Equations with Parameters: A Locus Approach

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This paper introduces technology-based teaching ideas that facilitate the development of qualitative reasoning techniques in the context of quadratic equations with parameters. It reflects on activities designed for and used with prospective secondary mathematics teachers in accord with standards for teaching and recommendations for teachers in North America. The main educational implications of the proposed didactics include an emphasis on using geometric constructions in the context of algebra, emergence of residual mental power that can be used in the absence of technology, and development of skills in problem posing.

INTRODUCTION

For over two decades, emerging advances in the computerization of mathematics education have opened exciting opportunities to revisit, enhance, and extend secondary school mathematics curricula. As mentioned by many authors (Stephens & Hartman, 2004; Kersaint, Horton, Stohl, &
Abramovich and Norton, 2003; Mistretta, 2002; Chamblee & Slough, 2002; Browning & Klespis, 2000), these opportunities can only be realized if teachers of mathematics are well prepared to appropriately incorporate computers into their teaching. Appropriate use of technology in this context may include the joint use of computing activities and cognitive follow-up tasks that are true reflections of these activities. This requires a deep understanding of mathematics—an appreciation of the subject matter as a rich interplay of concepts, ideas, and techniques, including those traditionally taught at the secondary school level. The Conference Board of the Mathematical Sciences (2001) recommends that prospective teachers of secondary school mathematics (referred to below as teachers) become familiar with the ways of exploring fundamental mathematical concepts taught at this level from an advanced standpoint. Furthermore, the Board suggests that effective mathematics teacher education programs should take full advantage of technological tools, which enable informal entries into grade-appropriate advanced mathematical ideas and facilitate movement from informal to formal reasoning. In other words, the teachers should become familiar with technology-enabled approaches to revealing hidden domains of secondary school mathematics curricula (Abramovich & Brouwer, 2003).

This paper explores one such domain associated with topics dealing with quadratic functions and corresponding equations. As an appropriate software tool, less generally described in the literature, the authors introduce the Graphing Calculator 3.2 (Avitzur, Gooding, Herrmann, Piovanelli, Robbins, Wales, & Zadrozny, 2002), referred to below as the GC. Its ability to graph a relation from any two-variable equation without the need to convert the latter into a form suitable for traditional “function grapher” software or a hand-held graphing calculator is an important advantage of the GC over other computer graphics software. This paper will show how mathematics teacher educators can draw on this powerful feature of the GC to develop effective techniques for the study of families of quadratic equations depending on parameters. Those techniques can also be extended to include the exploration of more complex algebraic structures.

The main up-to-date document of mathematics education reform in North America, *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), in which technology is elevated to the status of being a principle, suggests that all students in grades 9-12 should be proficient in using a variety of representations of functional relations including those depending on parameters. The ability to recognize and analyze change in a variety of contexts, and to judge the meaning of such a change in the manipulation of a parameter, are among the most useful
mathematical skills that one needs to live and function successfully in the 21st century. Within secondary school algebra, such skills can be developed through making connections between external, representational characteristics of functions and equations, and their internal, structural properties. Teachers are unlikely to facilitate such connections for their students unless they have been involved in exploring and solving problems with parameters. Furthermore, the appropriate use of graphing technology in this exploratory context provides a support system for teachers to learn new problem-solving strategies and develops a strong foundation for the study of mathematics from an applied perspective.

This paper introduces pedagogical ideas for developing qualitative reasoning techniques using the GC in the context of pre-service secondary mathematics teacher education. It reflects the first author’s work with teachers (mathematics majors) over the last decade. It focuses on the exploratory nature of technology-enhanced learning of algebra and shows how an alternative look at the familiar can create a highly inquiry-oriented learning environment suitable to current standards and recommendations of mathematics education reform. These recommendations include the importance of courses within which the teachers “could examine … [the] use of computer tools in exploring algebraic ideas” (Conference Board of the Mathematical Sciences, 2001, p. 41), the need for the teachers to understand “the ways that basic ideas of … algebraic structures underlie rules for operations on expressions, equations, and inequalities” (Conference Board of the Mathematical Sciences, p. 40), and the development of rich problems for the teachers that “convey important aspects of mathematical thinking” (Steen, 2004, p. 869).

