Solutions to practice problems 10/24/2016

1. Define a domino to be a $1 \times 2$ rectangle. In how many ways can an $n \times 2$ rectangle be tiled by dominoes?

**Solution:** Let $x_n$ be the number of tilings of an $n \times 2$ rectangle. We have $x_1 = 1$, $x_2 = 2$. For $n \geq 3$ we can place the bottom domino horizontally and tile the rest of the rectangle in $x_{n-1}$ ways, or we can place two vertical dominoes at the bottom and tile the rest in $x_{n-2}$ ways. So $x_n = x_{n-1} + x_{n-2}$. The solution is the shifted Fibonacci sequence, $x_n = F_{n+1}$.

2. Compute $\lim_{n \to \infty} \left\{ \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} \right\}$. Hint: Use integrals to estimate the sum from above and below.

**Solution:** Let $s_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$. Then (if $n \geq 2$)

$$s_n \leq \int_{n-1}^{2n-1} \frac{dx}{x} = \ln x|^{2n-1}_{n-1} = \ln(2n-1) - \ln(n-1) = \ln \left( \frac{2n-1}{n-1} \right)$$

while, on the other hand,

$$s_n \geq \int_{n}^{2n} \frac{dx}{x} = \ln x|^{2n}_{n} = \ln(2n) - \ln(n) = \ln 2.$$

Hence $\ln 2 \leq s_n \leq \ln[(2n-1)/(n-1)]$. Taking $n \to \infty$ gives $\ln 2$ for the right-hand side, hence the given limit is $\ln 2$.

3. Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

Hint: The numbers 8 and 9 form one such pair. Given a pair $(n, n+1)$ of consecutive square-fulls, find some way to build another pair of consecutive square-fulls.

**Solution:** If $n$ and $n+1$ are square-full, so are $4n(n+1)$ and $4n(n+1)+1 = (2n+1)^2$ (the latter is a square, so its prime factors appear in even powers).

4. Evaluate

$$I = \int_{2}^{4} \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} \, dx$$

Make the substitution $x = 3 + u$ and then $v = -u$. The result is $I = 1$. 

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