MATH 1525: Supplement Sheet for Lab 3

These are the functions: \( y = a(1 + r)^t = ab^t, y = ae^{kt}, N = M/(1 + Ae^{-kt}), N = A + Be^{kt}, y = \log_b x, y = \ln x. \)

The definition of an **EXPONENTIAL FUNCTION** is as follows:
For \( b > 0 \) and \( b \neq 1 \), the exponential function with base \( b \) is \( f(x) = b^x \). (usually the base we will use in this class is \( b = e \) where \( e = 2.71828 \). This function is called the natural exponential function \( y = e^x \))

The inverse function of the exponential function is the **LOGARITHMIC FUNCTION**. These two functions are related as follows: If \( m = b^d \) then \( \log_b m = d \) (this is read log to the base \( b \) of \( m \) is \( d \)). Again we usually use the natural base \( e \) so \( \log_e x = \ln x \).

The following properties are important to remember:

1) \((b^m)(b^n) = b^{m+n}\)
2) \(b^m/b^n = b^{m-n}\)
3) \(b^0 = 1\)
4) \(b^n = 1/b^{-n}\)
5) \(b^{ln b} = b\)
6) \(ln(nm) = ln n + ln m\)
7) \(ln (n^d) = d ln n\)
8) \(ln e = 1\)
9) \(ln 1 = 0\)
10) \(ln e^m = m\)
11) \(e^{ln m} = m\)

I.) The above properties can be used to simplify a problem before you try to solve it.

Example:
Given the function \( \ln[(x-2)(x+3)^4] \). You can simplify this function using the above rules to get \( \ln(x-2) + 4\ln(x+3) \).

Example:
\(\ln[(x-2)/(x+5)] = \ln(x-2) – \ln(x+5)\)

II.) The most important use of the above properties will be to solve equations for unknowns.

Example:
If \( 10 = 3e^t \), solve for \( t \). First divide both sides by 3 and then take the natural log of both sides as shown:
\[
\frac{10}{3} = e^t \\
\ln\left(\frac{10}{3}\right) = \ln(e^t) \\
\ln(10/3) = t(\ln e) \\
\ln(10/3) = t (1) \quad \text{so} \quad t = \ln(10/3). \quad \text{You can use a calculator to convert to a decimal.}
\]
Which rules were used to solve this equation?

Example:
If \( \ln(y) = x + 4 \) solve for \( y \). To solve this problem we will take the exponential of both sides.
\[
e^{\ln y} = e^{x+4} \\
y = e^x e^4. \quad \text{Since } e^4 \text{ is a constant, } y = (2.71828)^4 e^x. \quad \text{Which rules were used to solve this equation?}
\]

Practice:
1. Use the relationship that if \( m = b^d \) then \( \log_b m = d \) to solve for \( x \).
   a) \( \log_2 25 = 2 \) \quad b) \( \log_{56} x = 1/2 \) \quad c) \( \log_8 64 = x \) \quad d) \( \log_5 (4x-4) = 2 \) \quad e) \( \log_2 x(x+2) = 3 \)
2. Solve each of the following for \( x \) by either taking the natural log or exponential of both sides.
   a) \( e^x + 2 = 5 \) \quad b) \( \ln(2x+1) = 5 \) \quad c) \( 3^x = 4 \)
3. Simplify using the laws of Logarithms:
   \( \ln [x^3(x+1)/(x+2)^4] \)
4. Express as a single log form:
   \( \ln(3x) + \ln(x-4) – 5\ln(x^2 + 3) \)
EXPONENTIAL GROWTH AND DECAY:  \( y = ab^t \) and \( y = ae^{kt} \)
You have already learned to look at a table of data and determine if it can be modeled by an exponential function by looking at the percentage change in the output values. Now consider the following data from another point of view.