Through a number of illustrations (all of which have been tested successfully through appropriate educational applications at the tertiary level), the authors will demonstrate how re-conceptualization of graphing strategies can help teachers develop useful skills in using a combination of computer-enhanced visualization and qualitative reasoning for exploring advanced mathematical ideas. It will be argued that this approach provides occasion for teachers to gain mathematical power that eventually can partially replace the use of technology. The authors will also show how this approach can be put to work to develop teachers’ research-like experience in mathematics through technology-enabled problem posing.
Solving quadratic equations in one variable is a traditional topic in secondary school algebra. Students do not encounter insurmountable difficulties in this topic in which all their cognitive efforts are concentrated on either factoring trinomials or using quadratic formulas. Such treatment of the topic nurtures procedural skills alone and pays little attention to conceptual development. The reformed vision of secondary school algebra, however, goes beyond the need for students to remember formulas and master factorization techniques. This vision presents algebra as a dynamic, engaging, and conceptually-oriented subject matter. It defines algebraic activities as the exploration of patterns leading to conjecturing and, ultimately, to formal demonstration of mathematical propositions discovered. Such activities may be based on the blend of informal and formal reasoning and can be amplified by the use of technology (Fey, 1989; Dugdale, Thompson, Harvey, Demana, Waits, Kieran, McConnell, & Christmas, 1995; Kaput, 1995; Akst, 1998; Yerushalmy & Chazan, 2002). As demonstrated below, topics that deal with functions and equations that are dependent on parameters offer a perfect milieu for the activities.

Ursini and Trigueros (2004) reported that students, both at the secondary and tertiary levels, have difficulties working with parameters because parameters are meaningless to them. These authors suggested using geometric contexts to help learners comprehend the meaning of parameters. However, whereas their approach does create context in which parameters arise, it seems to be a traditional one from an algebraic perspective (i.e., focusing on the study of discriminants in the case of quadratic equations). As will be shown below, geometric representations of algebraic equations, made possible by the use of the GC when a parameter becomes one of the coordinates in the Cartesian plane, has the potential of creating context in which parameters can be both meaningfully interpreted and easily visualized.

The use of equations and functions depending on parameters in the context of in-service teacher education was reported by Zaslavsky (1995) who suggested utilizing mathematically-rich modifications of traditional secondary school algebra problems as a means to bridge the ease of a routine and complexity of challenge. It was found that open-ended tasks involving quadratic functions with parameters are useful intellectual tools for professional development of mathematics teachers. The purpose of Zaslavsky’s use of open-ended tasks in this context was to move away from mundane
problems with only “one correct answer” to parameterized problematic situations. Indeed, one is likely to gain an appreciation of how much information can be extracted from familiar situations presented by routine tasks if one is allowed to question the familiar in a meaningful way. Therefore, an open-ended approach to quadratics shifts the focus of classroom activities away from memorizing standard procedures and allows for one’s conceptual development and use of advanced mathematical thinking in the context of secondary school algebra.

Dugdale, Wagner, and Kibbey (1992) suggested using a technological paradigm to introduce interesting teaching ideas, downplaying the pedagogy of memorized rules in favor of conceptual understanding. These authors developed an alternative approach to the graphing of polynomials with a focus on using a computer-generated graph as an intermediate thinking device rather than seeing the graph as a final product of activities. In particular, the approach enables one’s comprehension of the effect of each term of a polynomial function on the behavior of the whole function through experiments with monomial graphs. It enhances one’s understanding of the role of a particular coefficient on the behavior of the graph and facilitates the transition to the graphing of functions with parameters. Furthermore, learning to use computer-generated graphs as cognitive tools can lead to the development of qualitative graphing skills that can be used in the absence of technology.

