Math 2534 Test 2A Fall 2011 Solutions

Problem 1:
1) \( a \in \{ \{a,\{a\}\} \} \), Determine if this is true or false and justify your conclusion.
2) \( P(A) = \{ \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset\} \} \), Find the elements that are in set A.
3) Find the symmetric difference, \( A \oplus B \), when \( A = \{a,c,m,h\} \) and \( B = \{b,c,h,a,k\} \)

Solutions: (9pts)
1) False The only element in this set is \( \{a,\{a\}\} \)
2) \( A = \{\emptyset, \{\emptyset\}\} \)
3) \( A = (A - B) \cup (B - A) = \{m\} \cup \{b, k\} = \{m, b, k\} \)

Problem 2:
Part A:
   a) Draw a Venn Diagram to illustrate the set \( A^C \cup B^C \)
   b) Draw a Venn Diagram to illustrate the set \( (A \cup B)^C \)

See Class Notes: (6 pts)

Part B: Find the mistake in the following Proof: (You should refer to Part A)
Theorem: For all sets A, B, \( A^C \cup B^C \subseteq (A \cup B)^C \)
Proof: Consider
\[ \forall x \in A^C \cup B^C \rightarrow x \in A^C \vee x \in B^C \]
by definition of union
\[ \rightarrow x \notin A x \notin B \]
by definition of complement
\[ \rightarrow x \notin (A \cup B) \]
by definition of union
\[ \rightarrow x \in (A \cup B)^C \]
by definition of complement

Therefore \( A^C \cup B^C \subseteq (A \cup B)^C \)

Solution (6 pts)
\[ \rightarrow x \notin A x \notin B \]
\[ \rightarrow x \notin (A \cup B) \]

This step is invalid since x not in A but could be in B or vice versa. Therefore x could be in the union.
Problem 3: Use SET ALGEBRA to prove the following: (Justify each step)
Theorem: For all sets A and B, \[(A \cup B)^C \cup A^C\] \(\cap B = B - A\)
Solution: (12pts)
Proof: \[(A \cup B)^C \cup A^C\] \(\cap B \equiv\) given
\[(A^C \cap B^C) \cup A^C\] \(\cap B \equiv\) by DeMorgan's Law
\[A^C\] \(\cap B \equiv\) by Absorption Law
\[B \cap A^C \equiv\] by Commutative Law
\[B - A\] Equivalent form of difference
\[\therefore [(A \cup B)^C \cup A^C] \cap B \equiv B - A\]

Problem 4: Prove the following using the properties of a Boolean Algebra.
Theorem: For any element \(b\) in a Boolean Algebra \(B\) where \(k\) is the identity for the
operation \(\heartsuit\) and \(h\) is the identity for the operation \(\bullet\), then \((b \heartsuit h) \bullet (b \bullet k) = b\)
Solution: (12pts)
Proof:
\[(b \heartsuit h) \bullet (b \bullet k) =\] given
\[b \heartsuit (b \bullet k) =\] since \(h\) is the identity element for \(\heartsuit\)
\[(b \heartsuit b) \bullet (b \heartsuit k) =\] by distribution law
\[(b) \bullet (b \heartsuit k) =\] by idempotent law
\[b \bullet (b) =\] since \(k\) is the identity element for \(\heartsuit\)
\[b\] by idempotent law

(there are other proofs that are acceptable as well)
Problem 5: Prove the following using PMI and clearly indicate where you use the inductive hypothesis and justify each inequality.

Theorem: For all natural numbers \( n \geq 2, \ 5^n + 9 < 6^n \)

Solution (20pts)

Proof by PMI

I will first verify that the hypothesis is true for at least one value of \( n \). Consider \( n = 2 \) and we have that \( 5^2 + 9 < 6^2 \) which evaluates to \( 34 < 36 \).

I will now assume that the hypothesis is true from \( n = 2 \) up to some arbitrary value \( k \), i.e. \( 5^k + 9 < 6^k \) and prove that the hypothesis is true for \( k+1 \), \( 5^{k+1} + 9 < 6^{k+1} \).

