7.1 Integration by Parts

What is it?

An exploitation of the product rule for derivatives to write an integral in terms of a simpler one. Integration by parts is usually only possible if you have a product of different functions (there are a few exceptions such as $\int \tan^{-1}(x)\,dx$, $\int \sin^{-1}(x)\,dx$, or $\int \ln(x)\,dx$, see below).

Formula: Indefinite- $\displaystyle\int u\,dv \;=\; uv \;-\; \int v\,du$

Definite- $\displaystyle\int_a^b u\,dv \;=\; uv\Big|_a^b \;-\; \int_a^b v\,du$

For definite integrals it is conventional to evaluate as you go. For example:

$$\int_0^{\pi} (x^2 - 2x)\sin(2x)\,dx \quad [\text{using } u = x^2 - 2x \Rightarrow du = (2x-2)dx;\; dv = \sin(2x)dx \Rightarrow v = -\frac{1}{2}\cos(2x)]$$

$$= (x^2 - 2x)(-\frac{1}{2})\cos(2x)\Big|_0^{\pi} - \int_0^{\pi} (2x-2)(-\frac{1}{2})\cos(2x)\,dx$$

$$= \Big[(\pi^2 - 2\pi)(-\frac{1}{2})\cos(2\pi) - 0\Big] + \int_0^{\pi}(x-1)\cos(2x)\,dx$$

$$= -\frac{\pi^2 - 2\pi}{2} + \int_0^{\pi}(x-1)\cos(2x)\,dx \quad [\text{using } u = x - 1 \Rightarrow du = dx;\; dv = \cos(2x)dx \Rightarrow v = \frac{1}{2}\sin(2x)]$$

$$= -\frac{\pi^2 - 2\pi}{2} + \frac{x-1}{2}\sin(2x)\Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin(2x)\,dx$$

$$= -\frac{\pi^2 - 2\pi}{2} + [0 - 0] + \frac{1}{4}\cos(2x)\Big|_0^{\pi}$$

$$= -\frac{\pi^2 - 2\pi}{2} + \frac{1}{4}[\cos(2\pi) - \cos(0)]$$

$$= -\frac{\pi^2 - 2\pi}{2}$$

Although integration by parts applies in a variety of different integrals, there are 3 classic cases that you need to immediately recognize as integrals that require integration by parts:

1. Integrals involving the product of a polynomial with a function that you can keep taking anti-derivatives of. Examples:

   $(a)\ \displaystyle\int (x^2 - 3x + 2)e^{-4x}\,dx$

   $(b)\ \displaystyle\int x^3\cos\left(\frac{x}{4}\right)\,dx$

   $(c)\ \displaystyle\int (x^{10} - 2x^9 + x^3 - 1)\sin(4x)\,dx$

   In each case above you let $u = \mathit{Poly}$ and $dv = (\mathit{everything\ else})\,dx$. You will get a new integral of the same form but the polynomial will be one degree lower. You apply parts again in the same manner and obtain a new integral with a polynomial of another degree lower. You keep going until your polynomial is no longer involved in the new integral. In (a) you apply parts 2 times, in (b) you apply it 3 times, and (c) you apply it 10 times.

   Try (a) and (b) on your own, the answers are:

   $$(a)\ \int (x^2 - 3x + 2)e^{-4x}\,dx = -\frac{x^2 - 3x + 2}{4}e^{-4x} - \frac{2x-3}{16}e^{-4x} - \frac{1}{32}e^{-4x} + C$$

   $$= -e^{-4x}\frac{8x^2 - 20x + 9}{32} + C$$

   $$(b)\ \int x^3\cos\left(\frac{x}{4}\right)dx = 4x^3\sin\left(\frac{x}{4}\right) + 48x^2\cos\left(\frac{x}{4}\right) - 384x\sin\left(\frac{x}{4}\right) - 1536\cos\left(\frac{x}{4}\right) + C$$