Another approach to reforming secondary school algebra is to integrate the pedagogy of open-ended tasks and the use of technology. As recommended by many authors (e.g., Kaput, 1995; Burril, 1998; Saul, 1998), the focus of technology-enhanced activities in the context of secondary school algebra should be shifted from supporting the traditional algebra curriculum to extending it to new cognitive activities made possible with the use of new tools of technology. One such suggestion was to create learning situations that highlight invariant relations among variables. The concept of invariance under a certain transformation is one of the most fundamental concepts in mathematics. The ease of observing variation made possible by the use of computer technology provides great opportunities for learners to gain insight into invariance. In Kaput’s (1992) words, “to recognize invariance—to see what stays the same—one must have variation” (p. 525). As will be shown below, the study of specific combinations of the x-intercepts of a parabola enhanced by a computational experiment, can demonstrate their invariance under a parameter variation. Once invariance is recognized, it can be used as a tool in developing a formula that solves a quadratic equation. Such a blend of computing activities and the use of deductive reasoning provides a clear-cut example of appropriate use of technology.
Significant curricular implications may result from shifting the focus of school mathematics activities from the study of specific functions to those dependent on parameters. Indeed, the introduction of parameter-dependent functions in the curriculum brings about a dynamic flavor in the seemingly static structure of pre-calculus and supports the reformed vision of school mathematics in general. When solving (or exploring) equations with parameters, one can shift focus from the search for numbers that solve a particular equation to the study of the structural properties of the family of equations (including global relations between parameters involved) that provide the solutions of a specified type (Fey, 1989). This kind of mathematical behavior, having its origin in that of pure and applied mathematics research, has great potential to reorganize mathematics classrooms to embody the vision expressed by the National Council of Teachers of Mathematics (2000): “Imagine a classroom … [where] technology is an essential component of the environment [and] students confidently engage in complex mathematical tasks chosen carefully by teachers” (p. 3). In other words, the reformed vision of a mathematics classroom manifests both the change of curricula and re-conceptualization of traditional teaching strategies.

More recently, Feurzeig, Katz, Lewis, and Steinbock (2000a, 2000b) proposed computer-based representations of properties of monic quadratic trinomials in the plane of their coefficients (parameters). These authors developed software allowing for the mapping of analytic properties of functions in the plane of corresponding parameters. For example, the property of having two equal, real, and complex roots can be represented, respectively, by a parabola, by a region inside the parabola, and by a region outside it. This approach to traditional topics in secondary school algebra was found to be conducive to promoting reflective thinking and encouraging skill in abstracting and generalizing. A similar (locus) approach for exploring quadratic equations with parameters using the GC is illustrated below.

**REMARKS IN SUPPORT OF USING THE GC IN A MATHEMATICS CLASSROOM**

Before proceeding with illustrations, it may be helpful to point out that the authors chose the GC as a tool for exploring algebraic ideas because of its ability to graph relations in a user-friendly setting. In fact, the authors argue that teachers do not need any prerequisite training in the use of this tool. To support this statement, note that when asked to reflect on their experience with the GC, several teachers acknowledged, “it is probably better
than anything else we have been using” and stated they are confident that “any student can easily use [it because] … the list of commands to master is straightforward and simple, unlike GSP which requires the book and quite a bit of experience to operate.” Implicitly addressing the Equity Principle (National Council of Teachers of Mathematics, 2000), one of the teachers affirmed “using the GC will enable more students to do meaningful explorations … [including those] who struggle with algebra.” Another implication of the above comments is that the use of the GC may be extended beyond mathematics teacher education programs due to the ease of the tool’s functioning and its great potential to enhance the development of algebraic reasoning of those with career paths outside of mathematics or mathematical education.

A LOCUS APPROACH TO QUADRATIC EQUATIONS WITH PARAMETERS

A locus is a set of points determined by a specified condition applied to a function. Consider the quadratic function \( f(x,c) = x^2 + x + c \) of variable \( x \) with parameter \( c \). One can say that the graph of the following equation

\[ x^2 + x + c = 0 \]  

is a locus defined by the zero value for the function \( f(x,c) \). One could use loci to study the effects of parameters in quadratic and higher degree functions, an approach referred to here as the locus approach. Equation (1) can be viewed as a relation and the use of the GC, capable of graphing relations, can facilitate the locus approach.

Each problematic situation (PS) shown below illustrates the advantage of using the concept of locus in the study of functions depending on a parameter, versus the traditional approach based on the graphing of a series graph. While both approaches are computer-based, the locus approach allows for much more impressive work on the part of the learners. As will be shown below, the same locus can be utilized as a familiar thinking device in solving qualitatively different problems with parameters. This suggests that the locus approach is conducive to the emergence of residual mental power that can be used in the absence of technology. According to Kieran (1993), the importance of developing such power was recognized by Waits and DeMane (1988) who studied students’ abilities to sketch graphs of complicated functions mentally as a function of training in graphing with technology. To begin, consider the following situation.
PS 1.1: For what values of parameter $c$ does Equation (1) have two positive roots?