Consider the \( k+1 \) term:
\[
5^{k+1} + 9 = 5(5^k) + 9 = (5^k + 9) + 4(5^k)
\]

\[
< 6^k + 4(5^k) \quad \text{by the inductive assumption}
\]

\[
< 6^k + 4(6^k) \quad \text{since} \ k \geq 2 \text{ and } 5 < 6
\]

\[
= 5(6^k)
\]

\[
< 6(6^k)
\]

\[
= 6^{k+1}
\]

Since I assumed true up to \( k \) and proved true for \( k+1 \), I have proved that my hypothesis is true for all natural numbers \( n \geq 1 \).

Problem 6: Prove the following using PMI and clearly indicate where you use the inductive hypothesis and state definitions when needed.

Theorem: If given that \( a_1 = 4, \ a_2 = 12, \) and \( a_n = a_{n-1} + a_{n-2}, \ n \geq 3, \) then \( a_n \) is always divisible by 4 for all natural numbers.

Solution (20pts)

Proof by PMI

I will first verify that the hypothesis is true for at least one value of \( n \). Consider \( n = 1 \) and \( a_1 = 4 \) which is clearly divisible by 4.

For \( n = 2 \) we have that \( a_2 = 12 = 4(3) \) and is divisible by 4. Finally I will consider \( n = 3 \) which will \( a_3 = a_2 + a_1 = 12 + 4 = 16 = (4)(4) \) which is divisible by 4.

I will now assume that the hypothesis is true from \( n = 3 \) up to some arbitrary value \( k \) that \( 4 | a_k \) and prove true for \( k + 1 \) that \( 4 | a_{k+1} \).

Consider the \( k+1 \) term:

\[
a_{k+1} = a_k + a_{k-1}
\]

\[
= 4p + 4q
\]

By the inductive assumption we know that each of these terms are divisible by 4 and by the definition of divisible they can be represented by \( 4p \) and \( 4q \) respectively where \( p, q \) are integers.
So we now have the following:

\[ a_{k+1} = a_k + a_{k-1} \]
\[ = 4p + 4q \]
\[ = 4(p + q) \]
\[ = 4m \quad \text{where } m = p + q \text{ is an integer} \]

Therefore by definition of divisible we have that the \( k + 1 \) term is divisible by 4.

Since I assumed true up to \( k \) and proved true for \( k+1 \), I have proved that my hypothesis is true for all natural numbers.

**Problem 7:** The Principle of Math Induction was used to prove that every letter or package that needs 8 cents or greater in postage (i.e. \( n \geq 8 \)) can be mailed using a combination of positive integer postage stamps of only 3 cents and 5 cents. The following proof write up is not complete. You must supply the missing components that will make this a valid proof. Rewrite the statements below (Do NOT change anything already written) inserting your additional remarks to present a well written and complete proof. (There are 3 missing components.)

Proof: I will first verify that the hypothesis is true for some natural number value. Consider \( n = 8 \), we have that \( 8 = 5 + 3 \)

Now assume true for \( k \) and then prove true for \( k+1 \).

Consider the value \( k + 1 \) which can be rewritten to be \( k + 1 = (k - 4) + 5 \). Using our inductive assumption, we have that \( k - 4 \) is in our truth set and is already expressed as a sum of 3 cents and 5 cents. Then \( (k - 4) + 5 \) is certainly a sum of threes and fives. Therefore my hypothesis is true for all natural numbers.

Solution: (15pts)

Proof: I will first verify that the hypothesis is true for at least one natural number value. Consider \( n = 8 \), we have that \( 8 = 5 + 3 \) we will also need to consider the following:

\( n = 9 \) and \( 9 = 3(3) \)
\( n = 10 \) and \( 10 = 2(5) \)
\( n = 11 \) and \( 11 = 5 + 2(3) \)
\( n = 12 \) and \( 12 = 4(3) \)

Now assume true up to some natural number \( k \) and then prove true for \( k+1 \).

Consider the value \( k + 1 \) which can be rewritten to be \( k + 1 = (k - 4) + 5 \). Using our inductive assumption, we have that \( k - 4 \) is in our truth set and is already expressed as a sum of 3 cents and 5 cents. Then \( (k - 4) + 5 \) is certainly a sum of threes and fives. Therefore my hypothesis is true for all natural numbers \( n \geq 8 \)
Pledge: I have neither given nor received help on this test
Signature:_____________________________