Applications of Partial Derivatives:

Extreme Values with No Constraints:

The process of finding relative extreme points of a three dimensional function is very similar to the process of finding the relative extrema for two-dimensional functions. Recall in the introduction to Project 9, you graphed the profit function \( P(x, y) = 140x + 200y - 4x^2 + 2xy - 12y^2 - 700 \) where \( P \) is the profit, in dollars, for \( x \) chairs and \( y \) tables produced and sold each month. The following is a duplicate of the graph that was generated from the profit function. (Remember to leave the legend on the graph when plotting three-dimensional functions.)

When you were dealing with a two-dimensional function \( y = f(x) \), it was easy to find the instantaneous rate of change in \( f(x) \) for any desired value of \( x \) by simply taking the derivative and evaluating at the desired point. Suppose, for your three-dimensional function \( z = P(x, y) \), you want to know the instantaneous rate of change in profit when seven chairs and five tables are produced and sold each month. How would you use the derivative in this three dimensional situation? You will recall that the derivative may be interpreted as the rate of change in output with respect to the input variable or the slope of the tangent line touching the graph at the point of interest. Since you are working with a three-dimensional function, you need to consider the derivative from two points of view - with respect to the \( x \)-axis and with respect to the \( y \)-axis.

First, consider the cross sectional model that results when the number of chairs produced and sold each month varies, but the number of tables remains fixed at five. In this case we are interested in the function \( P(x, 5) = 140x + 200(5) - 4x^2 + 2x(5) - 12(5)^2 - 700 = 150x - 4x^2 \). You can easily find the derivative of this function and evaluate at \((7,5)\) to find that the instantaneous rate of change in profit with respect to the number of chairs produced and sold is \$94\). This means if the factory holds the number of tables produced and sold each month constant at five and increases the number of chairs from seven to eight, they can expect approximately a \$94 increase in profit. The graph shown below on the left shows the cross sectional model \( P(x, 5) \) and the line tangent to the curve at \((7,5)\).

The above information, however, only considers the case when the number of tables is held constant at five per month and the number of chairs varies. What about the case when the number of chairs is held constant at seven and the number of tables produced and sold each month varies? The cross-sectional
model in this case is \( P(7, y) = 140(7) + 200y - 4(7)^2 + 2(7)y - 12y^2 - 700 = 84 + 214y - 12y^2 \). Finding the derivative with respect to \( y \) and evaluating at \( (7,5) \) results in $94. This means if the factory holds the number of chairs produced and sold each month constant at seven and increases the number of tables produced and sold each month from five to six, they can expect approximately a $94 increase in profit. The graph shown below on the right shows the cross sectional model \( P(7, y) \) and the line tangent to the curve at \( (7,5) \).

This information is clear and accurate, but you don't want to have to find two cross sectional models each time you want to find the two rates of change for a three-dimensional function. Think of the process that you actually used to find the values that represented each of these rates of change. When you found the rate of change in a three-dimensional function \( f(x, y) \) with respect to \( x \), you held \( y \) constant and took the derivative of \( f(x, c) \) with respect to \( x \) for some constant \( c \). This is called the partial derivative of \( f \) with respect to \( x \). When you were finding rate of change in \( f(x, y) \) with respect to \( y \), you held \( x \) constant and took the derivative of \( f(c, y) \) with respect to \( y \). This is the partial derivative of \( f \) with respect to \( y \).

By now you have studied the rules for finding partial derivatives and the various notation that is used to represent these values in your classroom lecture sessions. Use those rules to find the rates of change that you found above using your cross-sectional models:

**Example 1:**

Given the profit equation \( P(x, y) = 140x + 200y - 4x^2 + 2xy - 12y^2 - 700 \), find the rate of change in profit (a) with respect to \( x \) and then (b) with respect to \( y \) at the point \( (7,5) \).

(a) The derivative of \( P \) with respect to \( x \) is denoted by \( P_x \) or by \( \frac{\partial P}{\partial x} \). This value is given by:

\[
\frac{\partial P}{\partial x} = 140 + 0 - 8x + 2y - 0 - 0 = 140 - 8x + 2y \quad \text{so} \quad \frac{\partial P}{\partial x}_{(x,y)=(7,5)} = 140 - 8(7) + 2(5) = 94
\]

This means that total monthly profit increases by approximately $94 if the production and sales of chairs is increased from 7 to 8 and the number of tables is kept constant at 5.
The derivative of $P$ with respect to $y$ is denoted by $P_y$ or by $\frac{\partial P}{\partial y}$. This value is given by:

$$\frac{\partial P}{\partial y} = 0 + 200 - 0 + 2x - 24y - 0 = 200 + 2x - 24y$$
so
$$\frac{\partial P}{\partial y} \bigg|_{(x,y)=(7,5)} = 200 + 2(7) - 24(5) = 94$$

This means that total monthly profit increases by approximately $94 if the production and sales of tables is increased from 5 to 6 and the number of chairs is kept constant at 7.