To resolve PS 1.1 with the help of the GC, one does not need to construct a series of graphs $y = x^2 + x + c$ for different values of parameter $c$ (if using technology) or to carry out transformations of inequalities involving radicals (in the absence of technology). Quite the contrary, by graphing the locus of Equation (1), one can conclude immediately that because no horizontal line $c = constant$ intersects the locus to the right of the origin only such values of $c$ do not exist. The graph of the locus of Equation (1) is illustrated in Figure 1. Note also that hereafter, when typing an equation with a parameter in the equation window, the variable $y$ will be used in place of the parameter (in this case, $y$ is substituted for $c$) and, in order to avoid confusion, a comment about such a substitution will be created through the New Text Box feature available in the text menu of the GC. Furthermore, one can make use of the same locus (graph) and continue exploring Equation (1). To this end, consider the following situations.

PS 1.2: For what values of parameter $c$ does Equation (1) have two negative roots?

PS 1.3: For what values of parameter $c$ does Equation (1) have roots located by different sides of the origin?

PS 1.4: For what values of parameter $c$ does Equation (1) have two real roots?

Figure 1. Locus of Equation (1).
It should be noted that whereas the formula-oriented approach to PS 1.1 through 1.4 is based on memorization and the use of algebraic symbolism, the locus approach treats the locus of the family of quadratic equations as an object to be studied. This orientation emphasizes conceptual understanding of algebraic structures rather than the overuse of algebraic transformations. It leads naturally to more advanced learning situations such as the study of roots’ location about an arbitrary point (rather than about the origin only). Indeed, consider the following PS.

PS 1.5: For what values of parameter $c$ does Equation (1) have two different roots separated by the number 1?

A traditional approach to this inquiry involves the construction of the series of graphs of the function $y=x^2+x+c$ until the value of $c$ that provides an $x$-intercept that is equal to 1 is found. Alternatively, Kaput (1992) suggested that graphing a family of relations with one animated parameter in a single drawing enables students to recognize invariance through variation. Indeed, this kind of approach, too, has become more commonplace with the development of computer algebra tools. However, such an animation would only illustrate that the parameter $c$ raises and lowers the parabola, whose shape remains invariant (Figure 2).

In PS 1.5, the animation provides a seemingly continuous series of graphs with little benefit beyond the traditional approach. On the other hand, the locus approach involves the construction of the locus of Equation (1) and the straight line $x=1$ enabling one to see that the two graphs intersect at $c=-2$ (Figure 3). Furthermore, one can see that for all $c<-2$, the correspond-
ing points of the locus are located by different sides of the line $x=1$ implying that for all such $c$ the number 1 resides between the corresponding roots (the $x$-intercepts of points in common of the line $c=\text{const}$ and the locus). This concludes the exploration of PS 1.5. In much the same way, answers to PS 1.2 through 1.4 can be found.

![Figure 3. The locus approach to PS 1.5.](image)

While admitting the ease of manipulating a parameter by using a slider to animate the graph, it should be noted that this approach is computer-dependent. Generating the sequence of graphs in a non-computer environment using pure qualitative reasoning appears to be beyond one’s reach without technological amplification. The locus approach makes it possible to examine the structure of a quadratic equation by using essentially a single graph (Figure 3). Moreover, the locus approach enables one to extend the inquiry to other equations, such as $2x^2+x+c=0$, and resolve it by plugging $x=1$ in it to get the equation $2+1+c=0$ whence $c=-3$. One could conclude, without further use of technology that, for all $c<-3$, the last quadratic equation has roots with the property sought.

Apparently, the change of coefficient in $x^2$ (or in $x$) in Equation (1) yields a new problem; yet it does not require recourse to graphing. In fact, prospective teachers taught by the first author were encouraged to work on several tasks involving some type of locus, in order to develop and use its mental image as a thinking device. Just as knowledge of the commutative property of multiplication allows young children to avoid using a calcula-
tor in multiplying two numbers if they were already multiplied in different order, this mental image made it possible for the teachers to appreciate the notion of learning to graph in the absence of technology. It is in this sense that the locus approach has a potential to create a residual mental power that can be used by learners when graphing technology is not available. As mentioned in the beginning of the paper, this may be considered an appropriate use of technology because it involves the joint use of computer graphing and follow-up tasks that are true reflections of the graphing.