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<tbody>
<tr>
<td>Pop in millions</td>
<td>67.38</td>
<td>69.13</td>
<td>70.93</td>
<td>72.77</td>
<td>74.66</td>
<td>76.66</td>
<td>78.59</td>
</tr>
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Look at the growth ratio \( y_2/y_1 = \text{output}/(\text{previous output}) \) of the dependent values. \( G_1 = (69.13/67.38) = 1.026, G_2 = (70.93/69.13) = 1.026 \), etc. If the growth ratio is constant then the data is exponential. You can develop an equation for it as follows. Consider \( y = ab^t \). If you align the data above and plug the appropriate data in the formula you will have \( 67.38 = ab^0 \) which will become \( 67.38 = a \). So we may modify our formula to be \( y = 67.38 b^t \). It also turns out that \( b = \text{the growth ratio} \) so the formula now can be written as follows: \( y = 67.38(1.026)^t = 67.38 e^{(\ln 1.026)t} \) which has the form \( y = 67.38 e^{kt} \) where \( k = \ln 1.026 \).

**Example:** In 1960 the population was 10,000 and in 1980 it was 20,000. What will be the population in the year 2010? Assume exponential growth. (This statement usually means to use the equation \( y = ae^{kt} \))

The first step is to align the data. For \( t = 0 \) the population is 10,000 and for \( t = 20 \) the population is 20,000. If you put in the values when \( t = 0 \) then 10,000 = \( ae^0 \) and so \( a = 10,000 \). Now put in the values when \( t = 20 \) and the equation becomes 20,000 = 10,000\( e^{20k} \). Solve for \( k \) by dividing by 10000 and taking the log of both sides so \( k = (\ln 2)/20 \). Now we can answer our question. For the year 2010, \( t = 50 \) and \( y = 10,000e^{50\ln(2/20)} \).

Use your calculator to find the answer.

LOGISTIC GROWTH: \( N = M/(1 + Ae^{-kt}) \). This is a growth formula with a maximum \( y \)-value or upper bound (or decay with a lower bound).

**Example:** An Exercise Club wishes to increase its membership. But its facilities will support at most 800 members. One year ago they had 50 members. Now they have 300 members. How many members will they have in 3 years?

You will use the formula \( N = M/(1 + Ae^{-kt}) \) to solve this problem. Align your data so that for \( t = 0 \) there were 50 members and for \( t = 1 \) there were 300 members. Now find out how many members there will be when \( t = 4 \). \( M \) will equal the upper bound of 800. For \( t = 0, 50 = 800/(1 + Ae^0) = 800/(1 + A) \). Now solve to find that \( A = 15 \). For \( t = 1, 300 = 800/(1 + 15e^{-k}) \) and now solve for the quantity \( e^{-k} \) and it will equal \( 1/9 \). So the formula needed to answer the question is \( N = 800/(1 + 15(1/9)) \) now fill in the value \( t = 4 \) to get the final answer.

NEWTON'S LAW OF COOLING:
This equation, \( N = A + Be^{kt} \), deals with the rate at which the temperature of an object cools in relation to the surrounding temperature \( A \), but it also has applications in the business world as well. This equation is also bounded.

**Example:** A cake that is removed from the oven has a temperature of 300 degrees F. Three minutes later its temperature is 200 degrees F. The room temperature is 70 degrees F. What will be the temperature of the cake in ten minutes?

Align your data and let \( t = 0 \) when the temperature is 300. For \( t = 3 \) the temperature is 200. What is the temperature when \( t = 10 \)? In the equation above \( A \) is the surrounding temperature, so \( A = 70 \). When \( t = 0 \) then \( 300 = 70 + Be^0 \). Therefore you have \( 300 = 70 + B \). Now solve to find that \( B = 230 \). For \( t = 3, 200 = 70 \)

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and you need to solve for \( k \), \( k = \left[ \frac{\ln(130/230)}{3} \right] \). Use your calculator to simplify the calculation. 

and answer the question for \( t = 10 \). 

\[ N = 70 + 230e^{10\ln(130/230)/3} \] 

Is \( k \) a negative number?

**Practice Examples:**

1. Suppose five grams of a radioactive substance decreases to four grams in thirty seconds. How much will be left in fifty seconds? Assume exponential decay.