Using the same profit function, in Project 9 you "guessed" that the maximum profit for this function would occur at the point (20, 10, 1700), meaning that a maximum profit of $1,700 would occur when 20 chairs and 10 tables are produced and sold each month. You will now see how to use partial derivatives to determine if this guess was accurate.

When you studied the derivatives of two-dimensional functions, you learned that $f'(a) = 0$ means that $f(x)$ has a critical point at $x = a$. Therefore, $f(x)$ may have a relative maximum or a relative minimum at $x = a$. The same basic concept is true for the three-dimension functions. For a function $f(x, y)$, if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then $f(x, y)$ has a critical point at the point $(x, y) = (a, b)$. This indicates that $f(x, y)$ may have a relative extreme point at $(x, y) = (a, b)$ since relative maximum values and relative minimum values occur at critical points for three-dimensional functions just as they do for two-dimensional functions.

**Example 2:**

Given the profit function $P(x, y) = 700 + 12y^2 - 8140x - 2200y - 700$, verify that maximum profit occurs at the point (20, 10, 1700).

Recall, from Example 1, that $f_x = 140 - 8x + 2y$ and $f_y = 200 + 2x - 24y$. Use this information to verify that the first partial derivatives evaluated at (20, 10) are actually equal to zero:

$$f_x(20, 10) = 140 - 8(20) + 2(10) = 0$$
$$f_y = 200 + 2(20) - 24(10) = 0$$

Since both first partial derivatives are equal to zero at a given point, it is possible that maximum profit does occur at the point (20, 10). But how can you be sure?

To verify that this is indeed maximum profit you need to find the second derivatives and evaluate them at the point (20, 10) to determine whether the graph of the profit function is concave up or down at this point. The second partial derivatives for this function are:

$$\frac{\partial^2 P}{\partial x^2} = f_{xx} = -8$$ (This indicates that the function is concave down in x-axis direction)
$$\frac{\partial^2 P}{\partial y^2} = f_{yy} = -24$$ (This indicates that the function is concave down in the y-axis direction)
It seems likely, therefore, that the graph of the profit function is indeed concave down at the point (20, 10) and therefore the point (20, 10, 1700) is maximum, but you need to do one more test before you can be sure. To apply this test, first you need to find the derivative of \( f_x \) with respect to \( y \):

\[
\frac{\partial^2 p}{\partial y \partial x} = f_{xy} = 2
\]

Now use the partial derivatives that you have found for this function and the following formula to complete your verification:

**Second Partials Test:**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( f_x(a, b) = 0 ) and ( f_y(a, b) = 0 ) (i.e., ( (x, y) = (a, b) ) is a critical point for ( f(x, y) )), then for the equation</td>
<td>( D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - (f_{xy}(a, b))^2 )</td>
</tr>
<tr>
<td>1. If ( D(a, b) &gt; 0 ) and ( f_{xx}(a, b) &gt; 0 ), then ( f(x, y) ) has a relative minimum located at ( (x, y) = (a, b) )</td>
<td></td>
</tr>
<tr>
<td>2. If ( D(a, b) &gt; 0 ) and ( f_{xx}(a, b) &lt; 0 ), then ( f(x, y) ) has a relative maximum located at ( (x, y) = (a, b) )</td>
<td></td>
</tr>
<tr>
<td>3. If ( D(a, b) &lt; 0 ), then there is neither a relative maximum nor a relative minimum at ( (x, y) = (a, b) ). Instead, there is a saddle point at that location.</td>
<td></td>
</tr>
<tr>
<td>4. If ( D = 0 ), then the Second Partials Test gives no information.</td>
<td></td>
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Now finish Example 2 and apply the Second Partials Test to the profit function at \( (x, y) = (20, 10) \).

\[
D(20,10) = f_{xx}(20,10) \cdot f_{yy}(20,10) - (f_{xy}(20,10))^2
\]

\[
= (-8)(-24) - (2)^2 > 0
\]

Since \( D(20,10) > 0 \) and \( f_{xx}(a,b) < 0 \), maximum profit occurs at \( (x, y) = (20, 10) \). The maximum profit is \( P(20,10) = 1700 \) or maximum profit of $1,700 is reached when 20 chairs and 10 tables are produced and sold each month.

In Example 2, you started with a good guess for the maximum profit, but suppose that you have no idea what point might give a maximum or minimum value for your three-dimensional function. How would you find the point that you want? The following example details the steps that you should follow to find the relative extrema for your three dimensional function.

**Example 3:**

Find all relative extrema for the function \( f(x, y) = -3x^2 + 2xy - 2y^2 + 14x + 2y + 10 \).