Using the context of PS 1.1 through 1.4, one can be guided to many interesting conclusions such as the inequalities $\frac{1}{4} \geq c > 0$ are necessary and sufficient for both roots of Equation (1) to be negative. One might be able to answer many questions about the roots’ structural properties and the dependence of these properties on the parameters, without recourse to the quadratic formula. The graph becomes not just an alternative representation of an algebraic object but rather a problem-solving tool. Therefore, whereas the idea of a graph as a tool is primary, the idea of a graph as a representation is secondary. Using a graph as a tool that enables one to answer questions or solve problems not accessible otherwise, teachers can have a much better appreciation of the multiple representations that a computer environment can offer and learn how to utilize these representations in both computer and non-computer problem-solving settings.

To conclude this section, note that the locus approach is a powerful means of analysis in many mathematical situations. Therefore, the use of the loci in coordination with the GC for the teaching of algebra has great potential to contribute to the professional development of teachers. It helps one to go “beyond the information given” (Bruner, 1973, p. 218) and, as will be shown below, to develop important skills in problem posing (Silver, 1994; Goldenberg & Walter, 2003).

**TRANSITION TO A HYPERBOLA-LIKE LOCUS**

The problematic situations examined so far involved only one family of quadratic equations. Whereas the corresponding locus alone can provide a rich problem-solving milieu, the locus approach can be extended naturally to include other families of quadratic equations. To this end, consider the more complicated phenomena exhibited by the family of equations of the form

$$x^2 + bx + 1 = 0$$

(2)
Traditional activities with this family in the GC environment may involve the examination of the locus of the vertices of the graph \( y=x^2+bx+1 \) with parameter \( b \) animated as a slider value. Through this exploration, a parabolic path \( y=-x^2+1 \) of the vertices can be discovered (Figure 4). However, the main focus of suggested activities is on interpreting and questioning the locus of Equation (2) pictured in Figure 5.

![Figure 4. A parabolic path.](image1)

![Figure 5. A hyperbola-like locus.](image2)

Note that the locus of Equation (2) appears to be hyperbolic. The locus approach becomes especially advantageous over traditional graphing or pure analytic approaches in the case of hyperbola-like loci. Once teachers are familiar with the locus as a problem-solving tool, they can be introduced to new problematic situations:

PS 2.1: For what values of parameter \( b \) does Equation (2) have real roots?
PS 2.2: Is it possible to point out values of parameter \( b \) such that Equation (2) has roots of different signs?
PS 2.3: For what values of parameter \( b \) does Equation (2) have both roots greater (smaller) than 1?
PS 2.4: For what values of parameter \( b \) does the equation have both roots greater than 0.5?

Such questions might arise from teachers’ initial examinations of the locus and, in turn, encourage further examination of the locus. They might notice that, when the equation has two real roots, one root increases in value as the other decreases, but that the roots will always have the same sign.
Through algebraic analysis, this can be related to the quadratic formula, as well as the existence of real roots. One may experiment with different horizontal straight lines in a single drawing to see whether or not they meet the graph and solve problems such as PS 2.5. Furthermore, one may be asked whether the previous observations (especially the inverse relationship between the roots) provide implications for the product of the roots, deriving an analytic formula for solving Equation (2).

PS 2.5: Let \( x_1 \) and \( x_2 \) be real roots of Equation (2). For what values of parameter \( b \) does the inequality \( x_1 < x_2 < b \) (alternatively, \( b < x_1 < x_2 \) or \( x_1 < b < x_2 \)) hold true?

PS 2.6: What can be said about the product of the roots for Equation (2)?

PS 2.7: How can one use the graph of the straight line \( b = -2x \) in order to find an analytic formula for solving Equation (2)?

