M1 - Linear Approximation

The problems in this assignment seek to develop both geometric intuition and analytic techniques for determining whether the linearization of a function $f(x)$ centered at $a$ is a good approximation with specified accuracy over a given interval.

Some Preliminaries

Let $\epsilon$ be a specified error bound. A linear approximation $L(x)$ of a function $f(x)$ at center $a$ is considered an “$\epsilon$-good” approximation of $f(x)$ over a given interval, $(b, c)$, if $|f(x) - L(x)| < \epsilon$ for all $x$ values in that interval. If the error bound is not met on this interval, so the function is more than $\epsilon$ away from the linearization at some point in $(b, c)$, then the approximation is considered “$\epsilon$-bad.”

You are to use the Matlab tool lineartool2 to complete the following assignment. Given a function $f(x)$ and a center $a$, lineartool2 calculates the linearization function $L(x)$. When you run lineartool2, four windows will open: three graphs and a control panel.

- The graph titled “Size of Second Derivative” shows the graph of $|f''(x)|$. You will use this graph to solve problem 13.
- The graph “Overview” shows the function $f(x)$ is blue and a tangent line in $L(x)$ in pink, over a fixed large scale view.
- The graph “Zoom” shows the same things as “Overview” but zoomed into the interval $(a - \delta, a + \delta)$, for the value of $\delta$ specified in the control panel. The interval $(a - \delta, a + \delta)$ is also marked on “Overview” and “Size of Second Derivative” graphs with a red rectangle.

Notice that the Zoom graph also shows two green curves: one $\epsilon$ units above the graph, and one $\epsilon$ units below it. The region between the green curves is what we call an $\epsilon$-band around the graph. The $\epsilon$ value is specified in the control panel. If the graph of $L(x)$ lies within this $\epsilon$-band for all $x$ in a given $\delta$-interval $(a - \delta, a + \delta)$, then the approximation is considered $\epsilon$-good. If the graph of $L(x)$ leaves the $\epsilon$-band at any point on a given $\delta$-interval, then the approximation is $\epsilon$-bad.

The Assignment

For the following problems, let $f(x) = \cos(x^2)$ and let $\epsilon = 0.01$ be the specified error bound. Note that problems 4 and 5 require judgment and don’t have a single best answer.

1. Consider the linearization $L(x)$ of $f$ centered at $a = 0$. Does the graph of $L(x)$ fall completely inside the $\epsilon$-band over the $\delta$-interval for $\delta = 0.5$? If not, use the zoom button to determine how big a $\delta$ you could have so that as long as $x$ is in the $\delta$-interval around 0, $|f(x) - L(x)| < 0.01$. Give both a $\delta$-value and the $\delta$-interval of $x$-values for which this error bound is met.

2. Suppose that we move the center of the linearization to $a = 1.75$. How big a $\delta$ interval can you have at that point and still have $|f(x) - L(x)| < 0.01$ for all $x$ in your $\delta$-interval? Give both a $\delta$-value and the $\delta$-interval of $x$-values for which this error bound is met.

3. Based on your findings from questions 1 and 2, which center provides an “$\epsilon$-good” linear approximation of $f(x)$ over a larger $\delta$-interval of $x$-values?

4. A linear approximation will give an “$\epsilon$-good” estimate of $f(x)$ for larger $\delta$-intervals when the graph appears to be linear over a larger interval around the center of approximation. Identify the largest intervals on the graph where the function appears to be nearly linear. Choose an appropriate $x$-value for a center of approximation in each of these intervals.
5. A linear approximation will give an “$\epsilon$-good” estimate of $f(x)$ over smaller $\delta$-intervals when the graph appears nearly linear only over a very small interval around the center of approximation. Identify the places on the graph where the behavior of $f(x)$’s graph deviates most quickly from that of a straight line. State your answer as an $x$-value for a center of approximation in each of these locations.

6. Compare your answers to problems 4 and 5. At which of the centers that you found in problems 4 and 5 are the slopes of the tangent lines at $x$-values near $a$ changing slowly?

7. Compare your answers to problems 4 and 5. At which of the centers that you found in problems 4 and 5 are the slopes of the tangent lines at $x$-values near $a$ changing quickly?

8. What function, related to $f(x)$, measures how the slopes of the tangent lines of $f(x)$ change as $x$ changes?

9. If the magnitude (absolute value) of $f''(x)$ is small on a given $\delta$-interval with center $a$, are the slopes of the tangent lines changing slowly or quickly as $x$ increases over the $\delta$-interval?

10. If the magnitude of $f''(x)$ is large on a given $\delta$-interval with center $a$, are the slopes of the tangent lines changing slowly or quickly as $x$ increases over the $\delta$-interval?

11. If $|f''(x)|$ is small on a given $\delta$-interval with center $a$, is the approximation of $f(x)$ by $L(x)$ more likely to be $\epsilon$-good or $\epsilon$-bad? Justify your answer using arguments related to the shape of the graph.

12. If $|f''(x)|$ is large on a given $\delta$-interval with center $a$, is the approximation of $f(x)$ by $L(x)$ more likely to be $\epsilon$-good or $\epsilon$-bad? Justify your answer using arguments related to the shape of the graph.

13. Based on your above observations, describe how the magnitude of $f''(x)$ can be used to determine whether a linear approximation of $f(x)$ centered at $a$ is an “$\epsilon$-good” approximation of $f(x)$ over a given $\delta$-interval. Graph $|f''(x)|$ on the interval $[-3, 3]$, and discuss how the graph compares to your earlier assertions about the locations of centers that provide $\epsilon$-good approximations over longer $\delta$-intervals.