1. Find the minimum value of the function \(f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n\), where \(x_1, x_2, \ldots, x_n\) are positive real numbers such that \(x_1x_2\cdots x_n = 1\).

Solution:

\[
\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n} = 1.
\]

Hence \(f(x_1, x_2, \ldots, x_n) \geq n\). On the other hand \(f(1, 1, \ldots, 1) = n\), so the minimum value is \(n\).

2. If \(a, b, c \geq 0\), prove that \(\sqrt{3(a + b + c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}\).

Solution: Use the Schwarz inequality.

3. If \(p\) is an odd number not divisible by 3, then it is congruent to \(\pm 1\) modulo 6.

Solution: (a) Suppose that \(p = 2k + 1 = 3s + 1\). Then \(2k = 3s\), so \(s\) is even, \(s = 2p\). Then \(p = 6p + 1\) and so \(p \equiv 1 \pmod{6}\).

(b) Suppose that \(p = 2k + 1 = 3s + 2\). Then \(2k = 3s + 1\), which implies that \((k, s) = w(3, 2) + (-1, -1)\). So \(k = 3w - 1\) and so \(p = 6w - 1\). This is the case when \(p \equiv -1 \pmod{6}\).

4. Suppose \(p > 3\) is prime. Show that \(p^2 + 2\) cannot be prime. Hint: Use #3.

Solution: Note that if \(p = 2\), then \(p^2 + 2 = 6\) is not prime. However, if \(p = 3\), then \(p^2 + 2 = 11\) is prime, and then \(p^3 + 2 = 29\) is also prime and the statement is true. For \(p > 3\) we have that \(p\) is odd and not divisible by 3. So it is congruent to \(\pm 1\) modulo 6. This implies that \(p^2 + 2 \equiv 3 \pmod{6}\). So \(p^2 + 2\) is a multiple of 3 and cannot be prime.

5. Find the minimum value of

\[
(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v}\right)^2
\]

for \(0 < u < \sqrt{2}\) and \(v > 0\). Hint: Think geometrically. The given expression represents the square of the distance between two geometric objects.

Solution: Putnam 1984) Consider the distance between a point of the quarter-circle \(x^2 + y^2 = 2\) in the open first quadrant and a point of the half hyperbola \(xy = 9\) in that quadrant. The tangents to the curves at \((1, 1)\) and \((3, 3)\) separate the curves, and both are perpendicular to \(y = x\), so those points are at the minimum distance, and the answer is \((3 - 2)^2 + (3 - 1)^2 = 8\).

6. Prove that from ten distinct two-digit numbers, one can always choose two disjoint non-empty subsets, so that their elements have the same sum. Hint: pigeonhole principle.
Solution:

(IMO 1972) A set of 10 elements has $2^{10} - 1 = 1023$ non-empty subsets. The possible sums of at most ten two-digit numbers cannot be larger than $10 \cdot 99 = 990$. There are more subsets than possible sums, so two different subsets $S_1$ and $S_2$ must have the same sum. If $S_1 \cap S_2 = \emptyset$, we are done. Otherwise remove the common elements and we get two disjoint subsets with the same sum.