Figure 6 presents the application of the locus approach to PS 2.5. It shows the locus of Equation (2) consisting of two hyperbola-type branches, a horizontal line \( b = \text{constant} \) (in Figure 6, this constant is defined by the slider value \( n = 4 \)), three vertical lines defined by \( x = b \) and the roots of \( x^2 + 4x + 1 = 0 \) (whose projections on the \( x \)-axis are \( b, x_1, \) and \( x_2 \), respectively), and the graph of \( b = x \), which, being the bisector of the first and third quadrant, maps any value of parameter \( b \) from the \( b \)-axis on the \( x \)-axis so that the corresponding values of \( x_1, x_2, \) and \( b \) can be compared. Visualization suggests that the upper branch of the locus is located to the right of the bisector \( b = x \) and above the level \( b = 2 \). This implies that for all \( b > 2 \) the line \( b = \text{constant} \) meets two points on the locus before it hits the graph \( b = x \). For the lower branch of the locus (which is located below the level \( b = -2 \)) and the graph \( b = x \) the opposite situation can be observed. Finally, for \( |b| < 2 \) the line \( b = \text{constant} \) is either tangent to the locus or has no points in common with it. These geometric descriptions of the roots’ dependence on parameter \( b \) can be given the following algebraic formulations: for \( b > 2 \) the inequality \( x_1 < x_2 < b \) holds true; for \( b < -2 \) the inequality \( b < x_1 < x_2 \) holds true; and no \( b \) exists to satisfy the inequality \( x_1 < b < x_2 \). This completes the solution to PS 2.5.

In turn, PS 2.6 can be resolved through a computational experiment based on the use of the cursor pointer which, when applied to any point of constructed graph, enables the GC to display its coordinates in the top-left corner of the graph window (this feature is shown in Figure 3). For different values of \( b \), one can find (using any type of a calculator) the product of corresponding roots generated through cursor pointing to obtain a very close approximation to the free term in Equation (2). A similar computational experiment, explained in the next section, can be conducted in the context of PS 2.7.
The PS list can be continued. As Halmos (1975) noted, “The best way to teach teachers is to make them ask and do what they, in turn, will make their students ask and do” (p. 470). Therefore, it is useful to consider the equation $x^2 + bx + c = 0$ with two parameters and setting $y = b$ to produce its locus for different (slider) values of the parameter $c$. Then, setting $y = c$, draw the locus for different (slider) values of the parameter $b$. The same explorations can be carried out for the equation $ax^2 + bx + 1 = 0$, and so on. This approach generates many interesting questions that promote teachers’ active involvement into in-depth explorations of quadratic equations through visualization with minimal use of pure algebraic skills and ultimately can be utilized at the pre-college level. Indeed, students’ curiosity, higher-level thinking, and reasoning skills are strongly dependent on the education their teachers received themselves (Kilpatrick, 1987a).

**VIÉTE’S FORMULAS AND THEIR GRAPHICAL REPRESENTATION**

As mentioned above, one can explore specific combinations of the roots of quadratic equations. This was done by Viète, a 16th century French mathematician (Viète was also the first to introduce the use of letters in algebra). Using the GC, teachers can experimentally derive a formula to solve quadratic equations and then represent the dependence of a variable on a
parameter expressed though this formula in a graphic form. They can then use the locus to confirm the correctness of their experimental finding. As an example, consider the following problematic situation in the context of the equation

\[ x^2 + bx + c = 0 \quad (3) \]

**PS 3.1:** Knowing that Equation (3) has real roots \( x_1 \) and \( x_2 \), find algebraic formulas for these roots.

By constructing a series of loci for Equation (3) in the plane \((x, c)\) for different values of \( b \) (whose variation can be controlled by a slider), it can be demonstrated that a vertical line \( x = -b/2 \) is the line of symmetry for the locus of Equation (3). To this end, one can use the cursor pointer to discover that regardless of the value of \( c \), this line bisects a horizontal segment connecting two points on the locus. In other words, for each value of parameter \( c \), the corresponding roots, \( x_1 \) and \( x_2 \), can be described as being at the same distance \( d \) from the line of symmetry. Analytically, this means that \( x_1 = \frac{-b}{2} + d \) and \( x_2 = \frac{-b}{2} - d \). Therefore, \( x_1 + x_2 = -b \) (in Figure 7: \( c = 1, b = 3, x_1 = -3/2 + d, x_2 = -3/2 - d \)). Knowing the sum of roots, one can find their product. Indeed, because \( x_2 = -b - x_1 \), \( x_1 \cdot x_2 = x_1 \cdot (-b - x_1) = c \). On the other hand, \( x_1 \cdot x_2 = \frac{b^2}{4} - d^2 \). Therefore, \( d = \pm \sqrt{b^2/4 - c} \). This results in the formulas, \( x = \frac{-b - \sqrt{b^2 - 4c}}{2} \) and \( x = \frac{-b + \sqrt{b^2 - 4c}}{2} \), which can be interpreted as an equivalent form of Equation (3). One can visualize this equivalence by graphing these formulas with the GC in the plane \((x, c)\) with \( b \) being a slider value; the resulting graph (Figure 8) is indeed the locus of Equation (3) for \( b = n \). Similar constructions of Viète’s formulas can be carried out in the plane \((x, b)\), with \( c \) being a slider value, to see if this results in a hyperbola-like locus (cf. PS 2.7). In that way, the locus approach can make quadratic formulas come alive for teachers and, eventually, for their students.
Figure 7. The symmetry of locus implies the symmetry of roots.

