  denotes the localization at  $P_\lambda$ .)
6. If  $G$  is a finite group, prove that there exist fields  $K$  and  $L$  such that  $L$  is a Galois extension of  $K$  with Galois group isomorphic to  $G$ .
7. Show that for each positive integer  $n$ , there is a polynomial  $d \in \mathbb{C}[x_1, \dots, x_n]$  such that for each  $n \times n$  matrix  $A$  with complex entries,

$$\det A = d(\operatorname{tr}(A), \operatorname{tr}(A^2), \dots, \operatorname{tr}(A^n)).$$

( $\mathbb{C}$  denotes the field of complex numbers and  $\operatorname{tr}(A)$  denotes the trace of  $A$ .)

## Algebra Preliminary Exam, Spring 1981

1. Prove that a group of order  $p^n$ , where  $p$  is a prime and  $n \geq 1$ , has a nontrivial center.
2. Let  $G$  be a group of order  $pq$ , where  $p$  and  $q$  are primes and  $p < q$ . Prove that  $G$  has only one subgroup of order  $q$ .
3. (a) Show that if  $H$  is a subgroup of order 12 in a group  $G$  of order 36, then  $H$  is a normal subgroup of  $G$ . (Map  $G$  to the automorphisms of the set of right cosets of  $H$ .) NOTE: there are [examples](#) of groups of order 36 with subgroups of order 12 which are *not* normal.  
(b) Describe *all* abelian groups of order 36, up to isomorphism.
4. Prove that if  $p(x)$  is a polynomial irreducible over a field  $F$ , and if  $\alpha$  and  $\beta$  are roots of  $p(x)$  in some extension field of  $F$ , then  $F(\alpha)$  and  $F(\beta)$  are isomorphic. What happens if  $p(x)$  is reducible?
5. Let  $L$  be a normal extension of a field  $F$ , let  $G = \text{Gal}(L/F)$ , and let  $H$  be a subgroup of  $G$ . Prove that
  - (a) if  $K = \{x \in L \mid h(x) = x \text{ for all } h \in H\}$ , then  $K$  is a subfield of  $L$ ;
  - (b) if  $K$  is normal over  $F$ , then  $H$  is a normal subgroup of  $G$ .
6. (a) Let  $R$  be a commutative principal ideal domain (PID) with a 1. If  $a, b \in R$ , show that  $a$  and  $b$  have a greatest common divisor  $d \in R$  (i.e.  $d$  divides  $a$  and  $b$ , and if  $g$  divides  $a$  and  $b$ , then  $g$  divides  $d$ ).  
(b) Show that  $\mathbb{Z}[x]$  is not a PID.
7. Let  $R$  be a commutative ring with identity, and let  $I, J$  be ideals in  $R$ . Let  $IJ = \{x \in R \mid x = \sum_{i=1}^n a_i b_i \text{ for } a_i b_i \in I, n \in \mathbb{Z}^+\}$ .
  - (a) Show that  $IJ$  is an ideal of  $R$ .
  - (b) Show that if  $I + J = R$ , where  $I + J = \{a + b \mid a \in I, b \in J\}$ , then  $IJ = I \cap J$  (recall that  $R$  has a 1).
  - (c) Suppose further that  $R$  is a domain, and that  $IJ = I \cap J$  for all ideals  $I, J$  in  $R$ . Prove that  $R$  is a field. (Take principal ideals.)

## Algebra Preliminary Exam, Fall 1981

Instructions: do all eight problems

Notation:  $\mathbb{Z}$  = integers,  $\mathbb{Q}$  = rational numbers

- Let  $G$  be a finite group.
  - Let  $A$  and  $B$  be subgroups of  $G$  such that  $B \triangleleft G$  and  $AB = G$ . Prove that  $[A : A \cap B] = [G : B]$ .
  - Let  $H$  be a subgroup of  $G$  such that  $[G : H] = 2$  and let  $h \in H$ . If  $m$  is the number of conjugates of  $h$  in  $G$  and  $n$  is the number of conjugates of  $h$  in  $H$ , prove that either  $n = m$  or  $n = m/2$ .
- If the order of  $G$  is 105 and  $H$  is a subgroup of  $G$  of order 35, prove that  $H \triangleleft G$ .
- Let  $P$  be a nonnormal  $p$ -Sylow subgroup of the finite group  $G$ . If  $\mathbf{N}_G(P)$  is the normalizer of  $P$  in  $G$ , prove that  $\mathbf{N}_G(P)$  is nonnormal in  $G$ .
- Let  $F_1$  and  $F_2$  be finite fields of orders  $q_1$  and  $q_2$ .
  - Prove that  $q_i$  is a power of a prime, say  $q_i = p_i^{\alpha_i}$  for  $i = 1, 2$ .
  - If  $F_1 \subseteq F_2$ , prove that  $p_1 = p_2$  and that  $\alpha_1$  is a divisor of  $\alpha_2$ .
- Prove that  $x^4 - 2$  is irreducible over  $\mathbb{Q}$ .
  - Let  $K$  be the splitting field of  $x^4 - 2$ . Prove that the Galois group of  $K$  over  $\mathbb{Q}$ ,  $\text{Gal}(K/\mathbb{Q})$ , is of order 8.
  - Exhibit the correspondence (given by the Fundamental Theorem of Galois theory) between the subgroups of  $\text{Gal}(K/\mathbb{Q})$  and the intermediate fields between  $\mathbb{Q}$  and  $K$ .
- State Nakayama's lemma.
  - Let  $R$  be a local commutative ring with maximal ideal  $M$ . Let  $X$  be a finitely generated  $R$ -module. Show that if  $X/MX$  can be generated by  $n$  elements, then so can  $X$ .
- Construct an example of finitely generated nonzero abelian groups  $A$  and  $B$  so that  $\text{Hom}_{\mathbb{Z}}(A, B) = \text{Hom}_{\mathbb{Z}}(B, A) = A \otimes_{\mathbb{Z}} B = 0$ .
  - If  $A$  and  $B$  are finitely generated abelian groups such that  $\text{Hom}_{\mathbb{Z}}(A, B) \neq 0$  and  $\text{Hom}_{\mathbb{Z}}(B, A) = 0$ , prove that  $B \otimes_{\mathbb{Z}} \mathbb{Q} = 0$  and  $A \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$ .
- Let  $R$  be a commutative ring and let  $A$  be an ideal of  $R$ . Define the radical of  $A$ , denoted  $\sqrt{A}$ , by  $\sqrt{A} = \{r \in R \mid r^n \in A \text{ for some positive integer } n\}$ . You may assume that  $\sqrt{A}$  is an ideal of  $R$  and that  $A \subseteq \sqrt{A}$ .

- (a) If  $P$  is a prime ideal of  $R$  such that  $P \supseteq A$ , prove that  $P \supseteq \sqrt{A}$  and as a consequence show that  $\sqrt{P} = P$ .
- (b) Prove that  $\sqrt{A}/A$  is the set of nilpotent elements of  $R/A$ . (An element  $r$  is nilpotent if  $r^n = 0$  for some positive integer  $n$ .)



## Algebra Preliminary Exam, Fall 1982

Do all problems

1. Prove that a group of order  $135 = 3^3 \cdot 5$  has a normal subgroup of order 15.
2. (a) For a positive integer  $n$ , show that every ideal in  $\mathbb{Z}/(n)$  is principal.  
(b) Explain how one determines the number of ideals of  $\mathbb{Z}/(n)$  in terms of  $n$ .
3. (a) Calculate the Galois group of  $(x^2 - 2)(x^2 + 3)$  over  $\mathbb{Q}$ .  
(b) Explicitly state the correspondence between the subfields of the splitting field  $K$  of  $(x^2 - 2)(x^2 + 3)$  over  $\mathbb{Q}$  and the subgroups of  $\text{Gal}(K/\mathbb{Q})$ .
4. Let  $V$  and  $W$  be vector spaces over the field  $k$  and let  $W^*$  be the space of linear functions from  $W$  to  $k$ . Prove that the map  $\phi: V \otimes_k W^* \rightarrow \text{Hom}_k(W, V)$  defined by  $\phi(v \otimes f)(w) = f(w)v$  is
  - (a) Well defined.
  - (b) Linear.
5. Let  $K$  be a field of characteristic  $\neq 2$ . Suppose  $f(x) = p(x)/q(x)$  is a ratio of polynomials in  $K[x]$ . Prove that if  $f(x) = f(-x)$ , then there are polynomials  $p_0(x^2), q_0(x^2)$  such that  $f(x) = p_0(x^2)/q_0(x^2)$ . (HINT: look for a field automorphism of  $K(x)$  that fixes  $f(x)$ .)
6. Let  $G$  be a solvable group. Prove that if  $N \neq \{e\}$  is normal in  $G$  and contains no other non trivial subgroups which are normal in  $G$ , then  $N$  is abelian.
7. Let  $A, B, C, D$  be finite abelian groups such that  $A \times B \cong C \times D$  and  $B \cong D$ . Prove that  $A \cong C$ .

## Algebra Preliminary Exam, Spring 1984

Do all problems

1. Find the Galois group of  $x^6 - 1$  over  $\mathbb{Q}$ .
2. Show that a semisimple right Artinian ring without zero divisors is a division ring. (A ring has no zero divisors if  $ab = 0$  implies  $a = 0$  or  $b = 0$ .)
3. Show that there are no simple groups of order 300.
4. Let  $R$  be a commutative domain with field of fractions  $F$ . Prove that  $F$  is an injective  $R$ -module.
5. State and prove a structure theorem analogous to the Fundamental Theorem for modules over a PID, which describes finitely generated  $\mathbb{Z}/n\mathbb{Z}$ -modules. (You may assume the Fundamental Theorem for any argument.)
6. Assume that  $K/F$  is a Galois field extension and  $\alpha$  lies in an algebraic closure of  $K$ . Prove that  $|\text{Gal}(K/F)|$  divides  $|\text{Gal}(K(\alpha)/F(\alpha))| \deg \alpha$ , where  $\deg \alpha$  denotes the degree of  $\alpha$  over  $F$ .
7. Prove that a finite  $p$ -group with a unique subgroup of index  $p$  is cyclic. (Hint: first consider abelian  $p$ -groups.)
8. Let  $f: M \rightarrow N$  be a surjective homomorphism of left  $R$ -modules. Show that if  $P$  is a projective  $R$ -module, then the induced map  $\text{Hom}_R(P, M) \xrightarrow{f} \text{Hom}_R(P, N)$  is a surjection of abelian groups.
9. Let  $k$  be a field and let  $R = k[X_1, \dots, X_m]$  be the polynomial ring in  $m$  indeterminates. Prove that if  $M$  is a simple  $R$ -module, then  $\dim_k M = \infty$ .

## Algebra Preliminary Exam, Fall 1985

Do ALL problems

- Define what is meant by a prime ideal in a commutative ring.
  - Prove that a nonzero prime ideal in a principal ideal domain is always a maximal ideal.
- Prove that there is no simple group of order 56.
- Show that a finite field cannot be algebraically closed.
- Find all finitely generated abelian groups  $A$  with the property that for any subgroups  $B$  and  $C$ , either  $B \subseteq C$  or  $C \subseteq B$ .
- Let  $F$  be the splitting field of  $(x^3 - 2)(x^2 - 3)$  over  $\mathbb{Q}(i)$ . Describe the Galois group in as much detail as possible.
- Let  $R$  be an integral domain and let  $\theta$  be an element of the quotient field of  $R$ . Set  $I = \{r \in R \mid r\theta \in R\}$ .
  - Prove that  $I$  is an ideal of  $R$ .
  - Show that either  $\theta \in R$  or there exists a maximal ideal  $\mathfrak{M}$  of  $R$  such that  $\theta \notin R_{\mathfrak{M}}$ .
  - Conclude that  $R = \bigcap R_{\mathfrak{M}}$ , where the intersection is taken over all the maximal ideals of  $R$ .
- Prove that  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\mathbb{Q}$  are isomorphic as  $\mathbb{Z}$ -modules. (Here  $\mathbb{Q}$  denotes the rational numbers and  $\mathbb{Z}$  denotes the integers.)
- Let  $G$  be a group. Suppose that
  - $H$  and  $K$  are nilpotent groups,
  - there are homomorphisms  $\alpha: G \rightarrow H$  and  $\beta: G \rightarrow K$ , and
  - $\ker \alpha \cap \ker \beta \subseteq \mathbf{Z}(G)$ , the center of  $G$ .

Prove that  $G$  is nilpotent.

## Algebra Preliminary Exam, Spring 1986

1. Let  $F$  be the free group on the set  $X = \{x_i \mid i \in I\}$ , and let  $H$  be the normal subgroup of  $F$  generated by  $\{x_i x_j x_i^{-1} x_j^{-1} \mid i, j \in I\}$ . Prove that  $F/H$  is isomorphic to the free abelian group on  $X$ .
2. Let  $G$  be an abelian group of order  $p^6$ , and let  $H = \{x \in G \mid x^p = 1\}$ . Suppose that  $|H| = p^2$ . Give all possible such groups  $G$ .
3. Prove that  $S_4$  is solvable.
4. In  $S_5$ , how many Sylow subgroups of each type are there?
5. (a) Let  $R$  be a PID and let  $S$  be a multiplicatively closed subset. Show that  $S^{-1}R$  is a PID.  
(b) Give an example of a PID with exactly 3 nonassociate irreducible elements.
6. Let  $(M_\alpha)_{\alpha \in A}$  be the set of maximal ideals in a commutative ring  $R$  with identity. Set  $J = \bigcap_{\alpha \in A} M_\alpha$ . For  $r \in R$ , prove that  $r \in J$  if and only if  $1 + rs$  is a unit for all  $s \in R$ .
7. Let  $\alpha \in \mathbb{C}$  be a root of  $x^3 + 4x + 2$ .
  - (a) Find a basis for  $\mathbb{Q}(\alpha)$  over  $\mathbb{Q}$ . Justify your answer.
  - (b) Express  $(\alpha + 1)^{-1}$  in terms of the basis.
  - (c) Express  $(\alpha^2 + 3\alpha + 5)(2\alpha^2 - 4\alpha + 1)$  in terms of the basis.
8. Let  $K$  be a subfield of the field  $L$ , and let  $\alpha \in L$  such that  $[K(\alpha) : K]$  is odd. Prove that  $K(\alpha^2) = K(\alpha)$ .