**Step 1:** Find all critical points for the function

Find first partial derivatives:

\[
f_x(x, y) = -6x + 2y + 14
\]

\[
f_y(x, y) = 2x - 4y + 2
\]

Then set both first partial derivatives equal to zero and solve:

\[
f_x(x, y) = -6x + 2y + 14 = 0
\]

\[
-6x + 2y + 14 = 0
\]

\[
2y = 6x - 14
\]

\[
y = 3x - 7
\]
and \( f_y(x, y) = 2x - 4y + 2 = 0 \) so use \( y = 3x - 7 \) that you found for \( f_x \) and substitute:

\[
\begin{align*}
2x - 4(3x - 7) + 2 &= 0 \\
2x - 12x + 28 + 2 &= 0 \\
-10x &= -30 \\
x &= 3
\end{align*}
\]

Now you know a value for \( x \), so substitute one of your equations to find a value for \( y \):

\[
y = 3(3) - 7 = 2
\]

So the only critical point for this function occurs at \((x, y) = (3, 2)\)

Step 2: Find \( f_{xx}, f_{yy}, \) and \( f_{xy} \). Evaluate each at \((3, 2)\).

\[
\begin{align*}
f_{xx}(x, y) &= -6, \text{ so } f_{xx}(3, 2) = -6 \\
f_{yy}(x, y) &= -4, \text{ so } f_{yy}(3, 2) = -4 \\
f_{xy}(x, y) &= 2, \text{ so } f_{xy}(3, 2) = 2
\end{align*}
\]

Step 3: Find a value for \( D(3, 2) \).

\[
D(3, 2) = f_{xx}(3, 2)f_{yy}(3, 2) - (f_{xy}(3, 2))^2 = (-6)(-4)-2^2 > 0
\]

Step 4: Analyze results:

Since \( D(3, 2) > 0 \) and \( f_{xx}(3, 2) < 0 \), there is a relative maximum at \((3, 2)\). The maximum value is

\[
f(3, 2) = -3(3)^2 + 2(3)(2) - 2(2)^2 + 14(3) + 2(2) + 10 = 33
\]

Note: Whenever your setting your first partial derivatives equal to zero gives a system of linear equations, you can use matrices to solve for \( x \) and \( y \). This is the case in Example 3. The system of linear equations for this problem is:

\[
\begin{align*}
-6x + 2y &= -14 \\
2x - 4y &= -2
\end{align*}
\]

Recall that one way to solve a system of equations, especially if the coefficients are not easy to work with, is to use linear algebra or matrices. For this example, you could use the inverse method to solve the system of equations by letting \( A = \begin{bmatrix} -6 & 2 \\ 2 & -4 \end{bmatrix} \), \( X = \begin{bmatrix} x \\ y \end{bmatrix} \) and \( B = \begin{bmatrix} -14 \\ -2 \end{bmatrix} \).

**Extreme Values with One Constraint (Lagrange Multipliers):**

A function is sometimes limited by other factors. In these cases there is at least one additional equation that must be considered when looking for extreme values. The method of Lagrange multipliers can be used to find maximum and minimum values in such cases. In this section you will examine functions that have one constraint. The basic form of problems of this type is to maximize (or minimize) the function \( f(x, y) \) subject to the constraint \( g(x, y) = 0 \). You will be learning to solve these problems by hand in the lecture portion of this class.
This project will concentrate on examining utility functions subject to a given budget constraint. Recall from Project 9, the utility function \( U = f(x, y) \) represents the utility or satisfaction that is derived by a consumer from the consumption of two goods where the variables \( x \) and \( y \) are used to represent the quantities of these two goods consumed. Whenever an indifference curve (level curve) intersects the budget constraint, that utility level is attainable within the budget. The highest level of attainable utility corresponds to the indifference curve that touches the constraint at exactly one point (i.e., the constraint is tangent to the indifference curve).

**Example 4:**
Suppose that the utility function for two commodities is given by \( U = f(x, y) = yx^2 \) and that the budget constraint is \( 4x + 2y = 12 \). What values of \( x \) and \( y \) will maximize utility?

The level curves below are for the particular values of \( K = f(x,y) = 4, 8, 16, \) and \( 25 \), giving the two-dimensional equations \( y = \frac{4}{x^2} \), \( y = \frac{8}{x^2} \), \( y = \frac{16}{x^2} \), and \( y = \frac{25}{x^2} \). The constraint must converted to the form \( y = f(x) \) in order to plot the graph in the xy-plane, so this gives \( y = 6-2x \) for the constraint \( g(x) \).

Note that the constraint is tangent to the indifference curve \( K = 8 \). This indicates that the maximum attainable utility occurs when \( x = 2 \) and \( y = 2 \) (the point of tangency). Further, the constraint crosses the utility curve \( K = 4 \), so that utility level is attainable within the budget. But since the constraint never intersects the indifference curves \( K = 16 \) and \( K = 25 \), neither of those utility levels are attainable within the budget.