Figure 8. Graphing Viète’s formulas.
TECHNOLOGY AS A MEDIUM FOR PROBLEM FORMULATING

An important aspect of a secondary mathematics teacher education program is providing prospective teachers with experience in formulating problems. As mentioned by Ellerton and Clarkson (1996), this experience, when used appropriately, has great potential to influence the mathematics taught in schools. It has been argued that problem posing is an informed extension of problem solving (Kilpatrick, 1987b; Brown & Walter, 1990; Silver, Kilpatrick, & Schlesinger, 1990). Therefore, the two intellectual activities are closely related. The appropriate use of technology in general has great potential to provide teachers with this kind of experience.

It appears that, whereas the role of technology in problem solving is well understood (McArthur, Stasz, & Hotta, 1987; Abramovich & Brown, 1996; Jiang & McClintock, 2000; National Council of Teachers of Mathematics, 2000; Conference Board of the Mathematical Sciences, 2001), the role of technology in problem posing is a less explored domain. Kilpatrick (1987b), seeking to understand the sources of problem formulation, singled out computers as valuable tools in fostering problem-posing skills among pre-service teachers, allowing them to generate numerical and pictorial patterns that lead to new problems, as well as to vary the conceptual and syntactic structure of an existing problem statement. These computer activities can result in problematic situations manifesting different levels of mathematical complexity.

The Professional Standards for Teaching Mathematics (National Council of Teachers of Mathematics, 1991) has recognized the potential for technology to enhance problem posing:

There are a variety of ways technology may be used to enhance and extend mathematics learning and teaching. By far the most promising are in the areas of problem posing and problem solving in activities that permit students to design their own explorations and create their own mathematics. (p. 134)

However, no specific examples of how that could be done have been provided. With this in mind, having conducted an extensive search of the literature, the authors concluded that the few existing papers, describing the use of technology as a medium for problem formulating are mostly related to Logo and dynamic geometry environments. The use of Logo by pre-college students in experimental and collaborative problem-posing activi-
ties was probably first reported by Hoyle and Sutherland (1986) and Noss (1986). A general description of such activities in terms of cultural amplifiers was made by Pea (1987) by appreciating the possibility of discovering unknown theorems on the part of students through the use of Geometric Supposers. Describing the same tool as being utilized in the framework of specific classroom activities, Yerushalmy, Chazan, and Gordon (1993) suggested that teachers could provide students with the experience of mathematical discovery “by posing inquiry problems to explore” (p. 117).

In the context of Cabri-géomètre, Laborde (1995) found that “computers allow for the design of some new problems that are impossible in a paper and pencil environment” (p. 35). She argued, however, that such a qualitative change of a didactical milieu is one of the reasons teachers resist the use of computers in the classroom. This suggests that pre-service teachers of mathematics do need support in making the shift from using technology as a problem solving tool to using it as a medium for problem posing, and in doing so, to acquire what in the context of spreadsheets has been recently referred to as research-like experience (Abramovich & Brouwer, 2004). Similar experience for the teachers can be provided through the GC-enabled explorations of algebraic equations with parameters.

In fact, all of the problematic situations illustrated in this paper were developed by the teachers under the guidance of the first author within the context of a secondary mathematics education course. The second author has subsequently used many of these problematic situations in another secondary mathematics education course to demonstrate that technology can be used not only to enhance the curriculum, but also to change it. A characteristic feature of many of the problems is that they exemplify the use of technology, through the locus approach, as a medium for posing problems that are either too difficult to solve through a pure algebraic approach or too abstract to formulate without technology. Once such a problem is posed, its solution can be found through a combination of geometric and algebraic reasoning. In this way, formal operations on algebraic equations may become true reflections on graphing activities with the GC.