## Algebra Prelim, Fall 1986

- Let  $G$  be a finite abelian (multiplicative) group. Prove that if  $G$  is not cyclic, then there exists a positive integer  $n$  such that  $n < |G|$  and  $g^n = e$  for all  $g \in G$ .
  - Prove that the multiplicative group of a finite field is cyclic.
- Let  $G$  be a group of order  $5 \cdot 7^2 \cdot 17$ , and let  $H$  be a subgroup of order  $7^2 \cdot 17$ .
  - Prove that  $H$  is abelian.
  - Prove that  $H$  is normal in  $G$ .
  - Prove that  $G$  is abelian.
- Let  $R$  be an integral domain. For an element  $a \in R$ , prove the equivalence of the following two statements.
  - There exists an infinite chain  $(a_1) \subset (a_2) \subset \cdots$  of principal ideals of  $R$  with  $a = a_1$ .
  - There exists an infinite set  $\{b_i \mid i = 1, 2, \dots\}$  of nonunits of  $R$  such that  $b_1 b_2 \dots b_n$  divides  $a$  for each positive integer  $n$ .
- Let  $R$  be a commutative ring with identity and let  $M$  be a maximal ideal of  $R$ .
  - Prove that  $R[x]/M[x] \cong (R/M)[x]$ .
  - Conclude that  $M[x]$  is a prime ideal but not a maximal ideal in  $R[x]$ . Indeed argue that there are infinitely many prime ideals of  $R[x]$  which contain  $M[x]$ .
- Let  $k \subseteq K \subseteq L$  be fields such that  $K$  is a splitting field over  $k$ . If  $\sigma \in \text{Gal}(L/k)$ , prove that  $\sigma(K) = K$ .
- Let  $K/k$  be a finite extension and let  $\alpha \in K$  with  $f(x) = \text{Irr}(\alpha, k)$ . If  $n = \deg f(x)$ , prove that  $n \mid [K : k]$ .
- Let  $f(x) = x^n - 1$  and let  $K$  be a splitting field for  $f(x)$  over  $\mathbb{Q}$ . Prove that the Galois group  $\text{Gal}(K/\mathbb{Q})$  is abelian.
- Let  $M$  be a module. A submodule  $S$  of  $M$  is *small* if whenever  $S + N = M$  for any submodule  $N$  of  $M$ , then  $N = M$ . Suppose  $S$  is small in  $M$  and there exists an epimorphism  $f: P \rightarrow M/S$  where  $P$  is projective. Prove that there exists an epimorphism  $\psi: P \rightarrow M$ .

## Algebra Preliminary Exam, Spring 1987

Do any 8 problems

- Let  $H$  be a subgroup of the finite group  $G$ , and let  $p$  be a prime. Prove that two distinct Sylow  $p$ -subgroups of  $H$  cannot lie in the same Sylow  $p$ -subgroup of  $G$ .
  - Let  $n$  be a positive integer and let  $R$  be a ring with a 1. Show that  $R$  has characteristic  $n$  if and only if  $R$  has a subring (with the same 1) isomorphic to  $\mathbb{Z}/n\mathbb{Z}$  (i.e. the integers modulo  $n$ ).
- Let  $A$  and  $B$  be normal subgroups of the group  $G$ .
  - Prove that  $A \cap B$  is a normal subgroup of  $G$ .
  - Prove that  $G/(A \cap B)$  is isomorphic to a subgroup of  $(G/A) \times (G/B)$ .
  - If  $G$  is finite and  $G/(A \cap B) \cong (G/A) \times (G/B)$ , prove that  $AB = G$ .
- Let  $G$  be a simple group with a subgroup  $H$  of index 6. Prove that there exists a homomorphism  $\phi: G \rightarrow S_6$ .
  - Prove there are no simple groups of order 300.
- Let  $R$  be the ring  $\{a + b\sqrt{29} \mid a, b \in \mathbb{Z}\}$  (so  $R = \mathbb{Z}[\sqrt{29}]$ ). Define  $N: R \rightarrow \mathbb{Z}$  by  $N(a + b\sqrt{29}) = a^2 - 29b^2$  for  $a, b \in \mathbb{Z}$ .
  - Show that for  $\alpha, \beta \in R$ ,  $N(\alpha\beta) = N(\alpha)N(\beta)$ .
  - Let  $\alpha \in R$ . Show that  $\alpha$  is a unit if and only if  $N(\alpha) = \pm 1$ .
  - Show that  $a^2 - 29b^2 = \pm 2$  has no solution with  $a, b \in \mathbb{Z}$ .
  - Show that  $2, -2, 5 + \sqrt{29}$  and  $5 - \sqrt{29}$  are irreducible elements of  $R$ .
  - Deduce that  $R$  is not a UFD.
- Let  $R$  be a UFD and let  $S$  be a multiplicatively closed subset of nonzero elements of  $R$ .
  - If  $u$  is irreducible in  $R$ , prove that  $u$  is either irreducible or a unit in  $S^{-1}R$ .
  - Prove that  $S^{-1}R$  is a UFD.
- Let  $V$  be a vector space over  $\mathbb{R}$  and let  $T: V \rightarrow V$  be a linear transformation. Describe how  $V$  can be made into an  $\mathbb{R}[x]$ -module via  $T$ .

Suppose  $V$  has basis  $(e_1, e_2, e_3)$  and  $T$  is the linear transformation defined by  $T(e_1) = 2e_1$ ,  $T(e_2) = -4e_2 - 4e_3$ , and  $T(e_3) = 9e_2 + 8e_3$ .

  - Express  $V$  as a direct sum of two nonzero  $\mathbb{R}[T]$ -modules.

- (b) Calculate  $(x^2 - 4x + 4)e_2$ .
- (c) If  $V \cong \bigoplus_{i=1}^n \mathbb{R}[x]/(f_i)$  where  $f_1 | f_2 | \dots | f_n$  and  $f_1$  is not a unit, what are the possibilities for the ideals  $(f_i)$ ?
- (d) Express  $V$  as a direct sum of cyclic modules.
- (e) Does there exist  $v \in V$  such that  $V = \mathbb{R}[x]v$ ?
7. Let  $R$  be a PID, let  $p$  be a prime of  $R$ , and let  $M$  be the  $R$ -module  $R/Rp^{e_1} \oplus \dots \oplus R/Rp^{e_n}$  where the  $e_i$  and  $n$  are positive integers. Define  $M(p) = \{m \in M \mid pm = 0\}$  and  $pM = \{pm \mid m \in M\}$ .
- (a) Prove that  $M(p)$  and  $pM$  are submodules of  $M$ .
- (b) Prove that  $M/pM \cong M(p)$ .
- (c) In the case  $R = \mathbb{Q}[x]$  and  $p = x^2 + 1$ , give an example of a finitely generated  $R$ -module  $M$  such that  $M/pM \not\cong M(p)$ .
8. Let  $R = \mathbb{Z} \times \mathbb{Z}$  and  $S = \mathbb{Z}^\# \times 0$ , where  $\mathbb{Z}$  is the ring of integers and  $\mathbb{Z}^\# = \mathbb{Z} \setminus 0$ . Prove that  $S^{-1}R \cong \mathbb{Q}$ , where  $\mathbb{Q}$  is the field of rational numbers.
9. List without repetition all the abelian groups of order  $3^2 2^3$ . Which ones are cyclic?

## Algebra Preliminary Exam, Fall 1987

Instructions: do all problems

1. Let  $G$  be a group with 56 elements. Prove that  $G$  is not simple.
2. Let  $G$  be a group and let  $S \leq G$ . Prove that  $\langle x^{-1}Sx \mid x \in G \rangle \triangleleft G$ . Now suppose that  $G = HA$  where  $H$  and  $A$  are subgroups and  $A$  is abelian. Prove that there exists  $K \triangleleft G$  such that  $H \cap A \subseteq K \subseteq H$ . Deduce that if  $G$  is nonabelian simple, then  $G = \langle H, x^{-1}Hx \rangle$  for all  $x \in G \setminus H$ .
3. Let  $G$  be the group  $\mathbb{C} \setminus 0$  with the operation multiplication. Define  $\theta: G \rightarrow G$  by  $\theta(x) = x^2$ . Prove that  $\theta$  is a group homomorphism,  $|\ker \theta| = 1$ , and  $G/\ker \theta \cong G$ . Suppose  $L = \{x \in G \mid x^{(2^n)} = 1 \text{ for some positive integer } n\}$ . Is  $G/L \cong G$ .
4. If  $H$  is a subgroup of the group  $G$ , let  $\mathbf{N}(H)$  denote the normalizer of  $H$  in  $G$ . Suppose  $G$  is a finite group and  $P$  is a Sylow  $p$ -subgroup of  $G$ . Prove that  $\mathbf{N}(\mathbf{N}(P)) = \mathbf{N}(P)$ .
5. Let  $R$  be a commutative ring. If  $I$  and  $J$  are ideals of  $R$ , define  $(I : J) = \{x \in R \mid Jx \subseteq I\}$ . Prove that  $(I : J)$  is an ideal of  $R$ .  
Now suppose  $I \subseteq (a)$ ,  $a \notin I$ ,  $K = (I : (a))$  and  $R/I$  is a domain. Prove that  $K = I$  and  $aK = I$ . Deduce that  $I \subseteq \bigcap_{n=1}^{\infty} (a^n)$ . Does the final assertion remain true if the hypothesis  $a \notin I$  is dropped? ( $(a)$  denotes the ideal generated by  $a$ .)
6. Let  $K$  be a field. Prove that  $K[X]$  has infinitely many irreducible polynomials, no two of which are associates. (Consider  $p_1 p_2 \dots p_n + 1$ ). Suppose now  $f \in K[X]$ ,  $f \neq 0$ . Prove that there exists a homomorphism  $\theta$  from  $K[X]$  to a domain with nonzero kernel such that  $\theta(f) \neq 0$ .
7. Let  $R$  be a ring, let  $M$  be an  $R$ -module, and let  $\theta: M \rightarrow M$  be an  $R$ -module homomorphism. Prove that  $\ker \theta$  is a submodule of  $M$ .  
Now suppose every submodule of  $M$  is finitely generated. Prove there exists an integer  $n$  such that  $\bigcup_{i=1}^{\infty} \ker \theta^i = \ker \theta^n$ . Deduce that if  $\theta$  is onto, then  $\theta$  is an isomorphism.
8. Let  $V$  be a vector space over  $\mathbb{C}$  and let  $T: V \rightarrow V$  be a linear transformation. Describe how  $V$  can be made into a  $\mathbb{C}[X]$ -module via  $T$ .  
Now let  $\{e_1, e_2, e_3\}$  be a basis for  $V$  and suppose  $T(e_1) = -e_1 + 2e_2$ ,  $T(e_2) = -2e_1 + 3e_2$ ,  $T(e_3) = -2e_1 + 2e_2 + e_3$ . Find the Jordan canonical form for the matrix of  $T$ . Hence find the isomorphism type of  $V$  (as a  $\mathbb{C}[X]$ -module) as a direct sum of primary cyclic modules. Does there exist a  $\mathbb{C}[X]$ -module homomorphism of  $\mathbb{C}[X]$  onto  $V$ ?



