Math 2534 Solution to in class PMI problem

Theorem:
If \( f(n) = 2^n - 1 \) and \( a_1, a_2 = 3, a_n = 3a_{n-1} - 2a_{n-2} \), for all natural number \( n > 2 \),
Then \( f(n) = a_n \ \forall n \in N \).

Proof: We will first verify that the hypothesis is true for at least one value of \( n \in N \).
Consider \( n = 1 \) and notice that \( f(1) = 2^1 - 1 = 1 \) and \( a_1 = 1 \)
Now consider \( n = 2 \), \( f(2) = 2^2 - 1 = 3 \) and \( a_2 = 3 \)
I will also verify that the pattern hold for \( n = 3 \),
\( f(3) = 2^3 - 1 = 7 \) and \( a_3 = 3a_2 - 2a_1 = 3(3) - 2(1) = 7 \)

I will now assume that my hypothesis is true from \( n = 1 \) up to some arbitrary value \( k \in N \).
ie: \( f(k) = a_k \) and I will prove true for \( k + 1 \) by showing you that \( f(k + 1) = a_{k+1} \) or to be more exact, I will show that \( 2^{k+1} - 1 = a_{k+1} \)

For the body of proof we will consider that \( k + 1 \) term
\( a_{k+1} = 3a_k - 2a_{k-1} \) by the given definition of recursive sequence in our hypothesis
\( = 3f(k) - 2f(k - 1) \) by the inductive assumption
\( = 3(2^k - 1) - 2(2^{k-1} - 1) \) by the given definition of the function
\( = 3(2^k) - 3 - 2(2^{k-1}) + 2 \)
\( = 3(2^k) - 3 - 2^k + 2 \)
\( = 2(2^k) - 1 \)
\( = 2^{k+1} - 1 \)
\( = f(k + 1) \)
I have shown that \( a_{k+1} = f(k + 1) \)

In conclusion, I have assumed true up to \( k \) and proved true for \( k+1 \). Therefore the hypothesis is true of all natural numbers.