Consider, for example, PS 2.4. It refers to an unknown condition (values of the parameter) and a phenomenon (behavior of the roots) that are difficult to observe through traditional graphing and difficult to discover through algebraic means. Yet, having constructed the locus shown in Figure 5, the teachers were able to resolve this problematic situation without recourse to the quadratic formula. Ironically, one does not need to use the formula as a problem-solving tool in order to deepen one’s understanding of its structure. As the first author’s work with the teachers indicates, such understanding
results from one’s conceptualization of the locus as an equivalent representation of the quadratic formula, including geometric interpretation of what it means for a discriminant of a quadratic equation to be a factorable or non-factorable algebraic expression.

Indeed, it was the teachers’ constructions of loci, like the one illustrated in Figure 5, that provided occasion for them to participate in the formulating of PS 2.1 through 2.7. Each PS depended on the pre-service teachers’ visualization of such loci. For example, PS 2.3 may result from the observation that the values of the roots converge toward (or diverge from) 1 (or -1), as the value of $b$ increases. Teachers’ observations about this are facilitated by the visualization of the two components of the locus, which have respective turning points when $x=1$ and $x=-1$. Such observations may also lead to questions about the product of the roots, as introduced in PS 2.6. This, in turn, may yield a kind of quadratic formula, such as that derived from PS 3.1.

Problem posing provides an experience that is relevant to the teachers’ future careers. It also seems to be at least as rich a mathematical experience for them as is problem solving. To understand this, consider the kind of mathematical reasoning and understanding that is required for teachers to pose the problematic situations shared in this paper. Each one suggests particular mathematical insights, whether they are about the graphical effects of particular parameters (PS 1.1 through 1.4), geometric means of roots (PS 2.3), or the quadratic formula (PS 3.1). However, these insights may not have been made explicit to the teachers ahead of time. Instead, any PS that the teachers create might be more akin to conjectures that they can subsequently test in their problem-solving experiences.

**CONCLUDING REMARKS**

As was mentioned at the beginning of the paper, the teachers who were engaged in posing and solving each PS described above were also asked to evaluate the locus approach. Analysis of available evaluations from a mathematical perspective indicated that the teachers perceived advantages to using the approach, not only because it encourages students to “examine many questions,” but also because the students can form their own questions, “ensuring more student involvement and giving them control over their learning.” This suggests that problem posing is an activity that may be extended to students in a secondary mathematics classroom.

Although some teachers were concerned that introducing new representations (loci) might generate confusion, most of them noted that such
a representation “gives another dimension to the quadratic equation.” This kind of argument is enhanced and supported by mathematics education theory, especially by the work of Pea (1987), who described the role of cognitive technologies in transforming the qualities of thought. In his words, a cognitive technology is “any medium that helps transcend the limitations of the mind… [and] make[s] external the intermediate properties of thinking, which can then be analyzed, reflected upon, and discussed” (p. 91). In agreement with the Vygotskian cultural-historical perspective on cognition (Bruner, 1985), he assumes that intelligence is “a product of the relation between mental structures and [cultural] tools” (Pea, 1987, p. 91). Therefore, changes in the use of the tool result in changes in thought. In such a way, learners’ actions in working with loci leave a residue that is a kind of mental power.

Using loci as advocated in this paper certainly instantiates the use of a medium to make external the properties of thinking. It is not the GC that acts as a cognitive tool, but the loci that one can produce with the GC. The loci serve as externalized properties of thinking that one can visualize, manipulate, and analyze. In that loci serve as a medium through which teachers can pose problems and discuss solutions, they are also cultural tools. Of course, teachers must first construct meaning for the loci they produce, and their potential difficulty in doing so was their main concern about using the locus approach. However, meaning can be constructed through the same activities of visualizing, manipulating, and analyzing graphs.

Because loci are qualitatively different from the equations on which they are based, one should expect to experience qualitatively different thinking in working with them. So, using the locus approach does not simply save time or work for the learner; it introduces a new dimension to thinking about quadratic equations, “qualitatively changing the structure of the functional system” (Pea, p. 94). This is the kind of residual mental power that the teachers experienced. In this paper, the authors have demonstrated that this occurred for the teachers by providing examples of their creation of novel problems. The authors have suggested how such power might develop for teachers and their future students alike by presenting hypothetical solutions to these problems, solutions that would indicate new ways of thinking about quadratic equations and their graphs.

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