## Algebra Preliminary Exam, Spring 1988

Do all eight problems

- Let  $G$  be a group and define  $\phi: G \times G \rightarrow G$  by  $\phi(g, h) = gh^{-1}$ .
  - Find necessary and sufficient conditions on  $G$  such that  $\phi$  is a homomorphism.
  - Under the conditions determined for (a), prove that  $\Delta = \{(x, x) \mid x \in G\}$  is a normal subgroup of  $G$  and  $(G \times G)/\Delta \cong G$ .
- Let  $f: G \rightarrow H$  be a group homomorphism with  $H$  an abelian group. Suppose that  $N$  is a subgroup of  $G$  containing  $\ker(f)$ . Prove that  $N$  is a normal subgroup of  $G$ .
- Let  $G$  be a group of order 99.
  - Show that  $G$  is not a simple group.
  - Show that  $G$  contains a subgroup of order 33.
- Prove that a finite abelian group is either cyclic or has at least  $p$  elements of order  $p$  for some prime  $p$ .
- If  $S$  is a simple nonabelian group, prove that  $\text{Aut}(S)$  contains a subgroup isomorphic to  $S$ . (Hint: consider conjugation.)
- Let  $R$  be a commutative ring with identity. A *simple*  $R$ -module  $S$  is a module whose only submodules are  $0$  and  $S$ .
  - Prove that an  $R$ -module  $S$  is simple if and only if there is a maximal ideal  $\mathfrak{M}$  such that  $S$  is isomorphic to  $R/\mathfrak{M}$ .
  - Let  $R$  be a commutative ring with identity. Show that simple  $R$ -modules exist.
- Let  $R$  be a ring with identity and let  $I$  be a (two-sided) ideal in  $R$ . Let  $M$  and  $N$  be  $R$ -modules.
  - Show that  $R/I \otimes_R M$  is isomorphic to  $M/IM$  as left  $R/I$ -modules.
  - Show that  $(M \oplus N)/I(M \oplus N)$  is isomorphic to  $(M/IM) \oplus (N/IN)$ . You may use any results about tensor products you know.
- Let  $E$  be an extension field of  $F$  with  $[E : F] = 11$ . Prove that if  $x, y \in E$  with neither in  $F$  and if  $\theta$  is an  $F$ -automorphism of  $E$ , then

$$\theta(x) \neq x \text{ implies } \theta(y) \neq y.$$

## Algebra Preliminary Exam, Fall 1988

Instructions: do all problems

1. Prove that there are no simple groups of order 600.
2. Let  $R$  be a principal ideal domain and assume that  $A$ ,  $B$ , and  $C$  are finitely generated  $R$ -modules. Suppose that  $A \oplus B$  is isomorphic to  $A \oplus C$ . Prove that  $B$  is isomorphic to  $C$ .
3. Prove that the Galois group of a splitting field  $K$  of an irreducible polynomial  $p$  over the rational numbers  $\mathbb{Q}$  acts transitively on the roots of  $p$ . Show by examples that this theorem does not necessarily hold if either
  - (a)  $K$  is not a splitting field, or
  - (b)  $p$  is reducible over  $\mathbb{Q}$ .
4. Prove that a group of order 255 is cyclic.
5. Define what is meant by a solvable group. Prove that if  $H \triangleleft G$ , and  $H$  and  $G/H$  are solvable, then  $G$  is solvable.
6. Let  $T: V \rightarrow V$  be a linear map where  $V$  is a finite dimensional vector space over an algebraically closed field. Prove that if 0 is the only eigenvalue of  $T$ , then  $T^n = 0$  where  $n = \dim(V)$ .
7. Prove that if  $f: S \rightarrow T$  is a homomorphism between simple  $R$ -modules  $S$  and  $T$ , then either  $f$  is an isomorphism or  $f$  is the zero homomorphism. (Recall that a nonzero  $R$ -module is simple if 0 and the module itself are the only submodules.)
8. Let  $R$  be a commutative ring with a 1.
  - (a) Prove that if  $M$  is a cyclic  $R$ -module, then  $M$  is isomorphic to  $R/I$  for some ideal  $I$  of  $R$ .
  - (b) Prove that if  $M$  is a cyclic  $R$ -module and  $N$  is an arbitrary  $R$ -module, then  $M \otimes_R N$  is isomorphic to  $N/IN$  for some ideal  $I$  of  $R$ .

## Algebra Prelim, Spring 1989

Do all problems

1. Show that a group of order 540 cannot be simple.
2. Compute the Galois group of  $5x^5 + 3x^4 + 15$  over

(i)  $\mathbb{Z}/2\mathbb{Z}$       (ii)  $\mathbb{Q}$

3. Let  $R$  and  $S$  be domains and let  $\theta: R \rightarrow S$  be an epimorphism. Which of the following statements are true? (Prove or give a counterexample.)

- (a) If  $R$  is a PID, then  $S$  is a PID.
- (b) If  $R$  is a UFD, then  $S$  is a UFD.
- (c) If  $\ker \theta \neq 0$  and  $R$  is a PID, then  $S$  is a field.

4. Let  $K$  be a field and let  $f \in K[x]$  be a polynomial.

- (a) Let  $\alpha_1, \dots, \alpha_r$  be distinct zeros of  $f$  in  $K$ . Prove that there exists  $g \in K[x]$  such that

$$f = (x - \alpha_1) \dots (x - \alpha_r)g.$$

- (b) Let  $p$  be a prime number and let  $K = \mathbb{Z}/p\mathbb{Z}$  be the finite field with  $p$  elements. For each integer  $m$ , let  $\bar{m}$  denote its residue class in  $K$ . Prove that as polynomials in  $K[x]$ , we have

$$x^{p-1} - \bar{1} = \prod_{m=1}^{p-1} (x - \bar{m}).$$

Deduce that  $p$  divides  $(p-1)! + 1$ .

5. Let  $G$  be a group of finite order and let  $F$  be the intersection of all maximal subgroups of  $G$ .

- (a) Prove that  $F \triangleleft G$ .
- (b) If  $H \leq G$  and  $FH = G$ , prove that  $H = G$ .
- (c) If  $S$  is a Sylow subgroup of  $F$  and  $x \in G$ , prove that  $xSx^{-1} = fSf^{-1}$  for some  $f \in F$ . Deduce that  $G = F\mathbf{N}_G(S)$ .

6. Which of the following statements are true? (Prove or give counterexample.)

- (a) If  $K/F$  and  $E/K$  are finite Galois extensions, then  $E/F$  is a finite Galois extension.
- (b) Let  $f, g \in \mathbb{Q}[x]$  be irreducible, let  $\alpha$  be a root of  $f$ , and let  $\beta$  be a root of  $g$ . If  $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$ , then  $\text{Gal}(f/\mathbb{Q}) \cong \text{Gal}(g/\mathbb{Q})$ .

7. Let  $R$  be a commutative ring. Define what is meant by saying that an  $R$ -module is Noetherian.

Suppose  $R$  has the property that the  $R$ -modules  $R^n$  are Noetherian for all  $n \in \mathbb{N}$  (where  $R^n$  denotes the direct sum of  $n$  copies of  $R$ , and  $\mathbb{N} = \{0, 1, 2, \dots\}$ ). Let  $M$  be a finitely generated  $R$ -module.

- (a) Show that  $M \cong R^n/N$  for some  $n \in \mathbb{N}$  and some  $R$ -submodule  $N$  of  $R^n$ .
- (b) Deduce that if  $L$  is a submodule of  $M$ , then  $L \cong K/N$  where  $K$  is a submodule of  $R^n$  containing  $N$ .
- (c) Conclude that all finitely generated  $R$ -modules are Noetherian.

8. Determine the matrices in  $M_3(\mathbb{Q})$  commuting with  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ .

## Algebra Qualifying Exam, Fall 1989

Do six problems

1. Compute the Galois group of  $3x^2 + 7x + 21$  over

(a)  $\mathbb{Z}/2\mathbb{Z}$       (b)  $\mathbb{Q}$

2. By an “ $N$ -group”, we mean a finite group with the property that every nonidentity homomorphic image has a nonidentity center. Prove that maximal subgroups of  $N$ -groups are always normal.

3. Let  $R$  be an integral domain and let  $K$  be its field of fractions. Assume that if  $0 \neq x \in K$ , then either  $x \in R$  or  $x^{-1} \in R$ . Prove that

(a)  $R$  is a local ring.

(b)  $R$  is integrally closed in  $K$ .

4. Let  $R$  be a PID and let  $A, M$  be nonzero finitely generated  $R$ -modules.

(a) Show that if  $A$  is torsion free, then  $A \otimes_R M \neq 0$ .

(b) Provide a counterexample to the conclusion of 4a in the case  $A$  is not torsion free.

5. Assume that  $p$  and  $q$  are distinct primes. Show that a group of order  $p^2q$  cannot be simple.

6. Let  $k$  be a field. If  $f \in k[X_1, X_2, \dots, X_n]$ , define

$$V(f) = \{(a_1, \dots, a_n) \in k^n \mid f(a_1, \dots, a_n) = 0\}.$$

Prove that if  $f_1, f_2, \dots$  is a countable list of polynomials in  $k[X_1, \dots, X_n]$ , then there is a positive integer  $T$  such that

$$\bigcap_{j=1}^{\infty} V(f_j) = V(f_1) \cap V(f_2) \cap \dots \cap V(f_T).$$

7. Let  $k$  be a field. Prove that if  $A$  and  $B$  are two  $n \times n$ -matrices with entries in  $k$ , both of which have minimal polynomial  $X^{n-1}$ , then  $A$  and  $B$  are similar.

## Algebra Prelim, Spring 1990

Answer all questions

- Let  $M$  be a left  $R$ -module, and let  $A$  and  $B$  be Artinian submodules. Show that  $A + B$  is an Artinian  $R$ -submodule.
  - If  $R$  is also left Noetherian and  $M$  is finitely generated, show that  $M$  has a unique maximum Artinian submodule  $A(M)$  and that  $A(M/A(M)) = 0$ .
- Let  $A$  be an abelian group with no elements of infinite order. Suppose that every element of prime order is of order 3. Show that the order of every element is a power of 3. (Hint: do finitely generated abelian groups first.)
- Let  $G$  be a simple group of order 144.
  - Prove that a group of order 18 has exactly one Sylow 3-subgroup.
  - If  $H$  is a proper subgroup of  $G$ , show that  $|H| \leq 26$ .
  - If  $P$  and  $Q$  are distinct Sylow 3-subgroups of  $G$ , show that  $|P \cap Q| = 1$ . (If  $|P \cap Q| > 1$ , consider  $N_G(P \cap Q)$ ).
- Prove that a group of order 765 is abelian.
- Let  $f(x)$  in  $\mathbb{Q}[x]$  be an irreducible polynomial of degree 5. Suppose  $a$  and  $b$  are distinct roots and that  $\mathbb{Q}(a) = \mathbb{Q}(b)$ . Show that  $\mathbb{Q}(a)$  is a normal extension of  $\mathbb{Q}$ .
- Let  $S \subseteq \mathbb{Z}[x_1, x_2, \dots, x_n]$ . Prove that there is a smallest principal ideal containing  $S$ . If this ideal is generated by  $\alpha$ , show that  $\alpha\mathbb{Q}[x_1, x_2, \dots, x_n]$  is the smallest principal ideal in  $\mathbb{Q}[x_1, x_2, \dots, x_n]$  containing  $S$ .
- Let  $V$  be a vector space over  $R$ , and let  $T : V \rightarrow V$  be a linear transformation. Describe how  $V$  can be made into an  $R[x]$ -module.

Now suppose there are  $v_1, \dots, v_n$  in  $V$  such that  $\{T^i(v_j) \mid i = 0, 1, \dots, j = 1, 2, \dots, n\}$  span  $V$  as a vector space.

  - Prove that  $V$  is a finitely generated  $R[x]$ -module.
  - If  $T$  is onto, show that  $V$  cannot have a summand isomorphic to  $R[x]$ .
  - If  $T$  is onto, show that  $V$  is finite dimensional.

8. (a) Let  $A = \{\iota, (12)(34)\}$  and  $V = \{\iota, (12)(34), (13)(24), (14)(23)\}$  in  $S_4$ . Show that  $A$  is normal in  $V$  and  $V$  is normal in  $S_4$ , but  $A$  is not normal in  $S_4$ .
- (b) Give an example of fields  $F \subseteq K \subseteq L$  such that  $K$  a normal extension of  $F$  and  $L$  a normal extension of  $K$ , but  $L$  is not a normal extension of  $F$ .

## Algebra Prelim, Fall 1990

Answer all questions

1. Give a complete list of all non-isomorphic abelian groups of order  $200 = 2^3 \cdot 5^2$ .
2. Show that a group of order  $216 = 2^3 \cdot 3^3$  cannot be simple.
3. Let  $G$  be a finite  $p$ -group with  $|G| > p^2$ . Prove that  $G$  contains a normal abelian subgroup of order  $p^2$ .
4. (a) Show that if  $G$  is a subgroup of  $S_n$ , then either  $G \subseteq A_n$  or  $[G : G \cap A_n] = 2$ .  
(b) Show that if  $n \geq 5$  and  $G$  is a normal subgroup of  $S_n$ , then  $G = 1, A_n$  or  $S_n$ .  
(c) Show that if  $n \geq 5$ , then  $S_n$  has no subgroups of index 3.
5. Let  $G$  be an abelian group with 54 elements. Suppose that  $G$  cannot be generated by one element, but can be generated by two elements. Prove that  $G$  is isomorphic to  $\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ .
6. Let  $K$  be an extension field of  $F$  with  $[K : F] = 14$ . Let  $f(x) \in F[x]$  be a polynomial of degree 5. Suppose  $f(x)$  has no roots in  $F$  but has a root in  $K$ . What can you say about the factorization of  $f(x)$  into irreducibles in  $F[x]$  and  $K[x]$ ?
7. Let  $f(x)$  be irreducible over  $\mathbb{Q}$  with splitting field  $E$ , and let  $\alpha$  and  $\beta$  be roots of  $f$  in  $E$ . If  $E/\mathbb{Q}$  has an abelian Galois group, prove that  $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$ .
8. Let  $R$  be a commutative ring with identity, and let  $(M_\alpha)_{\alpha \in \Gamma}$  be the set of maximal ideals of  $R$ . Let  $A$  be an ideal of the polynomial ring  $R[x]$  such that  $A \subseteq \bigcup_{\alpha} M_\alpha[x]$ . Show that  $A \subseteq M_\beta[x]$  for some  $\beta \in \Gamma$ . (Hint: consider the set  $B = \{r \in R \mid r \text{ is a coefficient of some polynomial in } A\}$ .)



