1. (Putnam 2004) Let \( m \) and \( n \) be positive integers. Show that
\[
\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^n} \frac{n!}{n^n}
\]

2. (Inspired by Putnam 1968, B6) Prove that a polynomial with only real roots and all coefficients equal to \( \pm 1 \) has degree at most 3.

3. (Putnam 1984) Let \( n \) be a positive integer, and define
\[
f(n) = 1! + 2! + \cdots + n!.
\]
Find polynomials \( P(x) \) and \( Q(x) \) such that
\[
f(n+2) = P(n)f(n+1) + Q(n)f(n)
\]
for all \( n \geq 1 \).

4. (Putnam 1974) Call a set of positive integers “conspirational” if no three of them are pairwise relatively prime. What is the largest number of elements in any conspirational subset of integers 1 through 16?

5. (Putnam 1958) If \( a_0, a_1, \ldots, a_n \) are real numbers satisfying
\[
\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,
\]
show that the equation \( a_0 + a_1 x + \cdots + a_n x^n = 0 \) has at least one real root.

Hint: Consider an integral of \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \).