Solution to homework problems on Sets:

1) Using proofs by elements, Prove If A and B are sets then $A \cup B = B \cup A$

Proof:

$\forall x \in A \cup B \rightarrow x \in A \lor x \in B$ (by definition of Union)

$\rightarrow x \in B \lor x \in A$ (commutative in Logic)

$\rightarrow x \in B \cup A$ (by definition of Union)

We have shown that $A \cup B \subseteq B \cup A$

by reverse steps we can show that $B \cup A \subseteq A \cup B$

Therefore we have by definition of equal sets we have that $A \cup B = B \cup A$

2) Using proofs by elements, Prove If A, B and C are sets then

$A \cup (B \cap C) = (A \cup B) \cap C$

Proof:

$\forall x \in A \cup (B \cap C) \rightarrow x \in A \lor x \in (B \cap C)$ (by definition of Union)

$\rightarrow x \in A \lor [x \in B \land x \in C]$ (by definition of Intersection)

$\rightarrow (x \in A \lor x \in B) \land (x \in A \lor x \in C)$ (Distribution in Logic)

$\rightarrow (x \in A \cup B) \land (x \in A \cup C)$ (by definition of Union)

$\rightarrow x \in (A \cup B) \cap (A \cup C)$ (by definition of Intersection)

So we have shown that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

and by reverse steps we can show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Therefore we have by definition of equal sets we have

that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3) Using proofs by elements, Prove If A and B are sets then $(A \cap B)^C = B^C \cup A^C$

Proof:
\[ \forall x \in (A \cap B)^C \rightarrow x \notin A \cap B \text{ (Definition of complement)} \]
\[ \rightarrow \neg [x \in A \cap B] \]
\[ \rightarrow \neg [x \in A \land x \in B] \text{ (Definition of Intersection)} \]
\[ \rightarrow \neg [x \in A] \lor \neg [x \in B] \text{ (DeMorgan's Law in logic)} \]
\[ \rightarrow (x \notin A) \lor (x \notin B) \]
\[ \rightarrow (x \in A^C) \lor (x \in B^C) \text{ (Definition of Complement)} \]
\[ \rightarrow x \in (A^C \cup B^C) \text{ (Definition of Union)} \]

We have shown that \((A \cap B)^C \subseteq A^C \cup B^C\)

by reverse steps we can show that \(A^C \cup B^C \subseteq (A \cap B)^C\)

Therefore we have by definition of equal sets we have that \(A^C \cup B^C = (A \cap B)^C\)