## Qualifying Exam Algebra Spring 1991

1. Suppose that  $A, H$  are normal subgroups of a group  $G$  such that  $G/A$  is a simple group of order  $n$ .
  - (a) Prove that  $H/A$  is a normal subgroup of  $H$ .
  - (b) Prove that either  $H \subseteq A$  or  $H/(H/A)$  is a simple group of order  $n$ . (Hint: use an isomorphism theorem.)
2.
  - (a) Prove that a group of order 100 cannot be simple.
  - (b) Describe all abelian groups of order 100 up to isomorphism.
  - (c) Either show that every group of order 100 is abelian, or exhibit a nonabelian example.
3. Let  $G$  be a group and let  $f: G \rightarrow H$  be a group homomorphism. Prove that if  $H$  is a solvable group and if  $\ker(f)$  is abelian, then  $G$  is a solvable.
4. Let  $R$  be a PID.
  - (a) Prove that the intersection of two nonzero maximal ideals cannot be zero.
  - (b) Assume that  $R$  contains an infinite number of maximal ideals. Show that the intersection of all the nonzero maximal ideals of  $R$  equals zero.
5. Let  $R \subseteq S$  be rings with a 1 such that  $S/R$  is a free left  $R$ -module. Prove that if  $L$  is a left ideal of  $R$ , then  $LS/R = L$ . (Hint: write  $S$  as a direct sum of  $R$ -modules.)
6. Let  $R$  be a ring with 1. A nonzero left  $R$ -module  $S$  is simple if  $0$  and  $S$  are the only submodules of  $S$ . Let

$$0 \rightarrow S \xrightarrow{\alpha} M \xrightarrow{\pi} S \rightarrow 0$$

be a short exact sequence of  $R$ -modules which is *not* split, and such that  $S$  is a simple  $R$ -module. Show that the only nonzero submodules of  $M$  are  $\alpha(S)$  and  $M$ . (Hint: if  $0 = N \subsetneq M$  and  $\alpha(S) \cap N = 0$ , show that there is an isomorphism  $\sigma: S \rightarrow N$  such that  $\pi\sigma = 1_S$ .)

7. Suppose that  $F$  is a Galois extension of  $\mathbb{Q}$  with  $[F:\mathbb{Q}] = 25$ . What possible groups can occur as the Galois group of  $F$  over  $\mathbb{Q}$ ? In all cases, describe the intermediate fields between  $F$  and  $\mathbb{Q}$  in terms of inclusion and dimension over  $\mathbb{Q}$ . Which intermediate fields are Galois over  $\mathbb{Q}$ ?
8. Recall that a group  $G$  of permutations of a set  $S$  is called *transitive* if given  $s, t \in S$ , then there exists  $\sigma \in G$  such that  $\sigma(s) = t$ . Let  $f(x)$  be a separable polynomial in  $K[x]$  and let  $F$  be a splitting field of  $f(x)$  over  $K$ . Prove that  $f(x)$  is irreducible over  $K$  if and only if the Galois group of  $F$  over  $K$  is a transitive subgroup when viewed as permutations of the roots of  $f(x)$ .

## Algebra Prelim, Spring 1992

Answer all questions

- (a) Let  $G$  be a finite group of order  $m$ , and let  $p$  be the smallest prime which divides  $m$ . Prove that if  $H$  is a subgroup of index  $p$ , then  $H$  is a normal subgroup of  $G$ .

(b) Prove that any group of order  $p^2q$  is solvable, where  $p = q$  are primes. (Hint: consider separately the cases  $p = q$  and  $p \neq q$ ).
- List all groups of order 6 (up to isomorphism), and prove that they are the only ones.
- If  $A$  and  $B$  are finitely generated abelian groups with  $A \oplus A$  isomorphic to  $B \oplus B$ , prove that  $A$  and  $B$  are isomorphic.
- Suppose  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is a short exact sequence of modules over a ring  $R$ . Prove that if the sequence splits, i.e. there is an  $R$ -module homomorphism  $h: C \rightarrow B$  such that  $gh = 1_C$ , then  $B = A \oplus C$ .
- Let  $R$  be a commutative ring with a 1. If  $S$  is a multiplicative set (i.e.  $x, y \in S \implies xy \in S$ ) containing 1, but not 0, prove there exists a prime ideal  $P$  of  $R$  with  $P \cap S = \emptyset$ .
- Let  $A$  be an abelian group and let  $m \neq 1$  be an integer. Prove that  $A \otimes \mathbb{Z}/m\mathbb{Z} = A/mA$ .
- Let  $K/k$  be a Galois extension of fields and let  $f(x) \in k[x]$  be an irreducible polynomial which has a root in  $K$ . Prove that  $f(x)$  splits into linear factors in  $K[x]$ .
- Let  $K$  be a finite Galois extension of  $\mathbb{Q}$  with Galois group isomorphic to  $A_4$ . For each divisor  $d$  of 12, how many subfields  $L$  of  $K$  have  $[K:L] = d$ ? In each case give the isomorphism class of  $\text{Gal}(K/L)$ , and state whether or not  $L/\mathbb{Q}$  is a Galois extension. (Recall  $A_4$  is a counter-example to the converse of Lagrange's theorem.)

## Algebra Prelim, Fall 1992

Answer all questions

1. Prove that any group of order 765 is abelian.
2. Let  $G$  be a finite group and let  $N$  be a normal subgroup of  $G$ . Prove that  $G$  is solvable if and only if both  $N$  and  $G/N$  are solvable.
3. (a) Prove that if  $G$  is a subgroup of  $S_n$ , then either  $G \subseteq A_n$  or  $[G : G \cap A_n] = 2$ .  
(b) Prove that if  $n \geq 5$  and  $G$  is a normal subgroup of  $S_n$ , then  $G = \{e\}$ ,  $A_n$  or  $S_n$ .  
(c) Prove that if  $n \geq 5$ , then  $S_n$  has no subgroup of index 3.
4. Let  $f(x)$  be an irreducible polynomial over  $\mathbb{Q}$  with splitting field  $K$ . If the Galois group of  $K/\mathbb{Q}$  is abelian, prove that for any roots  $\alpha, \beta$  of  $f(x)$  in  $K$ , we have  $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta) = K$ .
5. Let  $K$  be a field and  $f(x) \in K[x]$  be a separable irreducible polynomial of degree 4, and let  $E$  be a splitting field for  $f(x)$  over  $K$ . If  $\alpha \in E$  is a root of  $f(x)$  and  $L = K(\alpha)$ , prove that there exists a subfield  $F$  of  $L$  with  $[F : K] = 2$  if and only if the Galois group of  $E/K$  is not isomorphic to either  $A_4$  or  $S_4$ .
6. (a) If  $G, H$  and  $K$  are finitely generated abelian groups with  $G \oplus H \cong G \oplus K$ , prove that  $H \cong K$ .  
(b) Give an example to show that part (a) is false if  $G$  is not finitely generated.
7. Let  $R$  be an integral domain. A nonzero element  $\pi$  of  $R$  is a *prime* if  $\pi|ab$  implies that either  $\pi|a$  or  $\pi|b$ . A nonzero element  $\pi$  is *irreducible* if  $\pi = ab$  implies that either  $a$  or  $b$  is a unit.  
(a) Prove that every prime is irreducible.  
(b) If  $R$  is a UFD, prove that every irreducible is prime.
8. Let  $R$  be a commutative ring with a 1, and let  $M$  be a cyclic  $R$ -module.  
(a) Prove that  $M$  is isomorphic to  $R/I$  for some ideal  $I$  of  $R$ .  
(b) If  $N$  is any  $R$ -module, prove that  $M \otimes_R N$  is isomorphic to  $N/IN$  for some ideal  $I$  of  $R$ .

## Algebra Preliminary Exam, Spring 1993

Do six problems

1. Let  $R$  be a principal ideal domain. Assume that  $M$  is a nonzero finitely generated  $R$ -module with the property that the intersection of any two nonzero submodules is nonzero. Prove that  $M = R/Rt$  where  $t$  is either zero or some power of an irreducible element in  $R$ .
2. Let  $G$  be a group which acts transitively on a finite set  $X$ . Assume that there is an element  $x_0 \in X$  whose stabilizer has no element of finite order other than 1.
  - (a) Show that if  $f \in G$  has finite order larger than 1, then  $f$  has no fixed points.
  - (b) Show that if the order of  $f \in G$  is a prime  $q$ , then  $|X| \equiv 0 \pmod{q}$ .
3. Let  $S$  be a commutative ring with prime ideals  $P_1, P_2, \dots, P_t$ . Show that if  $S/P_1 \times P_2 \times \dots \times P_t$  is a finite set, then each of the  $P_i$  is a maximal ideal.
4. Let  $p$  be a prime. Prove that if every nontrivial finite field extension of the field  $F$  has degree divisible by  $p$ , then every finite field extension of  $F$  has degree a power of  $p$ . (You may assume that  $\text{char } F = 0$ .)
5. Let  $R$  be a commutative ring with a 1. If  $M$  and  $N$  are  $R$ -modules, then  $\text{Hom}(M, N)$  denotes the set of all  $R$ -module homomorphisms from  $M$  to  $N$ . If  $f: N \rightarrow N$  is an  $R$ -module homomorphism, then  $f: \text{Hom}(M, N) \rightarrow \text{Hom}(M, N)$  is defined by  $f(g) = f \circ g$ , the composition of  $g$  followed by  $f$ . Prove that  $M$  is a projective  $R$ -module if and only if for all surjective  $f: N \rightarrow N$ , the function  $f$  is surjective.
6. Let  $D$  be a finite dihedral group, and let  $V$  be a finite dimensional complex vector space which is a  $D$ -module. (You may regard  $D$  as a group of linear transformations from  $V$  to itself.) Prove that if the only  $D$ -invariant subspaces of  $V$  are 0 and  $V$  itself (i.e.  $V$  is a simple or irreducible  $D$ -module), then  $\dim_{\mathbb{C}} V = 2$ .
7. Determine the Galois group of (the splitting field for) the polynomial  $X^{10} - 1$  over the rational numbers.

## Algebra Preliminary Exam, Fall 1993

Do all problems

1. Assume that  $R$  is a ring and  $e \in R$  has the property that  $e^2 = e$ . Prove that  $Re$  is a projective left  $R$ -module.
2. Let  $F$  be a field of characteristic zero and let  $K$  be a finite field extension of  $F$ .
  - (a) Explain why there is a polynomial  $p(X) \in F[X]$  such that  $K \cong F[X]/(p)$ .
  - (b) Prove that if  $c \in K[X]/(p)$  and  $c^2 = 0$ , then  $c = 0$ .
3. Let  $M_2(\mathbb{Q})$  denote the group of  $2 \times 2$  matrices with rational entries under addition, and let  $GL_2(\mathbb{Q})$  denote the group of invertible  $2 \times 2$  matrices with rational entries under multiplication.
  - (a) Prove that if  $M_2(\mathbb{Q})$  acts on a set, then all orbits are either infinite or singletons.
  - (b) Show that  $GL_2(\mathbb{Q})$  acts on  $\mathbb{Q}$  via  $g * \lambda = \frac{\det(g)}{|\det(g)|} \lambda$ , and that there exists a finite orbit which is not a singleton.
4. Let  $G$  be the direct product of the dihedral group of order 34 and the cyclic group of order 9. Suppose that  $L$  is a field and  $G$  is a group of automorphisms of  $L$ . Prove that there is a unique field  $K$  such that  $L^G \subseteq K$  and  $\dim_K L = 17$ . (You may assume that  $\text{char } L = 0$ .)
5. Let  $S$  be a commutative integral domain. Prove that if every prime ideal of  $S[X]$  is principal, then  $S$  is a field.
6. Let  $A$  be an abelian group.
  - (a) Show that the collection  $H$  of all homomorphisms from  $A$  to  $\mathbb{Z}$  is a group under addition of functions.
  - (b) Prove that if  $f_1, \dots, f_m \in H$ , then the subgroup generated by  $f_1, \dots, f_m$  is free (i.e. free as a  $\mathbb{Z}$ -module).
7. Let  $p$  be a prime, and let  $G$  be the group of invertible  $2 \times 2$  matrices under multiplication with entries in the field of integers modulo  $p$ . Let  $H$  be the subgroup consisting of all matrices of the form  $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ .
  - (a) Show that  $|G| = (p^2 - 1)(p^2 - p)$ .
  - (b) Find all values of  $p$  such that the number of conjugates of  $H$  in  $G$  is congruent to 8 mod  $p$ .

**Algebra Prelim Spring 1994**  
**Answer All Problems**

- (1) Let  $G$  be a group with exactly three elements of order two. Prove that  $G$  is not simple.
- (2) Let  $G = \langle x, y \mid x^6 = e, y^4 = e, yxy^{-1} = x^{-1} \rangle$ . Prove that  $G$  has a homomorphic image isomorphic to  $S_3$  (the symmetric group of degree 3), but is *not* isomorphic to  $S_3$ .
- (3) Let  $R$  be a PID which is *not* a field, and let  $M$  be a finitely generated  $R$ -module which is *not* a torsion module.
  - (i) Prove that the  $R$ -module  $R$  is isomorphic to a proper submodule of itself.
  - (ii) Prove that  $M$  is isomorphic to a proper submodule of itself.
- (4) Let  $R$  be a ring, and let  $A, B, C$  be  $R$ -modules.
  - (i) Prove that  $\text{Hom}_R(A, B \oplus C) \cong \text{Hom}_R(A, B) \oplus \text{Hom}_R(A, C)$  as abelian groups.
  - (ii) Prove that  $\text{Hom}_R(A, A \oplus A)$  is *not* isomorphic to  $\mathbb{Z}$ .
- (5) Let  $R$  be a commutative Noetherian ring.
  - (i) If  $S$  is a multiplicative subset of  $R$ , prove that  $S^{-1}R$  is Noetherian.
  - (ii) Prove that  $R[[X^{-1}, X]]$  (the Laurent Series ring in  $X$ ) is a Noetherian ring (you may assume that the power series ring  $R[[X]]$  is Noetherian).
- (6)
  - (i) Prove that  $X^4 + X^3 + X^2 + X + 1$  is irreducible in  $\mathbb{Q}[X]$ . (Set  $Y = X - 1$ .)
  - (ii) Let  $A \in M_n(\mathbb{Q})$ , and suppose no eigenvalue of  $A$  is equal to 1. Prove that if  $A^5 = I$ , then  $4 \mid n$  (where  $I$  denotes the identity matrix).

