Solutions to practice session 11/28/2016

1. (Putnam 2004) Let $m$ and $n$ be positive integers. Show that
\[
\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}
\]

**Solution:** The given inequality is equivalent to
\[
\frac{(m+n)!}{m!n!} m^n n^n = \binom{m+n}{n} m^n n^n < (m+n)^{m+n}
\]
which is obvious in view of the binomial expansion of $(m+n)^{m+n}$.

2. (Inspired by Putnam 1968, B6) Prove that a polynomial with only real roots and all coefficients equal to $\pm 1$ has degree at most 3.

**Solution:** We may assume that the leading coefficient is +1. The sum of the squares of the roots of $x^n + a_1 x^{n-1} + \cdots + a_n$ is $a_1^2 - 2a_2$. The product of the squares of the roots is $a_n^2$. By the Arithmetic Mean-Geometric Mean inequality we have
\[
\frac{a_1^2 - 2a_2}{n} \geq \sqrt[n]{a_n^2}
\]
Since the coefficients are $\pm 1$ that inequality is $(1 \pm 2)/n \geq 1$, hence $n \leq 3$. $n = 3$ is optimal; consider $x^3 - x^2 - x + 1$.

3. (Putnam 1984) Let $n$ be a positive integer, and define
\[
f(n) = 1! + 2! + \cdots + n!.
\]
Find polynomials $P(x)$ and $Q(x)$ such that
\[
f(n+2) = P(n)f(n+1) + Q(n)f(n)
\]
for all $n \geq 1$.

**Solution:** We have
\[
f(n+2) - f(n+1) = (n+2)! = (n+2)(f(n+1) - f(n))
\]
hence
\[
f(n+2) = (n+2)(f(n+1) - f(n)) + f(n+1) = (n+3)f(n+1) - (n+2)f(n)
\]
and we can take $P(x) = x + 3, Q(x) = -x - 2$.

4. (Putnam 1974) Call a set of positive integers “conspirational” if no three of them are pairwise relatively prime. What is the largest number of elements in any conspirational subset of integers 1 through 16?
**Solution:** A conspirational subset of $S = \{1, 2, \ldots, 16\}$ has at most two elements from $T = \{1, 2, 3, 5, 7, 11, 13\}$, so it has at most $2 + 16 - 7 = 11$ numbers. On the other hand all elements of $S \setminus T = \{4, 6, 8, 9, 10, 12, 14, 15, 16\}$ are multiples of either 2 or 3, so adding 2 and 3 we obtain the following 11-element conspirational subset:

$$\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}.$$

Hence the answer is 11.

5. (Putnam 1958) If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

show that the equation $a_0 + a_1 x + \cdots + a_n x^n = 0$ has at least one real root.

Hint: Consider an integral of $f(x) = a_0 + a_1 x + \cdots + a_n x^n$.

We have

$$\int_0^1 f(x) \, dx = \frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$

Now apply the Mean Value Theorem for integrals: $\exists \xi \in (0, 1)$ such that $f(\xi) = \int_0^1 f(x) \, dx = 0$. 

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