**Competitive and Complementary Products:**
Products A and B are considered competitive products if price increase for one product increases the demand for the other. An example of competitive products would be Coke and Pepsi. Products A and B are considered complementary products if the increase in demand for one product indicates an increase in demand for the other. Hot dogs and hot dog buns could be considered complementary products.

There is a simple test using partial derivatives to determine whether products are complementary, competitive, or neither.
Suppose that demand for product A and B are given by \( q_A = F(p_A, p_B) \) and \( q_B = G(p_A, p_B) \) where \( q_A \) and \( q_B \) are the quantities demanded of product A and product B, respectively, and \( p_A \) is the price of product A and \( p_B \) is the price of product B.

1. If \( \frac{\partial q_A}{\partial p_B} > 0 \) and \( \frac{\partial q_B}{\partial p_A} > 0 \), then products A and B are competitive.
2. If \( \frac{\partial q_A}{\partial p_B} < 0 \) and \( \frac{\partial q_B}{\partial p_A} < 0 \), then products A and B are complementary.

(See page 682 in your textbook for more details).

**Example 5:**
The weekly demand equations for the sale of butter and margarine in a supermarket are

\[
q_A = 8000 - 0.09(p_A)^2 + 0.08(p_B)^2
\]
\[
q_B = 15000 + 0.04(p_A)^2 - 0.3(p_B)^2
\]

where \( q_A \) is the weekly demand for butter (product A) and \( q_B \) is the weekly demand for margarine (product B). Determine whether the given products are competitive, complementary, or neither.

\[
\frac{\partial q_A}{\partial p_B} = 0.16p_B > 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} = 0.08p_A > 0,
\]

so the products are competitive.

*(Note: we can be sure that both partial derivatives in this problem are positive, since price is never negative.)*
Problems to turn in:
• Work through the example problems in the lab introduction before attempting these problems. Do not turn in the worked examples from the introduction. Also, do not hand in this document with your lab.
• Work all problems in order. Do not say “see attached” or other notation referring the grader to another location in the lab for part of a problem.
• Label problems and parts of problems with appropriate numbers and/or letters.
• Label graphs.
• Display data on the same page as the graph.
• Do not split graphs over two pages.
• No large full-page graphs that contain no data.
• For all graphs, make sure the graph is large enough to see good detail. Also adjust the angle of view, if necessary.
• Write your answers by hand only when instructed to do so. Otherwise, use Excel and type.

1. Given the utility function \( U = f(x, y) = x^2y^2 \) subject to the budget constraint \( 2x + 4y = 40 \),
(a) Given the utility levels \( K = 625, K = 1600, K = 2500, \) and \( K = 3600 \), find equations in the form of \( y = f(x) \) to represent the indifference curve at each of the given utility levels and another equation in the form \( y = g(x) \) for the budget constraint (5 equations in all).
(b) Use Excel to plot the indifference curves and the budget constraint from part (a) on the same set of axes. Make sure that you adjust your scale so that your curves are clearly distinguishable.
(c) What values of \( x \) and \( y \) will maximize utility?
(d) Which of the utility levels are attainable within the budget? Which levels are not attainable under the budget?

2. The demand for two related products, A and B, are given by the functions
\[ q_A = 2500 + \frac{600}{p_A + 2} - 40p_B \]
and \( q_B = 2500 - 100p_A + \frac{400}{p_B + 5} \).
(a) Determine if the two products are competitive, complementary, or neither. Since the symbols for this one do not exist in Excel, you may write this part into your report neatly by hand.
(b) Explain why your answer makes sense by interpreting these derivatives as rates of change in the context of this problem.
(c) Give an example of two commodities that are competitive in today's market and an example of two commodities that are complementary in today's market. (Do not use commodities that are used as examples in your text or in this project).

3. Given the function \( f(x, y) = x^3 - y^2 - 12x + 6y + 5 \), complete each of the following:
(a) Graph the surface on the intervals \(-4 \leq x \leq 6, -4 \leq y \leq 4\), with a step value of 0.5 for both x-values and y-values. Setting your z-axis minimum to -50, the maximum to 175, with a major axis scale of 25 works really well.
(b) Graph the level curves for the function. Adjust the scale to match your z-axis scale in part (a) if necessary by double clicking on the legend.
(c) Examine parts (a) and (b). Where (at what x- and y-values) does it appear that \( f(x, y) \) has relative maximum points, relative minimum points, or saddle points?
(d) Find all critical points of the function by hand. You may write your work neatly into your report by hand.
(e) Determine if each of the critical points is a relative maximum, a relative minimum, or a saddle point using the Second Partial test. (Again, you may write your work into your report neatly by hand.) Does this match your observation in part (c)?