## Algebra Prelim Fall 1994 Answer All Problems

- (1) Let  $p$  be a prime, let  $G$  be a finite  $p$ -group, let  $Z$  be the center of  $G$ , and let  $1 = H \triangleleft G$ .
- (i) Let  $x \in H \setminus Z$ , and let  $\mathcal{C}(x)$  denote the conjugacy class containing  $x$ . Prove that  $\mathcal{C}(x) \subseteq H$  and  $p$  divides  $|\mathcal{C}(x)|$ .
  - (ii) Prove that  $Z \setminus H = 1$ .
  - (iii) Let  $A$  be a maximal normal abelian subgroup of  $G$ . Prove that  $A$  is also a maximal abelian subgroup of  $G$ . (Apply (ii) with  $G = G/A$  and  $H$  the centralizer of  $A$  in  $G$ .)
- (2) Let  $G$  be a simple group of order 180.
- (i) Prove that the number of 5-Sylow subgroups of  $G$  is 36.
  - (ii) Prove that the normalizer of a 3-Sylow subgroup of  $G$  has order 18.
  - (iii) Prove that the 3-Sylow subgroup of a group of order 18 is normal in that group.
  - (iv) If  $A$  and  $B$  are distinct 3-Sylow subgroups of  $G$ , prove that  $A \cap B = 1$  (consider the centralizer in  $G$  of  $A \cap B$ ).
  - (v) Prove that there is no simple group of order 180.
- (3) Let  $R$  be a PID (principal ideal domain), and let  $M$  be a cyclic left  $R$ -module. Suppose  $M = A \oplus B$  where  $A$  and  $B$  are nonzero left  $R$ -modules. Prove that there exists  $r \in R \setminus 0$  such that  $rM = 0$ . Prove further that for such an  $r$ , there exist distinct primes  $p, q \in R$  such that  $pq$  divides  $r$ .
- (4) Let  $R$  be the ring  $\mathbb{Z}_{(2)}[X]/(X-2)$ , where  $\mathbb{Z}_{(2)}[X]$  denotes the power series ring in  $X$  over  $\mathbb{Z}_{(2)}$ , the localization of  $\mathbb{Z}$  at the prime 2.
- (i) If  $q$  is an odd integer, prove that  $q$  is invertible in  $\mathbb{Z}[X]/(X-2)$ .
  - (ii) Define  $\theta: \mathbb{Z}[X] \rightarrow \mathbb{Z}_{(2)}[X]$  by  $\theta(\sum a_i X^i) = \sum a_i X^i$ , and let  $\pi: \mathbb{Z}_{(2)}[X] \rightarrow R$  be the natural epimorphism. Prove that  $\pi\theta$  is surjective and deduce that  $R = \mathbb{Z}[X]/(X-2)$ .
  - (iii) Prove that  $R \cong \mathbb{Z}[X]/(X-2)$ .
- (5) Let  $p$  be a prime, let  $K \subseteq L$  be fields of characteristic  $p$ , let  $\alpha, \beta \in L$ , and let  $d$  be a positive integer. Suppose  $K(\alpha): K = d$ ,  $K(\beta): K = p$ ,  $\alpha$  is separable over  $K$ , and  $\beta$  is not separable over  $K$ .
- (i) Prove that  $K(\alpha) = K(\alpha^p)$  and  $\beta^p \in K$ .
  - (ii) Prove that  $K(\alpha) \subseteq K(\alpha \beta)$ .
  - (iii) Prove that  $[K(\alpha \beta): K] = pd$ .
- (6) Let  $p$  be a prime, and let  $K = \mathbb{F}_p(t)$ , the quotient field (field of fractions) of the polynomial ring  $\mathbb{F}_p[t]$ .
- (i) Prove that  $X^p - t$  is irreducible in  $K[X]$ .
  - (ii) Let  $L$  be the splitting field of  $X^p - t$  over  $K$ . Determine the Galois group of  $L$  over  $K$ .

## Qualifying Exam Algebra Spring 1995

1. Determine, up to isomorphism, all groups of order  $1127 = 7^2 \cdot 23$ .
2. Let  $G$  be a noncyclic nilpotent group. Show that there is a normal subgroup  $N$  of  $G$  such that  $G/N$  is a noncyclic abelian group.
3. Describe all finitely generated abelian groups  $G$  such that if  $A$  and  $B$  are subgroups of  $G$ , then either  $A \subseteq B$  or  $B \subseteq A$ .
4. Let  $R$  be a commutative ring with a 1. Recall that if  $I$  is an ideal, then  $\text{rad} I = \{x \mid x^m \in I \text{ for some } m \in \mathbb{N}\}$  is also an ideal. Let  $P_1, \dots, P_n$  be distinct prime ideals in  $R$ .
  - (a) Show that  $R/\text{rad} I$  has no nonzero nilpotent elements.
  - (b) Prove that  $\text{rad}(P_1 \cdots P_n) = P_1 \cap P_2 \cap \cdots \cap P_n$ .
  - (c) Prove that  $R/\text{rad}(P_1 \cdots P_n)$  is an integral domain if and only if there exists an  $i$  such that  $P_i$  is contained in every  $P_j$  for  $j = 1, \dots, n$ .
5. Suppose that  $K$  is a subfield of a field  $L$ . Assume that  $K = \mathbb{Q}(\omega)$ , where  $\omega = e^{\frac{2\pi i}{3}} = -1/2 + \sqrt{3}/2 \cdot i$ . Note that  $\omega^3 = 1$ . Let  $\alpha \in L$  such that  $[K(\alpha) : K] = 2^3$ . Prove that  $K(\alpha^3) = K(\alpha)$ .
6. Let  $R$  be a commutative ring with 1 and let  $a \in R$  be non-nilpotent. Let  $S = \{a^i \mid i \geq 0\}$ .
  - (a) Prove that there is a prime ideal  $P$  not containing  $a$ .
  - (b) Let  $K$  be the quotient field of  $R/P$ . Prove that there exists a ring homomorphism  $\phi: S^{-1}R \rightarrow K$ .
7. Let  $F$  be a finite Galois extension of  $K$  and suppose that  $\text{Gal}(F/K) = S_4$ .
  - (a) Show that there are at least 9 different proper intermediate fields between  $F$  and  $K$ .
  - (b) Show that there is a proper Galois extension  $E$  of  $K$  (in  $F$ ) and describe the Galois group of  $E$  over  $K$ .
8. Find the Jordan canonical form of  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$  viewed as matrices over the complex numbers. Find all  $3 \times 3$ -matrices with entries in  $\mathbb{C}$  that commute with this canonical form.



## Qualifying Examination Algebra Fall 1995

#1. Suppose that  $G$  is a group of order  $(35)^3$ . Show that  $G$  has normal Sylow 5- and 7-subgroups. Also show that  $G$  has a normal subgroup of order 25.

#2. Suppose that  $G$  is an abelian group isomorphic to  $\mathbb{Z}/(36) \times \mathbb{Z}/(45)$ . Let  $H$  be a subgroup of  $G$  of order 27. Up to isomorphism, describe  $H$ .

#3. Let  $f: G \rightarrow H$  be a group epimorphism with  $G$  finite, let  $g \in G$ , and let  $C$  denote the centralizer of  $f(g)$  in  $H$ .

a). Prove that if  $D$  is a conjugacy class in  $f^{-1}(C)$ , then  $f(D)$  is a conjugacy class in  $C$ .

b). Prove that the order of the conjugacy class of  $g$  in  $f^{-1}(C)$  is at most  $|\ker f|$ .

c). Prove that the order of the centralizer of  $g$  in  $G$  is at least  $|C|$ .

#4. Suppose that  $R$  is a unique factorization domain which is NOT a principal ideal domain.

a). Show that  $R$  must have at least two (nonassociate) prime elements.

b). Show that  $R$  must have a nonprincipal maximal ideal.

#5. Let  $R$  be a ring with a 1 and suppose that  $X$  is an  $R$ -module and  $N$  is a submodule of an  $R$ -module  $M$ . Let  $i: N \rightarrow M$  denote the inclusion map and let  $\sigma: N \rightarrow N \oplus X$  be the  $R$ -module homomorphism  $\sigma(n) = (n, 0)$ . Prove that if

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ \parallel & & \downarrow f \\ N & \xrightarrow{\sigma} & N \oplus X \end{array}$$

is a commutative diagram for some  $R$ -homomorphism  $f$ , then  $M$  is isomorphic to  $N \oplus M/N$ .

#6. Suppose that  $E$  is a Galois field extension of  $F$  with  $[E : F] = p^n$  for some prime  $p$  and positive integer  $n$ . Show that there are intermediate fields  $F = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = E$  so that  $[K_i : K_{i-1}] = p$  and  $K_i$  is Galois over  $F$  for  $i = 1, \dots, n$ .

#7. Prove that there is no finite field which is algebraically closed.

## Qualifying Examination Algebra January 1996

- (1) Let  $f: G \rightarrow H$  be a group epimorphism between the finite abelian groups  $G$  and  $H$ . Suppose that  $G$  has order  $2^3 \cdot 3^3 \cdot 5^2 \cdot 11^2$  and  $H$  has order  $2^3 \cdot 5^2$ . What is the order of  $\ker f$ ? Describe, up to isomorphism, the possible groups of that order.
- (2) Suppose that  $G$  is a group of order  $p^4 q^5$  where  $p$  and  $q$  are distinct primes. Suppose further that both a Sylow  $p$ -subgroup and a Sylow  $q$ -subgroup are normal in  $G$ .
  - (i) Prove that  $G \cong A \times B$ , where  $A$  and  $B$  are subgroups of orders  $p^4$  and  $q^5$  respectively.
  - (ii) Prove that  $G$  has a normal subgroup of order  $pq$ .
- (3) Prove that a group of order  $2^2 \cdot 3 \cdot 11^2$  is not simple.
- (4) Find the degree and a  $\mathbb{Q}$ -basis of  $\mathbb{Q}[\sqrt{2} + \sqrt{3}]$  over  $\mathbb{Q}$  where  $\mathbb{Q}$  is the rational numbers. Justify your answer.
- (5) Prove that in a principal ideal domain  $D$ , every nonzero prime ideal is a maximal ideal. Deduce that if  $K$  is an integral domain and  $f: D \rightarrow K$  is a ring epimorphism with  $\ker f \neq 0$ , then  $K$  is a field.
- (6) Let  $R$  be a commutative ring. Prove that  $R$  has no nonzero nilpotent elements if and only if  $R_{\mathfrak{P}}$  has no nonzero nilpotent elements for all prime ideals  $\mathfrak{P}$  of  $R$  (where  $R_{\mathfrak{P}}$  denotes the localization of  $R$  at the prime ideal  $\mathfrak{P}$ ). Is it true that  $R$  is a domain if and only if  $R_{\mathfrak{P}}$  is a domain for all prime ideals  $\mathfrak{P}$  of  $R$ ?
- (7) Suppose that  $M$  is an  $R$ -module with submodules  $A$  and  $B$  such that  $A \cap B = 0$ . Prove that the submodule of  $M$  generated by  $A$  and  $B$  is isomorphic to  $A \oplus B$  (the direct sum of  $A$  and  $B$ ).
- (8) Suppose that the Galois group of a Galois extension  $E$  over  $F$  is  $S_6$ .
  - (i) Show that there are at least 35 proper subfields between  $E$  and  $F$ .
  - (ii) Show that there is a subfield  $L$  between  $E$  and  $F$  such that  $L$  is Galois over  $F$ , but there is no subfield between  $E$  and  $L$  which is Galois over  $L$ .
  - (iii) What is the dimension of  $L$  over  $F$ ?

## Algebra Prelim, August 1997

Answer all questions

- Let  $R$  be a commutative ring with unity and let  $I, J$  be ideals of  $R$ .
  - Prove that the product  $IJ = \{x \in R \mid x = \sum_{i=1}^n a_i b_i \text{ with } a_i \in I \text{ and } b_i \in J\}$  is an ideal of  $R$ .
  - Prove that  $IJ \subseteq I \cap J$ .
  - If  $I \cap J = R$ , prove that  $IJ = I \cap J$ .
  - If  $IJ = I \cap J$  for all ideals of  $R$  and  $R$  is an integral domain, prove that  $R$  is a field. (Hint: let  $I = Ra$  where  $a \neq 0$  be a principal ideal of  $R$ .)
- Let  $F$  be a finite Galois extension of the field  $K$  with  $\text{Gal}(F/K) = S_5$ .
  - Show that there are more than 40 fields strictly between  $F$  and  $K$ .
  - Show that there is a unique proper subfield  $E$  of  $F$  with  $E \neq K$  such that  $E/K$  is a Galois extension. Determine  $[E:K]$  and describe  $\text{Gal}(E/K)$  up to isomorphism.
- Let  $G$  be a group of order 455.
  - Prove that  $G$  is not simple.
  - Prove that  $G$  is cyclic.
- Let  $R$  be a PID, let  $n$  be a positive integer, and let  $A$  and  $B$  be finitely generated  $R$ -modules. If  $A^n = B^n$ , prove that  $A = B$ . ( $A^n$  denotes the direct sum of  $n$  copies of  $A$ .)
- Let  $P$  be a finitely generated projective  $\mathbb{Z}$ -module. If  $P$  is also injective, prove that  $P = 0$ .
- Let  $A, B$  be abelian groups, and let  $m$  be a positive integer. Prove that  $A \otimes (B/mB) = (A \otimes B)/m(A \otimes B)$ .
- Prove that a group of order 588 is solvable.
- Let  $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ .
  - Determine  $[K:\mathbb{Q}]$ .
  - Compute  $\text{Gal}(K/\mathbb{Q})$ .