   In this case the growth rate will be negative since the initial amount is decreasing instead of increasing as in the problems above. You will use the formula \( M = Ne^{kt} \) (or \( y = ae^{kt} \)). You align your data so that for \( t = 0 \) there are 5g and for \( t = 30 \) there are 4g. From this information develop a formula to find how much is left when \( t = 50 \). For \( t = 0 \), \( N = 5 \) and for \( t = 30 \), \( 4 = 5e^{30k} \). Solve for \( k \) (divide by 5 and take the log of both sides). Notice that you get a negative value for \( k \). 

   Now determine your final answer when \( t = 50 \).

2. A rumor has started on the Va Tech Campus. If Tech beats UVA in football by 100 to 0 the students will get a week break to celebrate. There are 25000 students enrolled at Tech. Initially 100 students know the rumor. In 2 days 500 students know the rumor. How many students will know the rumor in 10 days?

   Using the Logistic formula, \( M = 25000 \) and for \( t = 0 \), \( N = 100 \) and for \( t = 5 \), \( N = 500 \). You need to solve for \( A \) (using \( t = 0 \)) and for \( e^{-kt} \) (using \( t = 5 \)) as in the problem above in order to find \( N \) for \( t = 10 \).

3. The Hokie Book Store found that \( t \) weeks after the end of a sales promotion the volume of sales was given by the function \( S = A + Be^{kt} \) for \( 0 < t < 4 \). \( A \) is always the usual volume of sales. Therefore \( A = 50,000 \) which is the average weekly volume of sales before the promotion. 

   The sales volumes at the end of the first and third weeks were $83,515 and $65,055, respectively. Assume that the sales volume is decreasing exponentially. Find the sales volume at the end of the fourth week.

   Align the data so that for \( t = 0 \), \( S = 83,515 \) and for \( t = 2 \), \( S = 65,055 \). Fill in the equation for \( t = 0 \) and you have \( 83,515 = 50,000 + Be^{0} = 50,000 + B \). Now solve for \( B \). Next put in the values for \( t = 2 \) and solve for \( k \). Finally answer the question for \( t = 3 \) (the fourth week).

**Practice:**

1. If $200 is put into an account for 6% interest **yearly** for 5 years then the final amount you have is given by the equation. 

   \[ S = P(1+r)^{t} \] 

   \( P \) is the initial amount invested, \( r \) is the interest rate and \( t \) is the number of years.

   a) Find the final amount after 5 years.

   b) You wish to have your investment compounded **continuously** at 6%, so now you need to use the formula \( S = Pe^{rt} \). Find the final amount after 5 years and compare values from a) and b).

2. Find the Present value of $1000 deposited in a bank at 10% interest compounded **continuously** for 8 years.

3. Because of an economic downturn, the population of an area declines at a rate of 1% per year. Initially the population was 100,000. What will the population be in 3 years?

4. On the Tech campus, whose population is 24000, an outbreak of flu occurs. When the infirmary begins its record-keeping there are 150 cases. One week later there are 300 cases. Assuming **logistic growth**, estimate the number of infected students two weeks after the record-keeping started.

5. According to estimates the percentages of households that own VCRs is given by the equation 

   \[ P = 68/(1 + 21.67e^{0.62t}) \] 

   for \( 0 < t < 12 \), where \( t \) is measured in years, with \( t = 0 \) the beginning of 1985. What percentage of households owned VCRs at the beginning of 1985? At the beginning of 1995?

6. A waterfront murder was committed, and the victim’s body was discovered at 3:15 am by the police. At that time the temperature of the body was 32° C. One hour later the body temperature was 30° C. According to the weather bureau, the temperature at the waterfront was 10° C from 10:00 pm until 5:00 am. At about what time did the murder occur? Remember that normal body temperature is 37 degree C.

7. The Cow Computer Company found that the monthly demand for its new line of home computers \( t \) months after placing the line on the market is given by the equation 

   \[ D = 2000 - 1500e^{-0.05t} \] 

   or \( t > 0 \)
Find the demand after one month, one year, five years. Graph $D(t)$ and determine the level where demand is expected to stabilize?