Answers to worksheet 12:

Problem 1)

a)  \( f(x) = 10x + 4 \) on \( \mathbb{R} \)

I)  \( f(x) \) is one to one if \( f(a_1) = f(a_2) \) then \( a_1 = a_2 \).

Assume that \( f(a_1) = f(a_2) \)

Then

\[
10a_1 + 4 = 10a_2 + 4
\]

\[
10a_1 = 10a_2
\]

\[
a_1 = a_2
\]

So \( f(x) \) is one to one

II)  \( f(x) \) is onto if for any \( b \) in the co-domain there exist an \( a \) in the domain such that \( f(a) = b \)

Answer below are abbreviated:

Consider \( a, b \) in the co-domain. Then assume \( b = 10a + 4 \) and solve for \( a \) to see if it does exist. So \( a = (b - 4)/10 \). Therefore \( f(x) \) is onto

b)  I) Determine if \( F(x) = \frac{6x - 1}{x + 3} \) is one to one using the same approach as before, by assuming

\[
\frac{6a_1 - 1}{a_1 + 3} = \frac{6a_2 - 1}{a_2 + 3}
\]

now simplify to get \( a_1 = a_2 \)

II) Determine if \( F(x) \) is onto using the same approach in part a)

Assume \( b = \frac{6a - 1}{a + 3} \) and solve for \( a = -\frac{3b - 1}{b - 6} = \frac{3b + 1}{6 - b}, b \neq 6 \)

So \( F(x) \) is not onto.

c)  I)  Prove “1-1”:  Assume that

\[
\ln(a_1 - 4) = \ln(a_2 - 4)
\]

\[
e^{\ln(a_1 - 4)} = e^{\ln(a_2 - 4)}
\]

\[
a_1 - 4 = a_2 - 4
\]

\[
a_1 = a_2
\]

II)  Now prove “onto”  Assume that
\[ b = \ln(a - 4) \]
\[ e^b = e^{\ln(a-4)} \]
\[ e^b = a - 4 \]
\[ a = e^b + 4 \]

There \( f(x) \) is onto.

Problem 2) Proofs are not properly written here. I am just giving you the body of the proof: Prove that \( g \circ f \) is one to one.

If \( c_1 = c_2 \) then \( a_1 = a_2 \) such that \( g \circ f (a_1) = c_1 \) and \( g \circ f (a_2) = c_2 \)

Proof: Assume that \( c_1 = c_2 \), then \( g \circ f (a_1) = g \circ f (a_2) \) for some \( a_1 \) and \( a_2 \).

\[
(g \circ f)(a_1) = (g \circ f)(a_2)
\]

\[
g(f(a_1)) = g(f(a_2))
\]

Since \( g \) is one to one we know that \( f(a_1) = f(a_2) \)

And Since \( f \) is also one to one, we know that \( a_1 = a_2 \)

(Try to prove the onto on your own)

Problem 3) R1 (yes), R2 (Transitive only), R3 (symmetric only), R4 (Yes)

Problem 4)
\[
f[0] \rightarrow [0]
\]
\[
f[1] \rightarrow [7] = [3]
\]
\[
\]
\[
f[3] \rightarrow [21] = [1]
\]

f(x) is one to one and onto.

Problem 5) Verify that \( zRy \) iff \( 4 \mid z - y \) is an equivalence relation.

Reflexive: Does \( aRa \)? Yes since \( 4 \) divides \( a - a = 0 \)

Symmetric: If \( aRb \), does \( bRa \)?

Yes, since \( 4 \mid a - b \) we have that \( 4q = a-b = -(b-a) \)

And \( 4(-q) = b-a \) and therefore \( 4 \) divides \( b-a \) and \( bRa \).

Transitive: If \( aRb \) and \( bRc \), does \( aRc \)? Yes, since we have the following; \( aq = b \) and \( bd = c \). By substituting \( b \) by \( aq \) we get \( (aq)d = c \) and so \( a(qd) = c \) and \( ak = c \) where \( k = qd \) and is an integer. By definition \( a \) divides \( c \).

Problem 6) Prove that \( R \) is an equivalence relation if \( (a,b) R (c,d) \) iff \( ad = bc \).

Reflexive: Does \( (a,b) R (a,b) \)? Yes, since \( ab = ba \)

ab = ab
Symmetric: If (a,b) R (c,d), does (c,d) R (a,b)?

Yes, since we have that ac = bd, by commutativity, we can get ca = db which give us (c,d)R(a,b)

Transitivity: If (a,b) R (c,d) and (c,d) R (e,f), does (a,b) R (e,f)?

Yes, since we have the following:
ac = bd and ce = df then ac/ce = bd=df since equals divided by equals are equal. This will reduce to a/e = b/f which gives af = be. Therefore (a,b)R(e,f).

Notice that the equivalence classes produced are classes containing equivalent rational numbers, such as \( \frac{2}{3} = \frac{4}{6} = \frac{8}{12} = etc \)

Problem 7) \( R = \{(1,1),(2,2),(1,2),(2,1),(3,3),(5,5),(3,5),(5,3),(4,4)\} \)

Problem 8) If f maps A to B and n(A) > n(B), must f be onto? Try this one on your own. Remember the definition for onto and think about the “image” of f in B.

Problem 9) answer is \( \left\lfloor \frac{500}{365} \right\rfloor +1 \)

Problem 10) a)no, b) yes, c) yes, d)--, e) no, f) no, g) no, h) yes, i) no, j)no

Problem 11) a)no, b) no, c) yes, d)yes, e) no, g) no, h)no, i) no

Problem 12) a)no, b) no inverse function but it does have an inverse relation, c) yes, d)no, e) yes, g) yes, h) no, i) yes

Problem 13) Is R an equivalence relation when ARB iff \( A \cap C = B \cap C \)

Reflexive: ARA? Yes, since \( A \cap C = A \cap C \)

Symmetric: If ARB, Does BRA? Yes, since \( A \cap C = B \cap C \) then \( B \cap C = A \cap C \) equals are certainly commutative.

Transitive: If ARB and BRD, Does ARD? Yes, since we have \( A \cap C = B \cap C \) and \( B \cap C = D \cap C \) Then we know that for all elements in \( A \cap C \), these elements are also in \( B \cap C \) and all the elements in \( B \cap C \) are in \( D \cap C \) therefore we have that \( A \cap C \subseteq D \cap C \). By a similar process we can show that \( D \cap C \subseteq A \cap C \), which gives us \( A \cap C = D \cap C \) and ARD.