## Algebra Prelim, January 1998

Answer all questions

- Let  $F$  and  $K$  be fields of characteristic 0 with  $K$  an extension of  $F$  of degree 21. Let  $f(x)$  be a polynomial in  $F[x]$  of degree 6 which has no roots in  $F$  and exactly two roots in  $K$ .
  - Describe the factorization of  $f(x)$  into irreducible polynomials in  $F[x]$ .
  - Describe the factorization of  $f(x)$  into irreducible polynomials in  $K[x]$ .
- Let  $G$  be a group of order  $1947 = 3 \cdot 11 \cdot 59$ . Prove that  $G$  is cyclic.
- Let  $G$  be a group of order  $p^n$  with  $n \geq 2$  and  $p$  prime. Prove that  $G$  has a normal abelian subgroup of order  $p^2$ .
- Let  $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$  where  $\omega = \cos(2\pi/3) + i\sin(2\pi/3)$  is a primitive cube root of unity.
  - What is  $[K : \mathbb{Q}]$ ?
  - Prove that  $K/\mathbb{Q}$  is a Galois extension.
  - Describe the Galois group of  $K/\mathbb{Q}$ .
- Let  $R$  be a PID with field of fractions (quotient field)  $F$ , let  $S$  be a subring of  $F$  which contains  $R$ , and let  $A$  be an ideal of  $S$ .
  - Prove that  $A \cap R$  is an ideal of  $R$ .
  - If  $A \cap R = Rd$ , prove that  $A = Sd$ . (Hint: if  $a/b \in S$  with  $(a, b) = 1$ , prove that  $1/b \in S$ .)
- Let  $p$  be a prime, let  $a, k$  be positive integers such that  $p$  does not divide  $k$ , and let  $G$  be a group of order  $p^a k$ . Let  $M$  be a normal subgroup of  $G$  and let  $P$  be a Sylow  $p$ -subgroup of  $G$ .
  - Prove that  $PM/M$  is a Sylow  $p$ -subgroup of  $G/M$ .
  - Let  $H/M$  be the normalizer of  $PM/M$  in  $G/M$  and let  $N$  be the normalizer of  $P$  in  $G$ . Prove that  $N \subseteq H$ .
  - Prove that the number of Sylow  $p$ -subgroups of  $G/M$  is a divisor of the number of Sylow  $p$ -subgroups of  $G$ .
- Give an example of a group of order  $3540 = 59 \cdot 60 = 59 \cdot 5 \cdot 3 \cdot 4$  which is not solvable.
  - Give an example of a group of order 3540 which is solvable but not cyclic.
- Let  $G$  and  $H$  be finitely generated abelian groups such that  $G \oplus G = H \oplus H$ . Prove that  $G = H$ .

## Algebra Prelim, January 1999

Answer all questions

1. Let  $R$  be a commutative ring with a  $1 \neq 0$ . If every ideal of  $R$  except the ideal  $R$  is a prime ideal, prove that  $R$  is a field.
2. Let  $p$  and  $q$  be distinct primes and let  $G$  be a group of order  $p^3q^3$ . If  $G$  has a normal  $p$ -Sylow subgroup, prove that  $G$  has a normal subgroup  $H$  of order  $p^3q$ .
3. Let  $R$  be a commutative ring with a  $1$ . Prove that  $R$  is isomorphic to a proper  $R$ -submodule of  $R$  if and only if there exists an element in  $R$  which is neither a zero divisor nor a unit. (A zero divisor is an element  $r$  such that there exists  $s \in R \setminus 0$  such that  $rs = 0$ . A unit in  $R$  is an element  $r$  such that there exists  $s \in R$  such that  $rs = 1$ .)
4. Let  $K$  be a subfield of the field  $L$  and let  $\alpha \in L$ . If  $[K(\alpha) : K]$  is odd, prove that  $K(\alpha^2) = K(\alpha)$ .
5. Let  $p$  be a prime and let  $G$  be an abelian group of order  $p^6$ . Suppose the set  $\{x \in G \mid x^p = 1\}$  has order  $p^2$ . Describe all possible groups  $G$  (up to isomorphism). Justify your answer.
6. Let  $k \subseteq K \subseteq L$  be fields such that  $K$  is a splitting field over  $k$ , and let  $\sigma \in \text{Gal}(L/k)$ . Prove that  $\sigma(K) = K$ .
7. Prove that there is no simple group of order 280.
8. Let  $n$  be a positive integer, let  $E$  be a field of characteristic zero, and let  $F$  be a subfield of  $E$  such that  $[E : F] = n$ . Prove that there are at most  $2^{n-1}$  fields between  $F$  and  $E$ .

## Algebra Prelim, August 1999

Answer all questions

- Let  $G$  be a simple group of order 480 with an abelian Sylow 2-subgroup.
  - If  $P$  and  $Q$  are distinct Sylow 2-subgroups of  $G$ , by considering  $C_G(P \cap Q)$ , prove that  $P \cap Q = 1$ .
  - Prove that there is no such group  $G$ .
- Let  $R$  be a UFD, let  $S$  be a multiplicatively closed subset of  $R$  such that  $0 \notin S$ , and let  $p$  be a prime in  $R$ . Prove that  $p/1$  is either a prime or a unit in  $S^{-1}R$ .
- Let  $k$  be the field  $\mathbb{Z}/2\mathbb{Z}$ . Classify the finitely generated projective  $k[X]/(X^3 - X)$ -modules up to isomorphism.
- Let  $R$  be a ring, let  $M$  be a Noetherian  $R$ -module, and let  $J$  denote the Jacobson radical of  $R$ . Prove that either  $MJ^n = 0$  for some positive integer  $n$ , or  $MJ^{n-1} \subsetneq MJ^n$  (strict inequality) for all positive integers  $n$ .
- Let  $R$  be a nonzero right Artinian ring (with a 1) with no nonzero nilpotent ideals and no nontrivial ( $= 0, 1$ ) idempotents. Prove that  $R$  is a division ring.
- Compute the character table of  $S_4$ .
- Let  $K$  be a splitting field of the polynomial  $X^4 - 2$  over  $\mathbb{Q}$ . Determine the order of  $\text{Gal}(K/\mathbb{Q})$ . Use this to show that  $K$  contains a subfield  $L$  such that  $[L:\mathbb{Q}] = 4$  and  $L$  is normal over  $\mathbb{Q}$ .

**ALGEBRA PRELIMINARY EXAMINATION:  
Fall 2000**

Do all problems

1. Let

$$1 \longrightarrow A \longrightarrow G \longrightarrow P \longrightarrow 1$$

be a short exact sequence of groups such that  $A$  is abelian,  $|P| = 81$ , and  $|A| = 332$ . Show that  $G$  has a nontrivial center.

2. Let  $F$  be a finite field of odd characteristic. Prove that the rings  $F[X]/(X^2 - \alpha)$  as  $\alpha$  ranges over all nonzero elements of  $F$  fall into exactly two isomorphism classes.
3. Let  $R$  be a finite-dimensional simple algebra and let  $M$  be a finite-dimensional left  $R$ -module. Prove that there is a positive integer  $d$  such that

$$\underbrace{M \oplus M \oplus \cdots \oplus M}_{d \text{ copies}}$$

is a free module.

4. Let  $B$  be a square matrix with rational entries. Show that if there is a monic polynomial  $f \in \mathbb{Z}[T]$  such that  $f(B) = 0$  then the trace of  $B$  is an integer.
5. Let  $k$  be a field. Compute the dimension over  $k$  of

$$k[X]/(X^m) \otimes_{k[X]} k[X]/(X^n)$$

and prove your assertion.

6. In this problem  $X, Y, Z$  are indeterminates. Define  $\sigma: \mathbb{C}(X, Y, Z) \rightarrow \mathbb{C}(X, Y, Z)$  by  $\sigma(h(X, Y, Z)) = h(Y, Z, X)$  for every rational function  $h$  in three variables. Prove or disprove: every member of  $\mathbb{C}(X, Y, Z)$  which is left unchanged by  $\sigma$  is a rational function of  $X + Y + Z$ ,  $XY + YZ + XZ$  and  $XYZ$ .



**ALGEBRA PRELIMINARY EXAMINATION:  
Winter 2001**

Do all problems. All rings should be assumed to have a 1.

1. Let  $R, A$  and  $B$  be commutative rings with  $R \subseteq A$  and  $R \subseteq B$ . Prove that if  $A$  is an integral extension of  $R$  and  $B$  is an integral extension of  $R$ , then the ring  $A \otimes_R B$  is also an integral extension of  $R$ .
2. For this problem all fields have characteristic 0. Let  $K/L$  be a Galois extension with Galois group  $G$  and let  $H$  be a subgroup of  $G$ . Prove that there exists some  $\beta \in K$  such that  $H$  coincides with

$$\{\sigma \in G \mid \sigma(\beta) = \beta\}.$$

3. Let  $S$  be a semisimple ring. Prove that  $S \times S$  is semisimple.
4. Let  $G$  be a finite group and assume that  $p$  is a fixed prime divisor of its order. Set  $K = \bigcap N_G(P)$  where the intersection is taken over all Sylow  $p$ -subgroups  $P$  of  $G$  and  $N_G(-)$  denotes the normalizer. Show that
  - (a)  $K \triangleleft G$ .
  - (b)  $G$  and  $G/K$  have the same number of Sylow  $p$ -subgroups.
5. Suppose  $A$  is an abelian group (written additively) of order  $p^M$  for some prime  $p$ . Prove that if  $n$  is a positive integer such that  $p^n A = 0$ , then

$$|\{a \in A \mid pa = 0\}| \geq p^{M/n}.$$

6. Let  $G$  be a finite group. Prove that if  $H$  and  $K$  are normal nilpotent subgroups of  $G$ , then so is  $HK$ .
7. Prove or disprove: let  $\mathbb{Z}_6$  denote the ring of integers modulo 6. Then every projective  $\mathbb{Z}_6$ -module is free.

## Algebra Qualifying Exam, Summer 2001

Instructions: do all problems.

- Let  $G$  be a group.
  - Show that if  $A$  and  $B$  are normal subgroups of  $G$ , then  $A \cap B$  is also a normal subgroup of  $G$ .
  - Suppose that  $N$  is a proper nontrivial normal simple subgroup of  $G$  and that  $G/N$  is also a simple group. Prove that either  $N$  is the only nontrivial proper normal subgroup of  $G$  or that  $G$  is isomorphic to  $N \times G/N$ .
- Let  $A$  be a finite noncyclic abelian group with two generators. Let  $p$  be a prime. Assume that for all primes  $q$  with  $q \neq p$ , there are no nonzero group homomorphisms from  $\mathbb{Z}/(q)$  to  $A$ . Describe the structure of  $A$  and prove that there is no nonzero group homomorphism from  $A$  to  $\mathbb{Z}/(q)$  for all primes  $q$  with  $q \neq p$ .
- Let  $R$  be a ring with a 1 and let  $0 \rightarrow P \rightarrow M \rightarrow Q \rightarrow 0$  be a short exact sequence of  $R$ -modules. Show that if  $P$  and  $Q$  are projective  $R$ -modules, then  $M$  is a projective  $R$ -module.
- Prove that every group of order 441 has a quotient which is isomorphic to  $\mathbb{Z}/(3)$ .
- Let  $R$  be a commutative ring with a  $1 \neq 0$ . Suppose that every ideal of  $R$  different from  $R$  is a prime ideal.
  - Prove that  $R$  is an integral domain.
  - Prove that  $R$  is a field.
- Prove that  $\mathbb{Q}(\sqrt[4]{2})$  and  $\mathbb{Q}(i\sqrt[4]{2})$  are isomorphic fields.
  - Prove that  $\mathbb{Q}(i, \sqrt[4]{2})$  is a Galois extension of  $\mathbb{Q}$ .
  - Find the Galois group  $\text{Gal}(\mathbb{Q}(i, \sqrt[4]{2})/\mathbb{Q})$  and prove your claim.
- Recall that a group  $G$  of permutations of a set  $S$  is called *transitive* if given  $s, t \in S$ , then there exists  $\sigma \in G$  such that  $\sigma(s) = t$ . Let  $K$  be a field. Let  $f(x)$  be a separable polynomial in  $K[x]$  and let  $F$  be a splitting field of  $f$  over  $K$ . Prove that  $f(x)$  is irreducible over  $K$  if and only if the Galois group of  $F$  over  $K$  is a transitive group when viewed as a group of permutations of the roots of  $f(x)$ .

8. An ideal in a commutative ring  $R$  with a 1 is called *primary* if  $I \neq R$  and if  $ab \in I$  and  $a \notin I$ , then  $b^n \in I$  for some positive integer  $n$ .
- (a) Prove that prime ideals are primary.
  - (b) Prove that if  $R$  is a PID, then  $I$  is primary if and only if  $I = P^n$  for some prime ideal  $P$  of  $R$ .

## Algebra Prelim, Summer 2002

Instructions: do all problems.

