Average Rate of Change:

The world is always in change. Concepts we need to study in this section is total change, average rate of change, percent change and percent rate of change. First we will consider the following profit function:

\[ P(q) = -2q^2 + 400q - 5000 \] where \( q \) represents quantity of items sold.

### Total Change:

If number of items sold is increased from \( q = 40 \) to \( q = 120 \), What is the total change in profit? (Often the notation \( \Delta P \) (delta \( P \)) is used to indicate change in \( P \))

**Answer:**

The total change in \( P = \Delta P = P(120) - P(40) \)

\[-2(120)^2 + 400(120) - 5000) - (-2(40)^2 + 400(40) - 5000)\]

\[-14400 + 7800 = 6400 \]

### Percent Change:

Often the actual total change does not give as good overall perspective of the final results as percentage information. The percent change gives a relative measurement in terms of the whole. We will now find the percent change in \( y \) over the interval \( q = 40 \) to \( q = 120 \).

Percent change in \( y \) is defined to be the ratio of the change in \( y \) to the initial value of \( y \). This helps us to determine if the change is large or small compared to the original value of \( y \) before the change. The definition of percent change is expressed as follows:

\[
\text{Definition of percent change in } F(x) \text{ over the interval } [x_1, x_2] \text{ is given to be } \\
\text{The percent change in } y = \frac{F(x_2) - F(x_1)}{F(x_1)}(100) = \frac{\Delta y}{y}(100)
\]

We will now revisit the problem above. This time we will find the percent change in \( y \) as \( q \) increases from \( q = 40 \) to \( q = 120 \).
Answer:
The percent change in \( P = \frac{\Delta P}{P} = \frac{P(120) - P(40)}{P(40)} \times 100 = \frac{6400}{7800} \times 100 = 82.1\% \)

Now it is easy to see the magnitude of the change in \( y \).

Average Rate of Change:
Another approach to help study change is a process called average rate of change. In the problem above we know that total change in profit is $6400 over a change in production from 40 items to 120 items. On the average what was the change in profit as production increase one unit at a time? In other words, what is the change in profit per unit change in production from \( q = 40 \) to \( q = 120 \)?

The definition of average rate of change is given as follows:

\[
\text{Definition of Average Rate of Change of } F(x) \text{ over the interval } [x_1, x_2] \text{ is given to be}
\]

\[
\text{The Average Rate of Change in } y = \frac{F(x_2) - F(x_1)}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \text{y units / one unit of } x
\]

We will return again to the original problem. We will now find the average rate of change in profit as production increases from \( q = 40 \) to \( q = 120 \) items.

Answers:
The average rate of change in profit over the interval \([40, 120]\) is found to be

\[
\frac{\Delta P}{\Delta q} = \frac{P(120) - P(40)}{120 - 40} = \frac{14200 - 7800}{80} = 6400 = 80\text{ which means that profit changes at a rate of }$80\text{ per unit change.}
\]

Percent Rate of Change:
Earlier we commented that percent information is sometimes more informative than the actual value information. Therefore we will consider the concept of percent rate of change per unit. The definition of the percent rate of change is given below.

\[
\text{Definition of Percent Rate of Change of } F(x) \text{ over the interval } [x_1, x_2] \text{ is given to be}
\]

\[
\text{The Percent Rate of Change in } y = \frac{F(x_2) - F(x_1)}{x_2 - x_1} \times 100 = \frac{F(x_1)}{F(x_1)}
\]
We will now find the percent rate of change for the profit \( P(q) = -2q^2 + 400q - 5000 \) over the interval \([40, 120]\).

\[
P(x_2) - P(x_1)
\]

The Percent Rate of Change in \( P = \frac{x_2 - x_1}{P(x_1)} = \frac{80}{7800} = .01025 \Rightarrow 1.02\% \) The change in profit is 1.02\% per unit change of \( q \).

The Average Rate of Change and the Slope of the Secant Line.

In the discussion above we discovered that the average rate of change in profit over the interval \([40, 120]\) is $80 per item. Did you notice that the formula for average rate of change is actually the slope formula for a straight line? It turns out to be the slope of a secant line that intersects the graph of the profit function at two points \((40, 7800)\) and \((120, 14200)\).

![Secant line with slope m = 80](image)

Definition for ‘Marginal’:

The term marginal means the average rate of change for a function \( f(q) \) over an interval of one unit change in \( q \) where \( q \) is quantity.

The term marginal is a term that you will see often in your business and economic classes. You will often see the statements marginal cost, marginal revenue, marginal profit and marginal demand. We will now return to the profit equation we discussed previously, \( P(q) = -2q^2 + 400q - 5000 \). The term marginal profit means the average rate of change in the profit over a one unit interval change in quantity. For instance, find the marginal profit over the interval \([20, 21]\).
Solution:
Marginal Profit = \( MP = \frac{P(21) - P(20)}{21-20} \) = 318, The meaning of this answer is as follows;
As the number of items increases from 20 to 21, the total profit will increase by $318.

The term *marginal* is interpreted in the same way when used with cost, revenue and demand equations. If the value is negative then you will have a decrease instead of an increase.