2. Integrals involving "complicated" functions (functions with unknown or complicated anti-derivatives), such as $\ln(x)$, $\sin^{-1}(x)$, or $\tan^{-1}(x)$. Examples:

$$(a) \int x^4 \ln(x)\, dx$$

$$(b) \int x^4 (\ln(x))^2\, dx$$

$$(c) \int \sin^{-1}(x)\, dx$$

If you are going to try integration by parts on an integral involving any of the "complicated" functions $\ln(x)$, $\sin^{-1}(x)$, or $\tan^{-1}(x)$ (or any other function with an unknown or complicated anti-derivative), you *usually* let u be the stuff involving the "complicated" function and $dv = (everything\ else)\ dx$. The intuition is that if you let dv be the "complicated" function, your new integral will be worse than what you started with. On the other hand if the derivative of the complicated function is relatively simple, it could make things better by letting u be the "complicated" function.
In (a) you let $u = \ln(x)$ and $dv = x^4\, dx$. After some simplification you should be able to do the new integral.
In (b) you let $u = (\ln(x))^2$ and $dv = x^4\, dx$. The new integral will involve a polynomial times $\ln(x)$. You apply parts again letting $u = \ln(x)$, $dv = (Poly)dx$ and after some simplification you should be able to do the new integral.
In (c) you let $u = \sin^{-1}(x)$, $dv = dx$. You should recognize how to do the new integral using a basic substitution.

Try these on your own, the answers are:

$$(a) \int x^4 \ln(x)\, dx = \frac{1}{5} x^5 \ln(x) - \frac{1}{25} x^5 + C$$

$$(b) \int x^4 (\ln(x))^2\, dx = \frac{1}{5} x^5 (\ln(x))^2 - \frac{2}{25} x^5 \ln(x) + \frac{2}{125} x^5 + C$$

$$(c) \int \sin^{-1}(x)\, dx = x \sin^{-1}(x) + \sqrt{1-x^2} + C$$

3. The 3rd classic case where integration by parts works is "recursive" type. For this type, after a few applications of integration by parts you end up with an integral of the same form that you started with and use algebra to solve for the integral. Examples:

$$(a) \int e^x \cos(x)\, dx,$$

$$(b) \int e^{ax} \cos(bx)\, dx, \text{ where } a,\ b \text{ are fixed numbers}$$

$$(c) \int e^{ax} \sin(bx)\, dx, \text{ where } a,\ b \text{ are fixed numbers}$$

$$(d) \int \sec^3(x)\, dx$$

The integrals (a), (b), and (c) use basically the same idea. I'll show the solution to (a). First apply integration by parts two times, letting $u = e^x$ each time:

$$\begin{aligned}
&\int e^x \cos(x)\,dx \qquad [\text{using } u = e^x \Rightarrow du = e^x\,dx;\; dv = \cos(x)dx \Rightarrow v = \sin(x)] \\
&= e^x \sin(x) - \int e^x \sin(x)\,dx \qquad [\text{using } u = e^x, \Rightarrow du = e^x\,dx;\; dv = \sin(x)dx \Rightarrow v = -\cos(x)] \\
&= e^x \sin(x) - \left(e^x\,(-\cos(x)) - \int e^x\,(-\cos(x))\,dx \right) \\
&= e^x \sin(x) - \left(-\,e^x \cos(x) + \int e^x \cos(x)\,dx \right) \\
&= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx
\end{aligned}$$

We are almost done but this is where things get a little weird. By applying integration by parts twice, both times letting $u = e^x$, we just proved the formula:

$$\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx\,.$$

Notice that the integral we are looking for appears on both sides of this equation. At this point we can actually solve for it using basic algebra. By adding the integral to both sides we obtain:

$$\begin{aligned}
& \int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx \\
\Rightarrow\quad & \int e^x \cos(x)\,dx + \int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) \\
\Rightarrow\quad & 2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) \\
\Rightarrow\quad & \int e^x \cos(x)\,dx = \frac{1}{2}\,(e^x \sin(x) + e^x \cos(x)) + C
\end{aligned}$$

Now we are done. This is strange because it seems like we were able to find an anti-derivative without actually evaluating anything. We just played around with integration by parts and used a little algebra. As mentioned earlier, (b) and (c) are the same idea. (d) is done on page 459 of the textbook and it is not obvious. You would not be expected to figure out (d) by yourself, but it is another example of "recursive" type.

You can try (b) and (c) on your own, the answers are:

$$(b)\ \int e^{ax} \cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}\,(\,b\sin(bx) + a\cos(bx)\,) + C$$

$$(c)\ \int e^{ax} \sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}\,(-b\cos(bx) + a\sin(bx)\,) + C$$