- Let  $G$  be a finite group.
  - Let  $H$  and  $Q$  be subgroups of  $G$ . Note that  $H$  acts on the set of conjugates of  $Q$  via conjugation. Let  $O_Q$  denote the orbit containing  $Q$  with respect to this action. Prove that if  $H \cap N_G(Q) = 1$ , then the orbit  $O_Q$  has  $|H|$  subgroups in it.
  - Now suppose that  $|G| = p^m q$  where  $p$  and  $q$  are distinct primes and  $m$  is a positive integer. Let  $Q$  be a Sylow  $q$ -subgroup of  $G$  and suppose that  $N_G(Q) = Q$ . Prove that  $G$  has a normal Sylow  $p$ -subgroup.
- Let  $K$  be a finite Galois extension of the rationals  $\mathbb{Q}$ . Suppose that  $\sqrt{2}$  and  $\sqrt{3}$  are both elements of  $K$ . Show that  $\text{Gal}(K/\mathbb{Q})$  has a normal subgroup  $N$  of index 4. Show further that if  $|N|$  is odd, then  $\sqrt[8]{2} \notin K$ .
- Let  $R$  be a ring. Let  $A$  and  $B$  be right  $R$ -modules and let  $C$  be a left  $R$ -module. Prove that  $(A \oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \otimes_R C)$ .
  - Let  $M$  be a finitely generated  $\mathbb{Z}$ -module. Prove that if  $M \otimes_{\mathbb{Z}} M = 0$ , then  $M = 0$ .
- Let  $R$  be a commutative Noetherian ring with unity and let  $M$  be a nonzero  $R$ -module. Given  $m \in M$ , set  $\text{Ann}(m) = \{r \in R \mid rm = 0\}$ . Show there exists some  $w \in M$  such that  $\text{Ann}(w)$  is a prime ideal of  $R$ .
- Let  $R$  be an integral domain. Prove that  $R$  is a field if and only if every  $R$ -module is projective.
- Denote the center of a group by  $Z(\cdot)$ . Let  $G$  be a finite group with identity element  $e$ . Define a sequence of subgroups of  $G$  inductively by  $Z_0 = \{e\}$  and
$$Z_{j+1} \text{ is the preimage in } G \text{ of } Z(G/Z_j).$$
Since  $Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \dots$ , there is a positive integer  $N$  such that  $Z_N = Z_{N+1} = Z_{N+2} = \dots$ . Prove that  $Z_N$  is equal to the intersection of all normal subgroups  $K$  in  $G$  such that  $Z(G/K)$  is the trivial group.
- Explicitly find a simple (i.e. minimal) left ideal of the following ring of  $2 \times 2$  matrices:  $M_2(\mathbb{Q}[x]/(x^2 - 1))$ .

## Algebra Prelim, Winter 2003

Instructions: do all problems.

1. Let  $K$  denote the splitting field over the rational numbers  $\mathbb{Q}$  of the polynomial  $f(x) = x^5 + x^4 + 3x + 3$ .
  - (a) What is  $[K : \mathbb{Q}]$ ?
  - (b) Determine the Galois group  $\text{Gal}(K/\mathbb{Q})$ .
2. Prove or disprove: If  $x$  and  $y$  are elements of a finite abelian group  $G$  with the same order, then there is an automorphism  $\theta$  of  $G$  such that  $\theta(x) = y$ .
3. Let  $H$  be a group of order 2002. Prove that the number of elements in the set  $\{h \in H \mid h^2 = e\}$  is even (where  $e$  is the identity of  $H$ ).
4. Let  $G$  be a group with the following property: for each  $g \in G - 1$ , there exists a normal subgroup  $K$  of  $G$  such that  $G/K$  is abelian and  $g \notin K$  (where  $1$  is the trivial subgroup). Prove that  $G$  is abelian.
5. Prove that if  $A$  and  $B$  are commutative Noetherian rings, then so is the cartesian product  $A \times B$ .
6. Let  $R$  be a commutative ring, let  $P$  be a projective  $R$ -module and let  $I$  be an ideal in  $R$ . Prove  $P/IP$  is a projective  $R/I$ -module.
7. Let  $k$  be a field and let  $I$  be an ideal of  $k[x]$  (where  $k[x]$  is the polynomial ring over  $k$  in the variable  $x$ ).
  - (a) Show that  $k + I$  is a subring of  $k[x]$ .
  - (b) Prove that if  $I \neq 0$  then  $k[x] \otimes_{k+I} k[x]/I$  is finite dimensional over  $k$ .

## Algebra Prelim, Fall 2003

Do all problems

1. Prove that a group of order  $2256 = 47 * 48$  cannot be simple.
2. Let  $G = \langle x, y \mid x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle$ .
  - (i) Prove that every element of  $G$  can be written in the form  $x^i y^j$  where  $i, j$  are non-negative integers.
  - (ii) Prove that  $G$  has order at most 21.
  - (iii) Prove that there is a homomorphism  $\theta: G \rightarrow S_7$  such that

$$\theta x = (1\ 2\ 3\ 4\ 5\ 6\ 7), \theta y = (2\ 3\ 5)(4\ 7\ 6).$$

- (iv) Prove that  $G$  has order 21.
3. Let  $R$  be a UFD. Suppose that for every coprime  $p, q \in R$ , the ideal  $pR + qR$  is principal. Prove that for every  $a, b \in R$ , the ideal  $aR + bR$  is principal. (Coprime means that the greatest common divisor of  $p, q$  is 1.)
  4. Let  $R$  be the ring  $\mathbb{Z}/4\mathbb{Z}$  and let  $M$  be the ideal  $2\mathbb{Z}/4\mathbb{Z}$ . Prove that  $M \otimes_R M \cong M$  as  $R$ -modules.
  5. Let  $R$  be a PID and let  $M, N$  be  $R$ -modules. Suppose  $M$  is finitely generated and  $M \oplus M \cong N \oplus N$ . Prove that  $M \cong N$ .
  6. Let  $K$  be a field of characteristic zero, let  $f \in K[x]$  be an irreducible polynomial, let  $L$  be a splitting field for  $f$  over  $K$ , and let  $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in L$  be the roots of  $f$ . Suppose  $[L : K] = 24$  (i.e.  $\dim_K L = 24$ ).
    - (i) If  $1 \leq i, j \leq 4$  are integers, prove that  $L \neq K(\alpha_i, \alpha_j)$ .
    - (ii) Prove that  $L = K[\alpha_1 + 2\alpha_2 + 3\alpha_3]$ .
  7. Let  $k$  be an algebraically closed field, let  $n$  be a positive integer, and let  $U, V$  be affine algebraic sets in  $k^n$  (so  $U$  is the zero set of a collection of polynomials in  $k[x_1, \dots, x_n]$ ). Suppose  $U \cap V = \emptyset$ . Prove that  $I(U) + I(V) = k[x_1, \dots, x_n]$  (where  $I(U)$  is the set of all polynomials in  $k[x_1, \dots, x_n]$  which vanish on  $U$ ).

## Algebra Prelim, Fall 2004

Do all problems

1. An automorphism of a group is an isomorphism of the group with itself. The set of automorphisms  $\text{Aut}(G)$  of a group  $G$  is itself a group under composition of functions. Find the group of automorphisms of the cyclic group  $C_{2p}$  of order  $2p$  where  $p$  is an odd prime.
2. Let  $G$  be a simple group of order  $4032 = 8!/10$ . Prove that  $G$  is not isomorphic to a subgroup of the alternating group  $A_8$ . Deduce that  $G$  has at least 216 elements of order 7.
3. An ideal  $I$  in a commutative ring  $R$  with unit is called *primary* if  $I \neq R$  and whenever  $ab \in I$  and  $a \notin I$ , then  $b^n \in I$  for some positive integer  $n$ . Prove that if  $R$  is a PID, then  $I$  is primary if and only if  $I = P^n$  for some prime ideal  $P$  of  $R$  and some positive integer  $n$ .
4. Let  $k$  be a field and let  $k[x^2, x^3]$  denote the subring of the polynomial ring  $k[x]$  generated by  $k$  and  $\{x^2, x^3\}$ . Prove that every ideal of  $R$  can be generated by two elements. Hint: if the ideal is nonzero, we may choose one of the generators to be a polynomial of least degree.
5. Let  $k$  be a field, let  $f \in k[x]$  be a polynomial of positive degree and let  $M$  be a finitely generated  $k[x]$ -module. Suppose every element of  $M$  can be written in the form  $fm$  where  $m \in M$ . Prove that  $M$  has finite dimension as a vector space over  $k$ .
6. Let  $R$  be an integral domain (commutative ring with  $1 \neq 0$  and without nontrivial zero divisors) and suppose  $R$  when viewed as a left  $R$ -module is injective. Prove that  $R$  is a field.
7. Let  $K$  be a splitting field over the rational numbers  $\mathbb{Q}$  of the polynomial  $x^4 + 16$ . Determine the Galois group of  $K/\mathbb{Q}$ .

## Algebra Prelim, January 2005

Do all problems

1. If  $p$  and  $q$  are distinct primes and  $G$  is a finite group of order  $p^2q$ , prove that  $G$  has a nontrivial normal Sylow subgroup.
2. Find the Galois group of  $K$  over the rationals  $\mathbb{Q}$  where  $K$  is the splitting field of the polynomial  $x^4 + 4x^2 + 2$ .
3. Show that  $\mathbb{Q}$  is not a projective  $\mathbb{Z}$ -module.
4. Let  $R$  be a commutative Noetherian ring with a 1 and let  $M$  be a finitely generated  $R$ -module. Show that if  $f: M \rightarrow M$  is a surjective  $R$ -module homomorphism, then it must also be injective. (Hint: consider the kernels of  $f^n$ .)
5. Suppose  $R$  is a principal ideal domain that is not a field, and that  $M$  is a finitely generated  $R$ -module. Suppose further that for every irreducible element  $p \in R$ , the  $R/pR$ -module  $M/pM$  is cyclic (has a single generator). Show that  $M$  is cyclic.
6. Let  $G$  be a finite group with a composition series of length 2. Prove that if  $M$  and  $N$  are distinct nonidentity proper normal subgroups of  $G$ , then  $G = M \times N$ .
7. Let  $R$  be the ring  $\mathbb{Q} + x^2\mathbb{Q}[x]$ , the collection of all polynomials with rational coefficients that have no  $x$  term.
  - (a) Show that if  $0 \neq f \in R$ , then  $R/fR$  is a finite dimensional vector space over  $\mathbb{Q}$ .
  - (b) Use part (a) to prove that every nonzero prime ideal of  $R$  is maximal.



## Algebra Prelim, Fall 2005

Do all problems

1. Show that there are exactly 5 nonisomorphic groups of order 18.
2. Let  $A$  be a commutative ring and set  $B = A[X, Y]/(X^2 - Y^2)$ . Prove that  $A$  is a Noetherian ring if and only if  $B$  is a Noetherian ring.
3. Let  $F$  be a field with more than 2 elements and let  $\text{GL}_2(F)$  denote the group of  $2 \times 2$  invertible matrices with entries in  $F$ . Consider the action of  $\text{GL}_2(F)$  on one-dimensional subspaces of  $F^2$ . Show that the stabilizer of a one-dimensional subspace is never simple.
4. Let  $R$  be the ring  $\mathbb{Z}[X]$  and set  $M = 2R + XR$ . Prove or disprove:  $M$  is a free  $R$ -module.
5. Let  $F$  be a field of characteristic zero. Suppose that  $K/F$  is finite Galois extension with Galois group  $G$ . Prove that if  $a \in K$  and  $\sigma(a) - a \in F$  for all  $\sigma \in G$ , then  $a \in F$ .
6. Let  $S$  be a simple algebra of finite dimension  $n$  over  $\mathbb{C}$ . Prove that there are  $\sqrt{n}$  maximal left ideals of  $S$  whose intersection is zero.
7. Recall that if  $F$  is a field, then the tensor product of two  $F$ -algebras (over  $F$ ) is another  $F$ -algebra. Let  $L$  be a finite field extension of  $F$  and let  $\overline{F}$  be the algebraic closure of  $F$ . Show that if  $\overline{F} \otimes_F L$  is a field, then  $F = L$ .

## Algebra Prelim, May 2006

Do all problems

1. Prove that there are no simple groups of order 1755.
2. Let  $P$  be a finite  $p$ -group. Prove that every subgroup of  $P$  appears in some composition series for  $P$ .
3. Let  $R$  be a principal ideal domain. Let  $A$  be a finitely generated  $R$ -module and let  $B$  be an  $R$ -submodule of  $A$ . Assume that there exist nonzero elements  $r$  and  $s$  of  $R$  such that  $\gcd(r, s) = 1$ ,  $rB = 0$ , and  $s(A/B)$  is a torsion-free  $R$ -module. Prove that  $A \cong B \oplus A/B$  as  $R$ -modules.
4. Let  $F = \mathbb{Q}(i)$  and let  $K$  be the splitting field of  $x^6 - 7$  over  $F$ .
  - (a) Determine  $[K : F]$  and write down a basis for  $K$  over  $F$ .
  - (b) Show that  $\text{Gal}(K/F)$  is a dihedral group.
5. Let  $R$  be a ring with unity 1. Let  $P$  be a projective  $R$ -module and let  $M$  be an  $R$ -submodule of  $P$ . Prove: if  $P/M$  is a projective  $R$ -module, then  $M$  is a projective  $R$ -module.
6. Let  $R$  be a commutative Noetherian ring with unity 1. Let  $M$  be a nonzero  $R$ -module. Given  $m \in M$ , set  $\text{Ann } m = \{a \in R \mid am = 0\}$  and note that  $\text{Ann } m$  is an ideal of  $R$ . Prove that there exists  $s \in M$  such that  $\text{Ann } s$  is a prime ideal in  $R$ . (Remember:  $R$  itself is not a prime ideal in  $R$ .)
7. Let  $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ .
  - (a) Prove that there is a well-defined multiplication on  $A$  that satisfies the distributive property such that
$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$
for all complex numbers  $a_1, a_2, b_1, b_2$ .
  - (b) Now assume that this multiplication makes  $A$  into a ring. Prove that  $A$  is not an integral domain.

## Algebra Prelim, August 2007

Do all problems

- How many elements of order 7 must there be in a simple group of order 168?
- Let  $\rho$  be a primitive 4th root of 1 over  $\mathbb{Q}$ .
  - Compute the Galois group of  $(x^4 - 2)(x^2 - 3)$  over  $\mathbb{Q}$  and  $\mathbb{Q}(\rho)$ .
  - Is  $\mathbb{Q}(\rho)$  Galois over  $\mathbb{Q}$ ? (Explain your answer.)
  - Are there any proper subfields of the splitting field of  $(x^4 - 2)(x^2 - 3)$  over  $\mathbb{Q}(\rho)$  that are Galois over  $\mathbb{Q}(\rho)$ ? (Explain your answer.)
- Suppose that  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is a split exact sequence of left  $R$ -modules, where  $R$  is a ring with a 1. If  $D$  is a right  $R$ -module, prove that  $1 \otimes f: D \otimes_R A \rightarrow D \otimes_R B$  is a monomorphism.
- Let  $R$  be a PID and let  $S$  be a multiplicatively closed subset of  $R$ . Assume that  $S$  is nonempty and that  $S$  does not contain 0. Prove that  $S^{-1}R$  is a PID.
- Prove that a group of order  $2^4 \cdot 11^2$  is solvable.
- Prove that  $f(x) = x^4 + 9x - 30$  is an irreducible polynomial over  $\mathbb{Q}$ .
  - Let  $g(x) = x^2 + 2$  and let  $I$  be the ideal in  $\mathbb{Q}[x]$  generated by the product  $f(x)g(x)$ . Show that  $\mathbb{Q}[x]/I$  is the product of two fields. What is the dimension over  $\mathbb{Q}$  of these fields?
- Let  $R$  be an integral domain. If  $X$  is an  $R$ -module, then let  $t(X)$  denote the subset  $\{x \in X \mid rx = 0 \text{ for some nonzero } r \in R\}$ .
  - Prove that  $t(X)$  is a submodule of  $X$ .
  - Prove that  $t(X/t(X)) = 0$ .
  - Prove that if  $X/t(X)$  is a nonzero cyclic  $R$ -module, then  $X$  is isomorphic to  $t(X) \oplus R$ .

## Algebra Prelim, December 2007

Do all problems

- Let  $p$  and  $q$  be distinct prime integers.
  - List all nonisomorphic abelian groups of order  $p^3q$ , listing only one for each isomorphism class.
  - Show that if  $G$  is an abelian group of order  $p^3q$  such that  $G$  cannot be generated by one element but  $G$  can be generated by two elements, then  $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{pq}$ .
- Let  $K$  be a finite field extension of  $k$ , let  $\alpha \in K$ , and let  $f(x)$  be the irreducible polynomial of  $\alpha$  over  $k$ . Prove that if  $n \mid \deg f(x)$ , then  $n \mid [K : k]$ .
- Let  $R$  be a UFD with quotient field  $Q$  and let  $f(x)$  be an irreducible polynomial of degree  $\geq 1$  in  $R[x]$ . Let  $I$  denote the ideal in  $Q[x]$  generated by  $f(x)$ . Prove that  $Q[x]/I$  is a field.
- Prove that  $S_4$  is solvable.
- Let  $R$  be a ring with a 1 and let  $P$  and  $Q$  be projective  $R$ -modules. Prove that if  $f: P \rightarrow Q$  is a surjective  $R$ -module homomorphism, then  $\ker f$  is a projective  $R$ -module.
- Let  $R$  be an integral domain with quotient field  $Q$ . Show that if  $V$  is a finite dimensional vector space over  $Q$ , then  $(Q \otimes_R Q) \otimes_Q V \cong V$  as vector spaces over  $Q$ .
- Let  $K$  be a Galois extension of a field  $F$  of order  $11^4$ . Prove that there are intermediate fields  $F = K_0 \subseteq K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4 = K$  such that  $[K_i : K_{i-1}] = 11$  and  $K_i$  is a Galois extension of  $F$ , for  $i = 1, 2, 3, 4$ .
- Let  $R$  be a local commutative ring with 1 and with maximal ideal  $M$ . Suppose  $I$  is an ideal such that  $0 \subsetneq I \subseteq M$ . Prove that  $R/I$  is not a projective  $R$ -module.

## Algebra Prelim, August 2009

Do all problems

1. Let  $p$  be a prime, let  $n$  be a non-negative integer, and let  $S$  be a set of order  $p^n$ . Suppose  $G$  is a finite group that acts transitively by permutations on  $S$  (so if  $s, t \in S$ , then there exists  $g \in G$  such that  $gs = t$ ) and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Prove that  $P$  acts transitively on  $S$ .
2. Prove that there is no simple group of order  $448 = 7 * 64$ .
3. (a) Prove that  $x^2 + 1$  is irreducible in  $\mathbb{Z}/3\mathbb{Z}[x]$ .  
(b) Prove that  $x^3 + 3x^2 - 9x + 12$  is irreducible in  $\mathbb{Z}[i][x]$ .
4. Let  $R$  be a PID, let  $M$  be a finitely generated right  $R$ -module, and let  $N$  be an  $R$ -submodule of  $M$ . Prove that there exists an  $R$ -submodule  $L$  of  $M$  and  $0 \neq r \in R$  such that  $L \cap N = 0$  and  $Mr \subseteq L + N$ .
5. Let  $R$  be a ring and suppose we are given a commutative diagram of  $R$ -modules and homomorphisms,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{j} & D \end{array}$$

where  $f$  is onto and  $j$  is one-to-one. Prove that there exists a unique  $R$ -module homomorphism  $k: B \rightarrow C$  such that the resulting diagram commutes (so  $kf = g$  and  $jk = h$ ).

6. Compute the isomorphism type of the Galois group of  $x^4 - 2x^2 + 9$  over  $\mathbb{Q}$ .
7. Let  $R$  be a commutative ring with a 1 and let  $I, J$  be ideals of  $R$ . Prove that  $R/I \otimes_R R/J \cong R/(I + J)$  as  $R$ -modules.

## Algebra Prelim, August 2010

Do all problems

1. Let  $G$  be a group of order 105 with a normal Sylow 3-subgroup. Show that  $G$  is abelian.
2. Find the Galois group of  $f(x) = x^4 - 2x^2 - 2$  over  $\mathbb{Q}$ , describing its generators explicitly as permutations of the roots of  $f$ .
3. Let  $p$  be a prime. Prove that the extension  $\mathbb{F}_{p^n} \supset \mathbb{F}_p$  has Galois group generated by the Frobenius automorphism  $\sigma: \mathbb{F}_{p^n} \rightarrow \mathbb{F}_{p^n}$  given by  $\sigma(a) = a^p$  for all  $a \in \mathbb{F}_{p^n}$ .
4. Show that there is no 3 by 3 matrix  $A$  with entries in  $\mathbb{Q}$ , such that  $A^8 = I$  but  $A^4 \neq I$ .
5. Let  $R$  be an integral domain. A nonzero nonunit element  $p \in R$  is *prime* if  $p \mid ab$  implies  $p \mid a$  or  $p \mid b$ . A nonzero nonunit element  $p \in R$  is *irreducible* if  $p = ab$  implies  $a$  or  $b$  is a unit. Show that
  - (a) Every prime is irreducible.
  - (b) If  $R$  is a UFD, then every irreducible is prime.
6. Let  $R$  be the ring  $\mathbb{Z}/6\mathbb{Z}$  and let  $I$  be the ideal  $3\mathbb{Z}/6\mathbb{Z}$ . Prove that  $I \otimes_R I \cong I$  as  $R$ -modules.
7. Let  $S$  be a multiplicatively closed nonempty subset of the commutative ring  $R$  with a 1. Assume that  $0 \notin S$ .
  - (a) Show that if  $R$  is a PID, then  $S^{-1}R$  is a PID.
  - (b) Show that if  $R$  is a UFD, then  $S^{-1}R$  is a UFD.
8. Let  $R$  be a commutative ring with a 1.
  - (a) Show that if  $x \in R$  is nilpotent and  $y \in R$  is a unit in  $R$ , then  $x + y$  is a unit in  $R$ .
  - (b) Let  $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R[x]$ . Show that  $f$  is a unit in  $R[x]$  if and only if  $a_0$  is a unit in  $R$  and  $a_i$  is nilpotent for  $i > 0$ .

## Algebra Prelim, August 2011

Do all problems

1. Prove that there is no simple group of order 380.
2. Let  $k$  be a field with  $|k| = 7$ , let  $n$  be a positive integer, and let  $f, g$  be coprime polynomials in  $k[x_1, \dots, x_n]$ . If  $f^3 - g^3 = h^3$  for some nonzero polynomial  $h \in k[x_1, \dots, x_n]$ , prove that there exists  $p \in k[x_1, \dots, x_n]$  and  $u \in k$  such that  $f - g = up^3$ . Hint: factor  $f^3 - g^3$  as a product of three polynomials, and note that these polynomials are pairwise coprime.
3. Let  $R$  be a PID, let  $p$  be a prime in  $R$ , and let  $M, N$  be finitely generated left  $R$ -modules such that  $pM \cong pN$ . Assume that if  $0 \neq m \in M$  or  $N$  and  $pm = 0$ , then  $Rm$  is not a direct summand of  $M$  or  $N$  respectively (i.e. there is no submodule  $X$  such that  $Rm \oplus X = M$  or  $N$ ). Prove that  $M \cong N$ .
4. Let  $f(x) \in \mathbb{Q}[x]$  be an irreducible polynomial of degree 9, let  $K$  be a splitting field for  $f$  over  $\mathbb{Q}$ , and let  $\alpha \in K$  be a root of  $f$ . Suppose that  $[K : \mathbb{Q}] = 27$ . Prove that  $\mathbb{Q}(\alpha)$  contains a field of degree 3 over  $\mathbb{Q}$ .
5. Let  $p$  be a prime and let  $A$  denote all  $p^n$ -th roots of unity in  $\mathbb{C}$ . Thus  $A$  is the abelian subgroup of the nonzero complex numbers under multiplication defined by  $\{e^{2\pi im/p^n} \mid m, n \in \mathbb{N}\}$ , in particular  $A$  is a  $\mathbb{Z}$ -module. Determine  $A \otimes_{\mathbb{Z}} A$ .
6. Let  $k$  be a field, let  $A \in M_3(k)$ , the 3 by 3 matrices with entries in  $k$ , and suppose the characteristic polynomial of  $A$  is  $x^3$ . Prove that  $A$  has a square root, that is a matrix  $B \in M_3(k)$  such that  $B^2 = A$ , if and only if the minimal polynomial of  $A$  is  $x$  or  $x^2$ .
7. Let  $R$  be an integral domain, let  $n$  be a positive integer, let  $S$  be a subset of the polynomial ring in  $n$  variables  $R[x_1, \dots, x_n]$ , and define  $Z(S) = \{(r_1, \dots, r_n) \in R^n \mid f(r_1, \dots, r_n) = 0 \text{ for all } f \in S\}$ , the zero set of  $S$ . Prove that there exists a finite subset  $T$  of  $S$  such that  $Z(S) = Z(T)$ .

## Algebra Prelim, January 2012

Do all problems

1. Recall that a proper subgroup of the group  $G$  is a subgroup  $H$  of  $G$  with  $G \neq H$ . Now suppose  $G$  is a finite cyclic group. Prove that  $G$  is not a union of proper subgroups.
2. Prove that a group of order  $6435 = 9 \cdot 5 \cdot 11 \cdot 13$  cannot be simple.
3. Prove that  $M_2(\mathbb{Q}) \otimes_{M_2(\mathbb{Z})} M_2(\mathbb{Q}) \cong M_2(\mathbb{Q})$  as  $(M_2(\mathbb{Q}), M_2(\mathbb{Q}))$ -bimodules ( $M_2(\mathbb{Q})$  indicates the ring of 2 by 2 matrices with entries in  $\mathbb{Q}$ ).
4. Let  $R$  be a PID which is not a field and let  $M$  be a finitely generated injective  $R$ -module. Prove that  $M = 0$ .
5. Let  $p$  be an odd prime and for a positive integer  $n$ , let  $\zeta_n = e^{2\pi i/n}$ , a primitive  $n$ th root of 1.
  - (a) Prove that  $\mathbb{Q}(\zeta_p) = \mathbb{Q}(\zeta_{2p})$ .
  - (b) Prove that  $1 + x^2 + x^4 + \cdots + x^{2p-2}$  is the product of two irreducible polynomials in  $\mathbb{Q}[x]$ .
6. Determine the isomorphism class of the Galois group of the polynomial  $x^5 - 5x - 1$  over  $\mathbb{Q}$ .
7. For  $n$  a positive integer, let  $\mathbb{A}^n$  denote affine  $n$ -space over  $\mathbb{Q}$ .
  - (a) Prove that every element of  $\mathbb{Q}[x, y]/(x^3 - y^2)$  can be written in the form  $(x^3 - y^2) + f(x) + yg(x)$  where  $f(x), g(x) \in \mathbb{Q}[x]$ .
  - (b) Prove that  $\mathbb{Q}[x, y]/(x^3 - y^2) \cong \mathbb{Q}[t^2, t^3]$ , the subring of the polynomial ring  $\mathbb{Q}[t]$  generated by  $t^2, t^3$ .
  - (c) Prove that  $\mathbb{Q}[t^2, t^3]$  is not a UFD.
  - (d) Let  $V$  denote the affine algebraic set  $\mathcal{Z}(x^3 - y^2)$ , the zero set of  $x^3 - y^2$  in  $\mathbb{A}^2$ . Determine the coordinate ring of  $V$ .
  - (e) Is  $V$  isomorphic to  $\mathbb{A}^1$  as affine algebraic sets? Justify your answer.



## Groups of order 36

Here we construct a group of order 36 which has a *nonnormal* subgroup of order 12. Let  $S_3$  denote the symmetric group of degree 3. Then  $G := S_3 \times S_3$  is a group of order 36 which has a normal subgroup  $K$  such that  $G/K \cong S_3$  (e.g., we could let  $K = \{(x, 1) \mid x \in S_3\}$ ). Now  $S_3$ , and hence also  $G/K$ , have a nonnormal subgroup of order 2. Using the subgroup correspondence theorem, we deduce that  $G$  has a nonnormal subgroup of order  $2 \cdot |K| = 12$  (which contains  $K$ ), as required.

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